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Voronovskaya-Type Theorems for Derivatives of the Bernstein-Chlodovsky Polynomials and the Szász-Mirakyan Operator

(Dedicated to the late Professor Wladyslaw Orlicz[1] *(1903-1990) in commemoration of his doctoral thesis of 1928 as well as to his Poznań Mathematical School)*

Abstract. This paper is devoted to a study of a Voronovskaya-type theorem for the derivative of the Bernstein–Chlodovsky polynomials and to a comparison of its approximation effectiveness with the corresponding theorem for the much better-known Szász–Mirakyan operator. Since the Chlodovsky polynomials contain a factor b_n tending to infinity having a certain degree of freedom, these polynomials turn out to be generally more efficient in approximating the derivative of the associated function than does the Szász operator. Moreover, whereas Chlodovsky polynomials apply to functions which are even of order $\mathcal{O}(exp(x^p))$ for any $p \geq 1$, the Szász–Mirakyan operator does so only for $p = 1$; it diverges for $p > 1$. The proofs employ but refine practical methods used by Jerzy Albrycht and Jerzy Radecki (in papers which are almost never cited) as well as by further mathematicians from the great Poznań school.

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[1] Władyslaw Roman Orlicz, born 1903 near Cracow, was a student of Banach, Steinhaus and Eustachy Zylinski at the Jan Kazimierz University at Lvov. He completed his doctoral thesis *Some problems in the theory of orthogonal series* in 1928, spent the academic year 1929/1930 in Göttingen, and 1931-1937 at Lvov Technical University, where he received the venia legendi in 1934. Already 1937 he became a Professor at Poznań University, spent the World War II years in Lvov, returning to Poznań in May 1945. There he also worked at the Polish Academy of Sciences.

1. Introduction.

1.1. Bernstein-type polynomial processes.

This paper is first concerned with the classical Bernstein-Chlodovsky operators

$$(1)\qquad (C_n f)(x) := \sum_{k=0}^{n} f\left(\frac{b_n}{n}k\right) p_{k,n}\left(\frac{x}{b_n}\right),$$

where f is a function defined on $[0,\infty)$ and bounded on every finite interval $[0,b] \subset [0,\infty)$ with a certain rate, with $p_{k,n}$ denoting as usual

$$(2)\qquad p_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad 0 \le x \le 1,$$

and $(b_n)_{n=1}^{\infty}$ is a positive increasing sequence of reals with the properties

$$(3)\qquad \lim_{n\to\infty} b_n = \infty \quad , \quad \lim_{n\to\infty} \frac{b_n}{n} = 0.$$

These polynomials were introduced by I. Chlodovsky [10] in 1937 in generalization of the Bernstein polynomials $(B_n f)(x)$, the case $b_n = 1$, $n \in \mathbb{N}_0$, which approximate the function f on the interval $[0,1]$ (or, suitably modified on any fixed finite interval $[-b,b]$).

In fact, if $M(b;f) := \sup\limits_{0\le x\le b} |f(x)|$, then Chlodovsky (see also Lorentz [19], p.36) showed that if

$$(4)\qquad \lim_{n\to\infty} \exp\left(-\alpha\frac{n}{b_n}\right) M(b_n;f) = 0$$

for every $\alpha > 0$, then $(C_n f)(x)$ converges to $f(x)$ at each point of continuity of f.

As a corollary he states that if a function f belonging to $C[0,\infty)$ is of order $f(x) = \mathcal{O}(\exp x^p)$ for some $p > 0$, and if the sequence $\{b_n\}$ satisfies the condition

$$b_n \le n^{\frac{1}{p+1+\eta}},$$

where $\eta > 0$, no matter how small, then $(C_n f)(x)$ converges to $f(x)$ at each point $x \in \mathbb{R}^+$.

Chlodovsky showed more, namely the simultaneous convergence of the derivative $(C_n f)'(x)$ to $f'(x)$ at points x where it exists, a result taken up by Butzer [9].

[1] Retiring in 1974, he still continued his mathematical seminar on Problems of Functional Analysis until a year before his death in 1990. He received many decorations, prizes and medals as well as three honorary doctorates for his seminal contributions to mathematics. Whereas Orlicz spaces appeared for the first time in 1932, the term itself in the sixties in the Math. Reviews (Section 4635) of the AMS. Orlicz was the author of 170 mathematical papers, three books, the supervisor of 40 doctoral dissertations, and had 180 descendants. Orlicz's name is associated with at least a dozen concepts, including the Orlicz-Pettis theorem, Orlicz theorem on unconditional convergence in L^p, Matuszewska-Orlicz indices, Musielak-Orlicz spaces. The first author is proud of the fact that Orlicz visited and lectured at his Aachen chair a handful of times in the sixties and seventies. See L. Maligranda and W.Matuszewska: A survey of Wladyslaw Orlicz's scientific work. In: Wladyslaw Orlicz Collected Papers, PWN, Warsaw 1988, xv-Liv (2 Vol. with 1754 pages).

A first basic question concerning Bernstein polynomials was the rate of approximation by the $(B_nf)(x)$ to $f(x)$, answered by Voronovskaya [30] in 1932. She showed that for bounded f on $[0,1]$, one has the asymptotic formula

$$\lim_{n\to\infty} n[(B_nf)(x_0) - f(x_0)] = \frac{x_0(1-x_0)}{2} f''(x_0) \tag{5}$$

at each fixed point $x_0 \in [0,1]$ for which there exists $f''(x_0) \neq 0$.

Thus the best rate of approximation by Bernstein polynomials is no better than $\mathcal{O}(n^{-1})$ (unless f is a linear function), and it cannot be improved by assuming the existence of higher order derivatives. The portion of this order, $n^{-1/2}$, is due to the existence of $f'(x)$, the other $n^{-1/2}$ to that of $f''(x)$. This follows from a result of T. Popoviciu [25] of 1935 on the degree of approximation in terms of the classical modulus of continuity $\omega(\delta, f; C[0,1])$,

$$|f(x) - B_nf(x)| \leq \begin{cases} \frac{5}{4}\omega(n^{-1/2}, f; C[0,1]) \\ \\ \frac{3}{4}n^{-1/2}\omega(n^{-1/2}, f'; C[0,1]) \end{cases},$$

(see Lorentz [19], pp 20-22). Here it must be recalled that the rate of approximation by Bernstein polynomials is half as good as for polynomials of best approximation (thus $\mathcal{O}(n^{-1})$ if $f' \in \mathrm{Lip}(1, C[0,1])$ and not $\mathcal{O}(n^{-2})$).

The counterpart of (5) for the Chlodovsky polynomials, due to Albrycht[2] and Radecki[3] [2], appeared 28 years later. This difficult result of 1960, which is not cited in the relevant literature, states that under assumption

$$\lim_{n\to\infty} \frac{n}{b_n} \exp\left(-\alpha \frac{n}{b_n}\right) M(b_n; f) = 0 \tag{6}$$

for every $\alpha > 0$,

$$\lim_{n\to\infty} \frac{n}{b_n}[(C_nf)(x) - f(x)] = \frac{x}{2} f''(x) \tag{7}$$

at each point $x \geq 0$ for which $f''(x)$ exists.

An approach, the aim of which possibly included the Voronovskaya-type formula (7), is due to A. Attalienti and M. Campiti [3]. In their introduction they write

[2]Professor Jerzy Albrycht came to Poznań together with Professor Orlicz and Andrzej Alexiewicz from Lvov after World War II. He began his work at the University of Poznań about 1947, his scientific advisor being W. Orlicz. He presented his doctoral thesis "The theory of Markinkiewicz-Orlicz spaces and some applications"in 1959. He worked continuously at the University for 30-35 years, successively as assistant, adjunct, docent and professor. He then became Professor at the Technical University of Poznań, and finally at the Academy of Economics in Poznań. He is the author of numerious publications in mathematics and mathematical economics.

[3]Dr. Jerzy Radecki also studied at the University of Poznań, receiving the Doctor of Mathematical Sciences degree in 1962, with the thesis Ón modified Landau and Bernstein polynomials". His supervisor was also Prof. Orlicz. He has passed away in the meantime.

that " as far as we know, [a Voronovskaya-type formula] cannot be stated for the classical C_n". For this purpose they introduced the "more flexible" polynomials

$$C_n^* f(x) = \sum_{k=0}^{n} f\left(\frac{c_n}{n}k\right)\left(\begin{array}{c} n \\ k \end{array}\right)\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

for which $b_n \le c_n$ for all $n \ge 1$, and $b_n \to 0$, $b_n/n \to 0$, with $b_n - c_n \to 0$, all as $n \to \infty$. They worked in the weighted (polynomial) space

$$C_\alpha := \left\{f \in C[0,\infty); \exists \lim_{x\to\infty} \omega_\alpha(x) f(x) = 0\right\}$$

$$\|f\|_\alpha := \sup_{x\ge 0} \omega_\alpha(x)\,|f(x)|\,,\ \ \omega_\alpha(x) := (1+x^\alpha)^{-1}$$

for some $\alpha > 0$, $C[0,\infty)$ being the space of functions continuous on $\mathbb{R}^+ := [0,\infty)$. Their main theorem stated that

$$\lim_{n\to\infty} \rho_n[C_n^* f(x) - f(x)] = axf''(x) + bxf'(x)$$

in the $C_\alpha - norm$ provided that $f \in C^2[0,\infty) \cap C_\alpha$ $(\alpha \ge 4)$ such that $f'' \in UC_B[0,\infty)$, and where $\{\rho_n\}$ is a divergent increasing sequence of reals such that $\rho_n c_n/n \to 2a$ and $\rho_n(c_n/b_n - 1) \to b$ as $n \to \infty$, $a, b \ge 0$. Above, $UC_B[0,\infty)$ is the space of functions bounded uniformly continuous on $[0,\infty)$. It is a pity these authors were not aware of the paper [2]. Of interest is that they worked with functions that can be unbounded but with a polynomial growth condition at infinity.

It was another 43 years until there appeared the extension of (5) to derivatives. The result of [4] states that for bounded f for which $f'''(x)$ exists at $x \in [0,1]$, one has

$$\lim_{n\to\infty} n[(B_n f)'(x) - f'(x)] = \frac{1-2x}{2} f''(x) + \frac{x(1-x)}{2} f'''(x). \tag{8}$$

The present paper is first of all devoted to the counterpart of (8) for the Chlodovsky polynomials. The theorem states that

$$\lim_{n\to\infty} \frac{n}{b_n}[(C_n f)'(x) - f'(x)] = \frac{f''(x) + xf'''(x)}{2} \tag{9}$$

at each fixed point $x \ge 0$ for which $f'''(x)$ exists, provided that again the growth condition (6) is satisfied. This will be Theorem 1 of Section 3. Section 2 is devoted to several foundational lemmas needed.

The Bernstein-Chlodovsky polynomials, based on functions defined on $[0,\infty)$, which are bounded on every $[0,b_n] \subset [0,\infty)$ with a certain rate, such as (4) and (6) are indeed true polynomials of degree n (in x/b_n), also having support $[0,b_n]$, with $\{b_n\}$ satisfying (3).

Thus they are a very natural polynomial process in approximating unbounded functions on the unbounded infinite interval $[0,\infty)$; but this approximation process is not so easy to handle.

1.2. Szász-Mirakyan operator.

In contrast there is the more elegant, also linear discrete approximation process, defined for functions $f \in C[0,\infty)$, continuous on $[0,\infty)$, namely

$$(S_n f)(x) := e^{-nx} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{(nx)^k}{k!} \quad (x \in [0,\infty)),$$

an operator usually attached with the names Szász-Mirakyan [29], [20].

Although $(S_n f)(x)$ is itself not a polynomial, it does transform polynomials into polynomials of the same degree. In fact, $(S_n t^N)(x) = x^N + q_{N-1}(x)$ with $q_{N-1}(x) \in P_{N-1}$ such that $\lim_{N\to\infty} q_{N-1}(x) = 0$ uniformly on compact subsets of $[0,\infty)$. The proof follows by noting that

$$(S_n t^N)(x) = \sum_{k=0}^{N-1} \binom{N-1}{k} \left(\frac{1}{n}\right)^{N-1-k} x\left(S_n t^k\right)(x).$$

The Voronovskaya result for this operator, thus the counterpart of (7), reads

Let f be bounded on every finite interval such that $f(x) = O(x^N)$, $x \to \infty$, for some $N \in \mathbb{N}$. Then

$$\lim_{n\to\infty} n\left[(S_n f)(x) - f(x)\right] = \frac{1}{2} x f''(x) \tag{10}$$

at each fixed point $x > 0$ for which f is twice differentiable.

This result can be said to have been remarkable for the time in view of the polynomial growth condition O. Szász used. Perhaps it can be regarded as a forerunner of approximation in weighted function spaces. Thus M. Becker [5] worked in such a space, namely

$$C_N := \{f \in C[0,\infty); \omega_N f \text{ uniformly continuous and bounded on } [0,\infty)\}$$

$$\|f\|_N := \sup_{x\geq 0} \omega_N(x)\,|f(x)|\,, \;\; \omega_N(x) := \left(1 + x^N\right)^{-1} \quad (N \in \mathbb{N})$$

in the case of the Szász-Mirakyan operator. However his chief aim was not a Voronovskaya-type result but direct and inverse theorems for this operator.

In this setting the anolog of the (true) pointwise Voronovskaya result (10) of Szász takes the form:

If $f \in C_N^2 := \{f \in C_N : f',\; f'' \in C_N\}$ with fixed $N \in \mathbb{N}_0$, then for each fixed $x_0 \in \mathbb{R}^+$,

$$\lim_{n\to\infty} n\left[(S_n f)(x_0) - f(x_0)\right] = \frac{x_0 f''(x_0)}{2}. \tag{11}$$

The result was established by Z. Walczak [31] for an operator which contains that of Szász-Mirakyan as a particular case.

The second aim of this paper is to study Voronovskaya-type theorems for the derivatives of this operator and to compare the effectiveness of this operator with the Bernstein-Chlodovsky polynomials in general. For this purpose one must first mention the convergence of the derivative $(S_n f)'(x)$ to $f'(x)$, a problem already considered in 1954 [9] under the same hypotheses as those used by Szász.

The second chief result of this paper reads:

If $f \in C_\beta^3$ *for some* $\beta > 0$, *then for every* $x \geq 0$,

$$\lim_{n\to\infty} n\left[(S_n f)'(x) - f'(x)\right] = \frac{f''(x) + xf'''(x)}{2},$$

and if $f \in C_\beta^4$ *for some* $\beta > 0$, *then for every fixed* $x \geq 0$,

$$\lim_{n\to\infty} n\left[(S_n f)''(x) - f''(x)\right] = \frac{1}{2}\left[2f^{(3)}(x) + xf^{(4)}(x)\right].$$

Above, C_β and C_β^r are the weighted exponential spaces

$$C_\beta \ : \ = \{f \in C[0,\infty); \omega_\beta f \text{ uniformly continuous and bounded on } [0,\infty)\}$$

$$C_\beta^r \ : \ = \left\{f \in C_\beta : f^{(k)} \in C_\beta,\ k = 1, 2, ..., r.\right\}$$

$$\|f\|_\beta \ : \ = \sup_{x\geq 0} \omega_\beta(x) \left|f(x)\right|,\ \omega_\beta(x) := e^{-\beta x}.$$

Thus, the foregoing results work not only for functions of polynomial growth but also in exponential weight spaces, where the weight $\omega_N(x)$ has been replaced by $\omega_\beta(x) := \exp(-\beta x)$ with $x \geq 0$, $\beta \geq 0$. This is left to Sections 5/6.

The Bernstein-Chlodovsky polynomials apply to functions f which can even be of order $f(x) = \mathcal{O}(\exp(x^p))$ for any $p \geq 1$, provided b_n is chosen suitably. Thus, if $p = 1$, then $b_n = n^{1/3}$; for $p = 2$, $b_n = n^{1/4}$; for $p = 3$, $b_n = n^{1/5}$, etc. This follows from condition (6).

As to the Szász-Mirakyan operator, it is of practical use only for functions which are at worst of order $f(x) = \mathcal{O}(\exp \beta x)$ for any $\beta > 0$ (a condition which can be slightly weakened; see [17]). In fact, $(S_n(\exp t^p))(x)$ diverges for any $p > 1$, for every fixed $x > 0$; see [15].

The only way to fully match the assertion of (9) is to work with the Szász-Chlodovsky operator

$$\exp\left(-\frac{nx}{b_n}\right)\sum_{k=0}^{\infty} f\left(\frac{kb_n}{n}\right)\left(\frac{nx}{b_n}\right)^k \frac{1}{k!} := (L_n f)(x),$$

carried out by Z. Stypinski[4] [28]. It seems that this operator deserves further study.

[4] Dr. Zenon Stypinski received the Doctor of Math. Sciences degree with the thesis Approximation problems in Orlicz spaces" in 1969. His supervisor at the University of Poznań was Professor Julian Musielak, in turn a doctoral student of Prof. Orlicz. He is no longer alive.

Finally, in Section 7 we take a short look at the Favard operator on $\mathbb{R}$.

2. Auxiliary results for Bernstein-Chlodovsky polynomials.
In this section we present certain results needed to prove our first main theorem.

LEMMA 2.1 *For* $(C_n t^s)(x),\ s=0,1,2,3,4,5,$ *one has for* $0 \le x \le b_n$

$$\begin{aligned}
(C_n 1)(x) &= 1\\
(C_n t)(x) &= x\\
(C_n t^2)(x) &= x^2 + \frac{x(b_n - x)}{n}\\
(C_n t^3)(x) &= x^3\left[\frac{n^2-3n+2}{n^2}\right] + x^2\left[\frac{3b_n(n-1)}{n^2}\right] + x\frac{b_n^2}{n^2}\\
(C_n t^4)(x) &= x^4\left[\frac{n^3-6n^2+11n-6}{n^3}\right] + x^3\left[\frac{6b_n(n^2-3n+2)}{n^3}\right]\\
&\quad + x^2\left[\frac{7b_n^2(n-1)}{n^3}\right] + x\frac{b_n^3}{n^3}
\end{aligned}$$

$$(C_n t^5)(x) = x^5\left[\frac{n^4-10n^3+35n^2-50n+24}{n^4}\right] \tag{12}$$

$$+x^4\left[\frac{10b_n(n^3-6n^2+11n-6)}{n^4}\right]$$

$$+x^3\left[\frac{25b_n^2(n^2-3n+2)}{n^4}\right] + x^2\left[\frac{15b_n^3(n-1)}{n^4}\right] + x\frac{b_n^4}{n^4}. \tag{13}$$

From (13) we easily find the following equalities:

$$(C_n(t-x))(x) = 0, \qquad (C_n(t-x)^2)(x) = \frac{x(b_n-x)}{n}.$$

LEMMA 2.2 *One has for* $p \le n,\ p,n \in N$ *and* $0 \le x \le b_n$

$$(C_n t^p)(x) = \frac{[n]_p}{n^p}x^p + a_{1,p}\frac{b_n}{n}\frac{[n]_{p-1}}{n^{p-1}}x^{p-1} + a_{2,p}\left(\frac{b_n}{n}\right)^2\frac{[n]_{p-2}}{n^{p-2}}x^{p-2} + \ldots + \left(\frac{b_n}{n}\right)^{p-1}x,$$

where the $a_{k,p}$ *are positive numbers satisfying the recurrence formulae*

$$a_{k,p} = \sum_{m=1}^{n-k} m a_{m-1,p-1}, \qquad a_{0,p} = a_{p-1,p} = 1,$$

and $[y]_p := y(y-1)\cdots(y-p+1)$, $[y]_0 := 1$, $y \in R$, *is the falling difference polynomial.*

LEMMA 2.3 *One has for* $f(t) = e^t$ *and* $0 \le x \le b_n$

$$(C_n e^t)(x) = \left[1 + \left(\exp\left(\frac{b_n}{n}\right) - 1\right)\frac{x}{b_n}\right]^n.$$

In particular,

$$\lim_{n\to\infty}(C_n e^t)(x) = e^x.$$

For proofs of Lemmas 2.1, 2.2 and 2.3 see Chlodovsky [10].

LEMMA 2.4 *For the central moments of order* $m \in N_0$

$$T^*_{n,m}(x) := \sum_{k=0}^{n} \left(\frac{kb_n}{n} - x\right)^m p_{k,n}\left(\frac{x}{b_n}\right)$$

one has

$$T^*_{n,0}(x) = 1, \; T^*_{n,1}(x) = 0, \; T^*_{n,2}(x) = \frac{x(b_n - x)}{n}, \; T^*_{n,3}(x) = \frac{x(b_n - x)(b_n - 2x)}{n^2},$$

$$T^*_{n,4}(x) = \frac{x(b_n - x)\left[(b_n - x)(b_n - 2x) + x(4x - 3b_n) + 3nx(b_n - x)\right]}{n^3},$$

and for any fixed $x \in [0, \infty)$,

$$\left|T^*_{n,m}(x)\right| \le A_m(x)\frac{x(b_n - x)}{b_n}\left(\frac{b_n}{n}\right)^{[(m+1)/2]} \quad (n \in \mathbb{N},\; n > b_n), \tag{14}$$

where $A_m(x)$ *denotes a polynomial in* x*, of degree* $[m/2] - 1$*, with non-negative coefficients independent of* n*, and* $[a]$ *denotes the integral part of* a*.*

PROOF Let us consider the well-known recurrence formula ([2],[19]):

$$W_{n,m+1}(t) = t(1 - t)\left[W'_{n,m}(t) + mnW_{n,m-1}(t)\right] \quad m \ge 1$$

for

$$W_{n,m}(t) := \sum_{k=0}^{n}(k - nt)^m p_{k,n}(t), \quad t \in [0, 1].$$

From this formula we easily obtain by mathematical induction the known representations:

$$W_{n,2s}(t) = \sum_{j=1}^{s} \alpha_{j,s,n}\,(t\,(1 - t))^j\, n^j,$$

$$W_{n,2s+1}(t) = (1 - 2t)\sum_{j=1}^{s} \beta_{j,s,n}\,(t\,(1 - t))^j\, n^j,$$

where $s \in \mathbb{N}$ and $\alpha_{j,s,n}$, $\beta_{j,s,n}$ denote real numbers independent of t and bounded uniformly in n (see e.g. [16], Corollary 3.7 (with $c = -1$)).

Taking in the above formulas $t = x/b_n$, $0 \le x \le b_n$, we easily deduce

$$T^*_{n,m}(x) = \frac{b_n^m}{n^m} W_{n,m}\left(\frac{x}{b_n}\right)$$

and consequently the representations:

$$\begin{aligned} T^*_{n,2s}(x) &= \left(\frac{b_n}{n}\right)^{2s} \sum_{j=1}^{s} \alpha_{j,s,n} \left(\frac{x}{b_n}\left(1-\frac{x}{b_n}\right)\right)^j n^j, \\ T^*_{n,2s+1}(x) &= \left(\frac{b_n}{n}\right)^{2s+1} \left(1-2\frac{x}{b_n}\right) \sum_{j=1}^{s} \beta_{j,s,n} \left(\frac{x}{b_n}\left(1-\frac{x}{b_n}\right)\right)^j n^j \end{aligned}$$

for $s \in \mathbb{N}$, where $\alpha_{j,s,n}$, $\beta_{j,s,n}$ are independent of x and bounded uniformly in n.

Hence, for sufficiently large n (such that $n > b_n$) we have

$$\begin{aligned} T^*_{n,2s}(x) &\le \left(\frac{b_n}{n}\right)^{2s} \sum_{j=1}^{s} |\alpha_{j,s,n}|\, x^j \left(1-\frac{x}{b_n}\right)^j \left(\frac{n}{b_n}\right)^j \\ &\le \left(\frac{b_n}{n}\right)^{2s} \frac{x}{b_n}\left(1-\frac{x}{b_n}\right) n \sum_{j=1}^{s} |\alpha_{j,s,n}|\, x^{j-1} \left(\frac{n}{b_n}\right)^{j-1} \\ &\le \frac{x(b_n-x)}{b_n} \left(\frac{b_n}{n}\right)^{s} \sum_{j=0}^{s-1} \gamma_{j,s} x^j. \end{aligned}$$

Analogously, observing that $|1-2x/b_n| \le 1$ if $0 \le x \le b_n$, we get

$$\left|T^*_{n,2s+1}(x)\right| \le \frac{x(b_n-x)}{b_n} \left(\frac{b_n}{n}\right)^{s+1} \sum_{j=0}^{s-1} \eta_{j,s} x^j.$$

In the above estimations $\gamma_{j,s}$ and $\eta_{j,s}$ denote non-negative numbers independent of n and x.

Consequently there follows (14), whereby $A_m(x)$ denotes a polynomial in x, of degree $[m/2]-1$, with non-negative coefficients independent of n. ∎

The first part of the next lemma is due to Chlodovsky [10].

Lemma 2.5 *For $t \in [0,1]$ the inequality*

$$0 \le z \le \frac{3}{2}\sqrt{nt(1-t)}$$

implies

$$\sum_{|k-nt| \ge 2z\sqrt{nt(1-t)}} p_{k,n}(t) \le 2\exp\left(-z^2\right). \tag{15}$$

In particular, for $0 < \delta \le x < b_n$ and sufficiently large n,

$$\sum\nolimits_1^{*} := \sum_{\left|\frac{kb_n}{n}-x\right| \ge \delta} p_{k,n}\left(\frac{x}{b_n}\right) \le 2\exp\left(-\frac{\delta^2}{4x}\frac{n}{b_n}\right). \tag{16}$$

The proof of (16) is given in [2].

According to (1), (2) and (15), there follow by differentiation the two fundamental representations for $(C_n f)'(x)$, also needed,

$$(C_n f)'(x) = \frac{1}{x(b_n - x)} \sum_{k=0}^{n} f\left(\frac{k}{n} b_n\right) (k b_n - n x)\, p_{k,n}\left(\frac{x}{b_n}\right), \quad (0 < x < b_n) \tag{17}$$

and

$$(C_n f)'(x) = \frac{n}{b_n} \sum_{k=0}^{n-1} \left[f\left(\frac{k+1}{n} b_n\right) - f\left(\frac{k}{n} b_n\right) \right] p_{k,n-1}\left(\frac{x}{b_n}\right). \tag{18}$$

3. Voronovskaya-Type Theorem for $(C_n f)'(x)$.

THEOREM 3.1 *Let a function f, defined on $[0, \infty)$, satisfy the growth condition (6) for every $\alpha > 0$, $\{b_n\}$ being a positive sequence satisfying (3). Then there holds,*

$$\lim_{n\to\infty} \frac{n}{b_n} [(C_n f)'(x) - f'(x)] = \frac{f''(x) + x f'''(x)}{2}, \tag{19}$$

at each point $x \geq 0$ at which $f'''(x)$ exists.

PROOF Firstly, the asymptotic formula (19) is valid for $x = 0$. Since $(C_n f)'(0) = (n/b_n)\,[f(b_n/n) - f(0)]$ in view of (18), it suffices to show provided $f''(x)$ *and* $f'''(x)$ exist that

$$\lim_{n\to\infty} \frac{n}{b_n} \left\{ \left(\frac{n}{b_n}\right) \left[f\left(\frac{b_n}{n}\right) - f(0) \right] - f'(0) \right\} = \frac{f''(0)}{2}.$$

Indeed, Taylor's formula (see below) readily yields if $f'''(0)$ exists

$$\frac{n}{b_n} \left\{ \left(\frac{n}{b_n}\right) \left[f\left(\frac{b_n}{n}\right) - f(0) \right] - f'(0) \right\} = \frac{f''(0)}{2} + \frac{b_n}{n} \left[\frac{f'''(0)}{6} + h\left(\frac{b_n}{n}\right) \right],$$

where $h(b_n/n) \to 0$ as $n \to \infty$. Note that $(C_n f)'(0) \neq f'(0)$, all $n \in \mathbb{N}$.

So let $b_n > x > 0$. By Taylor's formula we have

$$\begin{aligned} f\left(\frac{k}{n} b_n\right) &= f(x) + \left(\frac{k}{n} b_n - x\right) f'(x) + \left(\frac{k}{n} b_n - x\right)^2 \frac{f''(x)}{2} \\ &\quad + \left(\frac{k}{n} b_n - x\right)^3 \left[\frac{f'''(x)}{6} + h\left(\frac{k}{n} b_n - x\right) \right], \end{aligned} \tag{20}$$

where $h(y)$ converges to zero with y. Substituting (20) into the representation (17), we can write:

$$(C_n f)'(x) = \frac{n}{x(b_n - x)} \sum_{k=0}^{n} f\left(\frac{k}{n} b_n\right) \left(\frac{k}{n} b_n - x\right) p_{k,n}\left(\frac{x}{b_n}\right) \tag{21}$$

$$\begin{aligned} &= \frac{nf(x)}{x(b_n-x)}\sum_{k=0}^{n}\left(\frac{k}{n}b_n-x\right)p_{k,n}\left(\frac{x}{b_n}\right)+\frac{nf'(x)}{x(b_n-x)}\sum_{k=0}^{n}\left(\frac{k}{n}b_n-x\right)^2 p_{k,n}\left(\frac{x}{b_n}\right)\\ &+\frac{nf''(x)}{2x(b_n-x)}\sum_{k=0}^{n}\left(\frac{k}{n}b_n-x\right)^3 p_{k,n}\left(\frac{x}{b_n}\right)\\ &+\frac{nf'''(x)}{6x(b_n-x)}\sum_{k=0}^{n}\left(\frac{k}{n}b_n-x\right)^4 p_{k,n}\left(\frac{x}{b_n}\right)\\ &+\frac{n}{x(b_n-x)}\sum_{k=0}^{n}h\left(\frac{k}{n}b_n-x\right)\left(\frac{k}{n}b_n-x\right)^4 p_{k,n}\left(\frac{x}{b_n}\right). \end{aligned}$$

According to Lemma 2 we have for (21)

$$(C_nf)'(x)=f'(x)+\frac{nf''(x)}{2x(b_n-x)}T^*_{n,3}(x)+\frac{nf'''(x)}{6x(b_n-x)}T^*_{n,4}(x)+R_n(x), \tag{22}$$

where

$$R_n(x):=\frac{n}{x(b_n-x)}\sum_{k=0}^{n}h\left(\frac{k}{n}b_n-x\right)\left(\frac{k}{n}b_n-x\right)^4 p_{k,n}\left(\frac{x}{b_n}\right).$$

Again by Lemma 2, we can rewrite (22) in the form

$$\begin{aligned} \frac{n}{b_n}\left[(C_nf)'(x)-f'(x)\right] &= \frac{f''(x)}{2}\left(1-\frac{2x}{b_n}\right)+\frac{f'''(x)}{2}x\left(1-\frac{x}{b_n}\right)\\ &+\frac{f'''(x)}{6n}\left[b_n-6x+\frac{6x^2}{b_n}\right]+\frac{n}{b_n}R_n(x). \end{aligned}$$

Now the first two terms on the right hand side as $n\to\infty$ tend to $f''(x)/2$ and $xf'''(x)/2$, respectively, the third to zero, it being of order $o_x(b_n/n)$.

In order to complete the proof, we have to prove

$$\lim_{n\to\infty}\frac{n}{b_n}R_n(x)=0.$$

So we consider now $R_n(x)$. For any $\varepsilon>0$ there exists a $\delta>0$ such that $|h(y)|<\varepsilon$ for $|y|\le\delta$, and we choose δ so small that $\delta\le x$. So we split the sum $R_n(x)$ in two parts as follows:

$$\begin{aligned} R_n(x) &= \frac{n}{x(b_n-x)}\sum_{\left|\frac{kb_n}{n}-x\right|<\delta}h\left(\frac{k}{n}b_n-x\right)\left(\frac{k}{n}b_n-x\right)^4 p_{k,n}\left(\frac{x}{b_n}\right)\\ &+\frac{n}{x(b_n-x)}\sum_{\left|\frac{kb_n}{n}-x\right|\ge\delta}h\left(\frac{k}{n}b_n-x\right)\left(\frac{k}{n}b_n-x\right)^4 p_{k,n}\left(\frac{x}{b_n}\right)\\ &=: R_{n,1}(x)+R_{n,2}(x) \end{aligned}$$

say. According to Lemma 2 once more, one has for $R_{n,1}(x)$ the inequality

$$\begin{aligned} |R_{n,1}(x)| &\leq \varepsilon \frac{n}{x(b_n-x)} T^*_{n,4}(x) \leq \varepsilon \frac{n}{x(b_n-x)} \frac{x(b_n-x)}{b_n} A_4(x) \left(\frac{b_n}{n}\right)^2 \\ &= \varepsilon \frac{b_n}{n} A_4(x), \end{aligned}$$

which implies that

$$\lim_{n\to\infty} \frac{n}{b_n} R_{n,1}(x) = 0. \tag{23}$$

In order to estimate the term $R_{n,2}(x)$, we rewrite representation (20) in the form

$$\begin{aligned} \left(\frac{k}{n}b_n - x\right)^3 h\left(\frac{k}{n}b_n - x\right) &= f\left(\frac{k}{n}b_n\right) - f(x) - \left(\frac{k}{n}b_n - x\right) f'(x) \\ &\quad - \left(\frac{k}{n}b_n - x\right)^2 \frac{f''(x)}{2} - \left(\frac{k}{n}b_n - x\right)^3 \frac{f'''(x)}{6}, \end{aligned}$$

and hence one obtains

$$\begin{aligned} |R_{n,2}(x)| &= \left| \frac{n}{x(b_n-x)} \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} h\left(\frac{k}{n}b_n - x\right)\left(\frac{k}{n}b_n - x\right)^3 \right. \\ &\qquad \left. \left(\frac{k}{n}b_n - x\right) p_{k,n}\left(\frac{x}{b_n}\right) \right| \\ &\leq \frac{n}{x(b_n-x)} \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left| f\left(\frac{k}{n}b_n\right)\right| \left|\frac{k}{n}b_n - x\right| p_{k,n}\left(\frac{x}{b_n}\right) \\ &\quad + |f(x)| \frac{n}{x(b_n-x)} \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left|\frac{k}{n}b_n - x\right| p_{k,n}\left(\frac{x}{b_n}\right) \\ &\quad + |f'(x)| \frac{n}{x(b_n-x)} \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left|\frac{k}{n}b_n - x\right| \left|\frac{k}{n}b_n - x\right| p_{k,n}\left(\frac{x}{b_n}\right) \\ &\quad + \left|\frac{f''(x)}{2}\right| \frac{n}{x(b_n-x)} \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left(\frac{k}{n}b_n - x\right)^2 \left|\frac{k}{n}b_n - x\right| p_{k,n}\left(\frac{x}{b_n}\right) \\ &\quad + \left|\frac{f'''(x)}{6}\right| \frac{n}{x(b_n-x)} \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left|\frac{k}{n}b_n - x\right|^3 \left|\frac{k}{n}b_n - x\right| p_{k,n}\left(\frac{x}{b_n}\right) \\ &=: \frac{n}{x(b_n-x)} \sum\nolimits_2^*(n) + \sum\nolimits_3^*(n) + \sum\nolimits_4^*(n) + \sum\nolimits_5^*(n) + \sum\nolimits_6^*(n), \end{aligned}$$

say.

In view of the Cauchy-Schwarz inequality,

$$\begin{aligned}\sum_{2}^{*}(n) \quad &:= \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta}\left\{\left|f\left(\frac{kb_n}{n}\right)\right|\sqrt{p_{k,n}\left(\frac{x}{b_n}\right)}\left|\frac{kb_n}{n}-x\right|\sqrt{p_{k,n}\left(\frac{x}{b_n}\right)}\right\}\\ &\leq \left\{\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta}\left|f\left(\frac{kb_n}{n}\right)\right|^2 p_{k,n}\left(\frac{x}{b_n}\right)\right\}^{1/2}\\ &\quad\left\{\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta}\left|\frac{kb_n}{n}-x\right|^2 p_{k,n}\left(\frac{x}{b_n}\right)\right\}^{1/2}\\ &=: \left(\sum_{2,1}^{*}\right)\left(\sum_{2,2}^{*}\right),\end{aligned}$$

say. Since $\sqrt{\sup_{0\leq x\leq a}|f(x)|^2} = M(a;f)$,

$$\begin{aligned}\sum_{2,1}^{*}(n) \quad &\leq M(b_n;f)\left\{\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} p_{k,n}\left(\frac{x}{b_n}\right)\right\}^{1/2}\\ &\leq \sqrt{2}M(b_n;f)\exp\left(-\frac{\delta^2}{8x}\frac{n}{b_n}\right).\end{aligned}$$

As to the second product term, noting Lemma 3 and inequality (14) (with $m = 4$), we have

$$\begin{aligned}\frac{n}{x(b_n-x)}\sum_{2,2}^{*}(n) \quad &\leq \frac{n}{x(b_n-x)}\left\{\frac{1}{\delta^2}\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta}\left|\frac{kb_n}{n}-x\right|^4 p_{k,n}\left(\frac{x}{b_n}\right)\right\}^{1/2}\\ &\leq \frac{n}{x(b_n-x)}\left\{\frac{1}{\delta^2}T^*_{n,4}(x)\right\}^{1/2}\\ &\leq \frac{n}{x(b_n-x)}\frac{1}{\delta}\sqrt{A_4(x)}\sqrt{\frac{x(b_n-x)}{b_n}}\left(\frac{b_n}{n}\right)\\ &= \frac{1}{\delta}\frac{\sqrt{A_4(x)}}{\sqrt{x(1-x/b_n)}}.\end{aligned}$$

Thus, altogether we have

$$\lim_{n\to\infty}\frac{n}{b_n}\left[\frac{n}{x(b_n-x)}\sum_{2}^{*}(n)\right] \leq \lim_{n\to\infty}\frac{\sqrt{2A_4(x)}}{\delta\sqrt{x\,(1-x/b_n)}}M(b_n;f)\exp\left(-\frac{\delta^2}{8x}\frac{n}{b_n}\right) = 0.$$

So it remains to show that

$$\lim_{n\to\infty} \frac{n}{b_n} \sum_i^* (n) = 0 \tag{24}$$

is valid for $i = 3, 4, 5, 6$. Indeed

$$\begin{aligned}
\sum_3^* (n) \quad : &= |f(x)| \frac{n}{x(b_n - x)} \sum_{\left|\frac{kb_n}{n} - x\right| \geq \delta} \left|\frac{k}{n} b_n - x\right| p_{k,n}\left(\frac{x}{b_n}\right) \\
&\leq \frac{|f(x)|}{x(b_n - x)} \frac{n}{\delta^5} \sum_{\left|\frac{kb_n}{n} - x\right| \geq \delta} \left|\frac{k}{n} b_n - x\right|^6 p_{k,n}\left(\frac{x}{b_n}\right) \\
&\leq \frac{|f(x)|}{x(b_n - x)} \frac{n}{\delta^5} \frac{x(b_n - x)}{b_n} \left(\frac{b_n}{n}\right)^3 A_6(x) \\
&= \frac{|f(x)|}{\delta^5} \left(\frac{b_n}{n}\right)^2 A_6(x)
\end{aligned}$$

for all $n \in \mathbb{N}$. Hence,

$$\lim_{n\to\infty} \frac{n}{b_n} \sum_3^* (n) \leq \lim_{n\to\infty} \frac{|f(x)|}{\delta^5} A_6(x) \left(\frac{b_n}{n}\right) = 0.$$

Now to the next term, again for $n \in \mathbb{N}$,

$$\begin{aligned}
\sum_4^* (n) \quad &\leq \frac{|f'(x)|}{x(b_n - x)} \frac{n}{\delta^4} \sum_{\left|\frac{kb_n}{n} - x\right| \geq \delta} \left|\frac{k}{n} b_n - x\right|^6 p_{k,n}\left(\frac{x}{b_n}\right) \\
&\leq |f'(x)| \frac{A_6(x)}{\delta^4} \frac{b_n^2}{n^2},
\end{aligned}$$

■

establishing (24) for $i = 4$.

Furthermore, following the same procedure, we have

$$\sum_5^* (n) \leq \frac{|f''(x)|}{2} \frac{A_6(x)}{\delta^3} \frac{b_n^2}{n^2},$$

yielding (24) for $i = 5$.

Since the case for $i = 6$ is similar, this finally establishes the theorem.

Observe that if f would be bounded on the whole $[0, \infty)$, then $|h(y)| \leq M$ for all y, so that

$$|R_{n,2}(x)| \leq M \frac{b_n}{n} A_4(x) \to 0 \quad (n \to \infty);$$

then the above proof would already be complete shortly after relation (23). This beginning part had as its model the proof of Theorem 3.5 [4].

Corollary 3.2 *Let f belong to $C[0,\infty)$ and let it be of order $f(x) = O(exp(x^p))$ on R^+ with some constant $p > 0$. If b_n satisfies condition*

$$b_n = n^{1/(p+1+r)}$$

for any $r > 0$, no matter how small, then the asymptotic formula (19) holds at each point $x \in R^+$ at which $f'''(x)$ exists.

In particular, the assertion holds for $b_n = n^{1/(p+2)}$. Of course, the polynomial C_n does not apply to $f(x) = (exp(e^x))$.

4. Kantorovich-Type Chlodovsky Polynomials.

Given a function f locally integrable on the interval $[0,\infty)$ we define the Kantorovich variant of the Chlodovsky-Bernstein polynomials as

$$(K_n f)(x) := \frac{n+1}{b_{n+1}} \sum_{k=0}^{n} \int\limits_{\frac{kb_{n+1}}{n+1}}^{\frac{(k+1)b_{n+1}}{n+1}} f(u)du\; p_{k,n}\left(\frac{x}{b_{n+1}}\right) \text{if } 0 \le x \le b_{n+1}.$$

If F denotes the indefinite integral of f, i.e., $F(x) = \int\limits_0^x f(u)du$, then $(C_{n+1}F)'(x) = (K_n f)(x)$. Of course, the relation $(C_{n+1}F)'(x) = (K_n f)(x)$ holds for all $x \in [0, b_{n+1}]$ (as for the classical Kantorovich polynomials). Under some assumptions of f one can proof that $\lim_{n\to\infty} K_n f(x) = f(x)$ for almost all x, i.e. for every x at which $F'(x) = f(x)$.

We set

$$M^I(b; f) := \sqrt{\int\limits_0^b |f(u)|^2\, du}.$$

The following result is a corollary of Theorem 3.1.

Corollary 4.1 *If one has*

$$\lim_{n\to\infty} \frac{n}{\sqrt{b_n}} \exp\left(-\alpha \frac{n}{b_n}\right) M^I(b_n; f) = 0$$

for every $\alpha > 0$, then

$$\lim_{n\to\infty} \frac{n+1}{b_{n+1}} \left[(K_n f)(x) - f(x)\right] = \frac{f'(x) + xf''(x)}{2}$$

at each fixed point $x \ge 0$ for which $f''(x)$ exists.

The proof follows by noting

$$M(b_n; F) \le \int_0^{b_n} |f(u)|\, du \le M^I(b_n; f)\sqrt{b_n}.$$

5. Preliminary Results for the Szász-Mirakyan operator.

The basic tools needed are the counterparts of Lemmas 2.4-2.5 in the case of the Szász-Mirakyan operators. They follow in Lemma 5.1, see e.g. M. Becker [5], Stypinski [28].

Lemma 5.1 *For the* $p_{k,n}(x) := \exp(-nx)(nx)^k/k!$, $k \in N_0$, $x \in [0,\infty)$ *and the central moments of order* $m \in N_0$,

$$L_{n,m}(x) := \sum_{k=0}^{\infty} \left(\frac{k}{n} - x\right)^m p_{k,n}(x), \tag{25}$$

there hold

$$(S_n t^0)(x) = \sum_{k=0}^{\infty} p_{k,n}(x) = 1, \quad (S_n t)(x) = \sum_{k=0}^{\infty} \frac{k}{n} p_{k,n}(x) = x,$$

$$L_{n,0}(x) = 1, \quad L_{n,1}(x) = 0, \quad L_{n,2}(x) = \frac{x}{n}, \tag{26}$$

$$L_{n,3}(x) = \frac{x}{n^2}, \qquad L_{n,4}(x) = \frac{3x^2}{n^2} + \frac{x}{n^3}, \tag{27}$$

$$L_{n,5}(x) = \frac{10x^2}{n^3} + \frac{x}{n^4}, \quad L_{n,6}(x) = \frac{15x^3}{n^3} + \frac{25x^2}{n^4} + \frac{x}{n^5}. \tag{28}$$

For any fixed $x \in [0,\infty)$ *one has in addition*

$$0 \le L_{n,m}(x) \le B_m(x)(\frac{1}{n})^{[(m+1)/2]}, \quad (n \to \infty), \tag{29}$$

where $B_m(x)$ *is a polynomial of degree* $[m/2]$, *with non-negative coefficients independent of* n, *and* $[a]$ *denotes the integral part of* a.

Proof At first, we obtain a recurrence formula which we will use in the proof. Differentiating (25) with respect to x, we deduce

$$L'_{n,m}(x) = \sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^m p'_{k,n}(x) - mL_{n,m-1}(x).$$

Since

$$\frac{d}{dx} p_{k,n}(x) = \frac{k - nx}{x} p_{k,n}(x),$$

there follows the recurrence relation

$$\frac{x}{n}\left\{L'_{n,m}(x)+mL_{n,m-1}(x)\right\}=L_{n,m+1}(x). \tag{30}$$

This yields by induction

$$L_{n,m}(x)=\frac{1}{n^m}\sum_{j=1}^{[m/2]}\alpha_{j,m,n}(nx)^j,$$

where $\alpha_{j,m,n}$ are non-negative numbers independent of x and bounded uniformly in n (see [16], Corollary 3.7 (with $c=0$)).

This establishes estimate (29) with $B_m(x)$ given there. ■

In regard to exponential weights the following results will play a central role. The space in question is C_β, defined in Section 1.2. The background to this approach is the basic work of Rolf J. Nessel and his students at Aachen, especially of Michael Becker [5] in polynomial and exponential weight spaces. His work was followed up by many mathematicians, especially those from Poland. Their chief aims were "direct" and the difficult "inverse" approximation theorems with rates and with weights. In the exponential weight case we recall the fundamental properties in the following lemma (see e.g. also Ditzian [12])

Lemma 5.2 *For any $\beta>0$, $n\in N$, set*

$$\beta_n := n\left[\exp(\beta/n)-1\right]. \tag{31}$$

a) $S_n\left(e^{\beta t}\right)(x)=\exp(\beta_n x)$,

b) $S_n\left(te^{\beta t}\right)(x)=x\exp(\beta/n)\exp(\beta_n x)$,

c) $S_n\left(t^2e^{\beta t}\right)(x)=\left[x^2\exp(\beta/n)+\frac{x}{n}\exp(\beta/n)\right]\exp(\beta_n x)$,

d) $S_n\left((t-x)^2e^{\beta t}\right)(x)=\left[\left(\frac{\beta_n x}{n}\right)^2+\frac{x}{n}\exp(\beta/n)\right]\exp(\beta_n x)$,

e) Given $0<\beta<\gamma$, then for any $f\in C_\beta$ one has

$$\|S_nf\|_\gamma\le\|f\|_\beta$$

provided $\gamma\ge\beta_n$, thus if $n\ge n_0(\beta,\gamma)$ with $n_0>\beta/\log(\gamma/\beta)$. The operator S_n does indeed not transform C_β into itself.

Important for the proof of Lemma 5.4 is also the following lemma communicated to the authors by Dr. A. Ciupa (Cluj), see also [11].

Lemma 5.3 *Given $0<\beta<\gamma$, then*

$$w_\gamma(x)S_n\left((t-x)^2e^{\beta t}\right)(x)\le C_{\beta,\gamma}\frac{x+1}{n}\qquad(x\in\mathbb{R}_0),$$

$C_{\beta,\gamma}$ being a constant.

Proof The sequence (β_n) of (31) is decreasing with $\lim_{n\to\infty}\beta_n = \beta$, noting that $\beta_n - \beta \le \beta^2 \exp(\beta/n)$, with

$$\beta < \beta_n \le \beta \exp(\beta/n) \le \beta e^{\beta}. \tag{32}$$

As observed in e),

$$n > n_0 \Rightarrow \gamma > \beta_n. \tag{33}$$

Indeed, $n > n_0$ implies with $n_0 > \beta/\log(\gamma/\beta)$,

$$e^{\beta/n_0} < e^{\beta/\log(\gamma/\beta)/\beta} = \gamma/\beta,$$

so that by (32)

$$\gamma > \beta e^{\beta/n_0} > \beta_{n_0} > \beta_n.$$

Applying d), observing that by (33)

$$x \exp(-x(\gamma - \beta_n)) \le x \exp(-x(\gamma - \beta_{n_0})) \le \frac{1}{\gamma - \beta_{n_0}},$$

there follows when noting again (32)

$$\begin{aligned} e^{-\gamma x} S_n\left((t-x)^2 e^{\beta t}\right)(x) &= \left[\left(\frac{\beta_n x}{n}\right)^2 + \frac{x}{n}\exp(\beta/n)\right]\exp(-x(\gamma-\beta_n)) \\ &\le \frac{x}{n^2}\beta_n^2 \frac{1}{\gamma-\beta_{n_0}} + \frac{1}{n}\exp(\beta/n)\frac{1}{\gamma-\beta_n} \\ &\le \frac{x}{n^2}\frac{\beta^2\exp(2\beta)}{\gamma-\beta_n} + \frac{1}{n}\frac{e^{\beta}}{\gamma-\beta_n} \\ &< C_{\beta,\gamma}\frac{x+1}{n}. \end{aligned}$$

■

The "direct" approximation theorem with rates of [8] is also a basic element in the proofs of Theorems 6.1,6.2. For a generalized Szász-type operator it is to be found in [11], for example.

Lemma 5.4 *Let $f \in C_\beta$ with some $\beta > 0$, $\gamma > \beta$, and let n_0 be a fixed natural number with $n_0 > \beta/\log(\gamma/\beta)$. Then there exists a constant $C_{\beta,\gamma} > 0$, depending only on β, γ, such that one has for $S_n f$, for all $x \ge 0$ and $n \ge n_0$,*

$$w_\gamma(x)\left|S_n f(x) - f(x)\right| \le C_{\beta,\gamma}\ \omega\left(f, C_\beta; \left(\frac{x+1}{n}\right)^{1/2}\right),$$

where

$$\omega(f, C_\beta; \delta) := \sup_{0<h\le\delta} \|f(.+h) - f(.)\|_\beta .$$

In particular, $S_n f$ converges to f(x) for each $x \ge 0$, and uniformly on any compact subinterval of $[0,\infty)$.

6. Voronovskaya-Type Theorems for $(S_nf)'(x)$ and $(S_nf)''(x)$.

This section is devoted to the second chief result of this paper, thus the counterpart of that of Z. Walczak [31], namely, (11) for derivatives and exponential weight spaces. A similar result for a certain Szász-Mirakyan type operator is due to L. Rempulska-M. Skorupka [27].

This Voronovskaya-type theorem for the derivative $(S_nf)'(x)$ reads

THEOREM 6.1 *Let $f \in C^3_\beta$ for some $\beta > 0$. Then for every $x \geq 0$,*

$$\lim_{n\to\infty} n\left[(S_nf)'(x) - f'(x)\right] = \frac{f''(x) + xf'''(x)}{2}.$$

PROOF As in the proof of Theorem 3.1, we will show that the asymptotic formula is valid for $x = 0$. Since

$$(S_nf)'(x) = n\sum_{k=0}^{\infty}\left[f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right]p_{k,n}(x),$$

so that $(S_nf)'(0) = n\left[f(1/n) - f(0)\right]$, it suffices to show provided $f''(x)$ *and* $f'''(x)$ exist that

$$\lim_{n\to\infty} n\left\{n\left[f\left(\frac{1}{n}\right) - f(0)\right] - f'(0)\right\} = \frac{f''(0)}{2}.$$

Indeed, Taylor's formula yields if $f'''(0)$ exists

$$n\left\{n\left[f\left(\frac{1}{n}\right) - f(0)\right] - f'(0)\right\} = \frac{f''(0)}{2} + \frac{1}{n}\left[\frac{f'''(0)}{6} + h\left(\frac{1}{n}\right)\right],$$

where $h(1/n) \to 0$ as $n \to \infty$. Note that $(S_nf)'(0) \neq f'(0)$, all $n \in \mathbb{N}$.

So let $x > 0$ and note that since

$$(S_nf)'(x) = \frac{1}{x}\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)(k - nx)p_{k,n}(x), \text{ for } x > 0,$$

a comparison with

$$(S_n(t-x)f(t))(x) = \sum_{k=0}^{\infty}\left(\frac{k}{n} - x\right) f\left(\frac{k}{n}\right)p_{k,n}(x)$$

gives

$$(S_nf)'(x) = \frac{n}{x}(S_n(t-x)f(t))(x). \tag{34}$$

Substituting now Taylor's formula into (34), namely

$$f(t) = \sum_{k=0}^{3}(t-x)^k\frac{f^{(k)}(x)}{k!} + (t-x)^3h(t;x),$$

where $h(.;x) \in C_\beta$ with $\lim_{t\to x} h(t;x) = 0$, there follows

$$\begin{aligned} (S_n f)'(x) &= \frac{n}{x}\left(S_n(t-x)\left(\sum_{k=0}^{3}(t-x)^k\frac{f^{(k)}(x)}{k!} + (t-x)^3 h(t;x)\right)\right)(x) \\ &= \frac{n}{x}\sum_{k=1}^{3}\frac{f^{(k)}(x)}{k!}L_{n,k+1}(x) + \frac{n}{x}\left(S_n h(t;x)(t-x)^4\right)(x). \end{aligned}$$

Consequently, since $nf'(x) = (n^2/x)f'(x)x/n$,

$$n\left[(S_n f)'(x) - f'(x)\right] = \frac{n^2}{x}f'(x)\left\{L_{n,2}(x) - \frac{x}{n}\right\}$$

$$+\frac{n^2}{2x}f''(x)L_{n,3}(x) + \frac{n^2}{6x}f'''(x)L_{n,4}(x) + \frac{n^2}{x}\left(S_n h(t;x)(t-x)^4\right)(x).$$

Now the function $h^2(t;x)$ satisfies the hypotheses of Lemma 7. Hence

$$\lim_{n\to\infty} S_n\left(h^2(t;x)\right)(x) = h^2(x;x) = 0.$$

Further, in view of inequality (29) with $m = 8$, we have by the Cauchy-Schwarz inequality for $n \to \infty$,

$$n^2\left|S_n\left(h(t;x)(t-x)^4\right)(x)\right| \le \sqrt{S_n\left(h^2(t;x)\right)(x)}\sqrt{n^4 L_{n,8}(x)} \to 0,$$

Also recalling that (26), (27) and (28), with

$$\frac{n^2}{6x}L_{n,4}(x) = \frac{n^2}{6x}\frac{3n^2x^2 + nx}{n^4} \to \frac{x}{2} \qquad (n\to\infty),$$

the proof of the theorem follows by collecting the results. ∎

A more sophisticated version of the proof of Theorem 6.1 yields a Voronovskaya-type theorem for the second derivative $(S_n f)''(x)$ in case of exponential weights,

THEOREM 6.2 *Let $f \in C_\beta^4$ for some $\beta > 0$. Then for every fixed $x \ge 0$*

$$\lim_{n\to\infty} n\left[(S_n f)''(x) - f''(x)\right] = \frac{1}{2}\left[2f^{(3)}(x) + xf^{(4)}(x)\right].$$

PROOF Again, we first consider the case $x = 0$. Since

$$(S_n f)''(x) = n^2\sum_{k=0}^{\infty}\left[f\left(\frac{k+2}{n}\right) - 2f\left(\frac{k+1}{n}\right) + f\left(\frac{k}{n}\right)\right]p_{k,n}(x),$$

one has $(S_n f)''(0) = n^2\left[f(2/n) - 2f(1/n) + f(0)\right]$. It suffices to show provided $f^{(4)}(x)$ exists that

$$\lim_{n\to\infty} n\left\{n^2\left[f(2/n) - 2f(1/n) + f(0)\right] - f''(0)\right\} = f^{(3)}(0).$$

Taylor's formula again yields if $f^{(4)}(0)$ exists

$$n\left\{n^2\left[f(2/n)-2f(1/n)+f(0)\right]-f''(0)\right\}=f^{(3)}(0)+\frac{14}{n}\left[\frac{f^{(4)}(0)}{24}+h\left(\frac{1}{n}\right)\right],$$

where $h\left(1/n\right)\to 0$ as $n\to\infty$. Note that $(S_nf)''(0)\neq f''(0)$, all $n\in\mathbb{N}$.

So let $x>0$. Differentiating $p_{k,n}(x)$ twice,

$$p'_{k,n}(x)=\frac{k-nx}{x}p_{k,n}(x)\quad,\quad p''_{k,n}(x)=\frac{(k-nx)^2}{x^2}p_{k,n}(x)-\frac{k}{x^2}p_{k,n}(x).$$

Thus we have

$$\left(S_nf\right)''(x)=\frac{1}{x^2}\sum_{k=0}^{\infty}f\left(\frac{k}{n}\right)(k-nx)^2\,p_{k,n}(x)-\frac{1}{x^2}\sum_{k=0}^{\infty}kf\left(\frac{k}{n}\right)p_{k,n}(x),$$

and a comparison with

$$\left(S_n(t-x)^2f(t)\right)(x)=\sum_{k=0}^{\infty}\left(\frac{k}{n}-x\right)^2f\left(\frac{k}{n}\right)p_{k,n}(x)$$

as well as

$$\left(S_ntf(t)\right)(x)=\sum_{k=0}^{\infty}\frac{k}{n}f\left(\frac{k}{n}\right)p_{k,n}(x)=\left(S_n(t-x)f(t)\right)(x)+x\left(S_nf(t)\right)(x)$$

yields the identity, basic below,
(35)

$$\left(S_nf\right)''(x)=\frac{n^2}{x^2}\left(S_n(t-x)^2f(t)\right)(x)-\frac{n}{x^2}\left[\left(S_n(t-x)f(t)\right)(x)+x\left(S_nf(t)\right)(x)\right].$$

Substituting Taylor's formula into (35), namely

$$f(t)=\sum_{k=0}^{4}(t-x)^k\frac{f^{(k)}(x)}{k!}+(t-x)^4h(t;x),$$

where $h(.;x) \in C_\beta$ with $\lim_{t\to x} h(t;x) = 0$, there follows

$$\begin{aligned}
(S_n f)''(x) &= \frac{n^2}{x^2}\left(S_n (t-x)^2\left(\sum_{k=0}^{4}(t-x)^k\frac{f^{(k)}(x)}{k!} + (t-x)^4 h(t;x)\right)\right)(x)\\
&\quad -\frac{n}{x^2}\left(S_n (t-x)\left(\sum_{k=0}^{4}(t-x)^k\frac{f^{(k)}(x)}{k!} + (t-x)^4 h(t;x)\right)\right)(x)\\
&\quad -\frac{n}{x}\left(S_n \left(\sum_{k=0}^{4}(t-x)^k\frac{f^{(k)}(x)}{k!} + (t-x)^4 h(t;x)\right)\right)(x)\\
&= \frac{n^2}{x^2}\sum_{k=0}^{4}\frac{f^{(k)}(x)}{k!}L_{n,k+2}(x) + \frac{n^2}{x^2}\left(S_n h(t;x)(t-x)^6\right)(x)\\
&\quad -\frac{n}{x^2}\sum_{k=0}^{4}\frac{f^{(k)}(x)}{k!}L_{n,k+1}(x) - \frac{n}{x^2}\left(S_n h(t;x)(t-x)^5\right)(x)\\
&\quad -\frac{n}{x}\sum_{k=0}^{4}\frac{f^{(k)}(x)}{k!}L_{n,k}(x) - \frac{n}{x}\left(S_n h(t;x)(t-x)^4\right)(x).
\end{aligned}$$

Consequently

$$\begin{aligned}
(S_n f)''(x) &= f(x)\left[\frac{n^2}{x^2}L_{n,2}(x) - \frac{n}{x^2}L_{n,1}(x) - \frac{n}{x}L_{n,0}(x)\right]\\
&\quad f'(x)\left[\frac{n^2}{x^2}L_{n,3}(x) - \frac{n}{x^2}L_{n,2}(x) - \frac{n}{x}L_{n,1}(x)\right]\\
&\quad +\frac{f''(x)}{2}\left[\frac{n^2}{x^2}L_{n,4}(x) - \frac{n}{x^2}L_{n,3}(x) - \frac{n}{x}L_{n,2}(x)\right]\\
&\quad +\frac{f'''(x)}{6}\left[\frac{n^2}{x^2}L_{n,5}(x) - \frac{n}{x^2}L_{n,4}(x) - \frac{n}{x}L_{n,3}(x)\right]\\
&\quad +\frac{f^{(4)}(x)}{24}\left[\frac{n^2}{x^2}L_{n,6}(x) - \frac{n}{x^2}L_{n,5}(x) - \frac{n}{x}L_{n,4}(x)\right]\\
&\quad +\frac{n^2}{x^2}\left(S_n h(t;x)(t-x)^6\right)(x) - \frac{n}{x^2}\left(S_n h(t;x)(t-x)^5\right)(x)\\
&\quad -\frac{n}{x}\left(S_n h(t;x)(t-x)^4\right)(x).
\end{aligned}$$

Also recalling (26)-(28), one has

$$n\left[(S_n f)''(x) - f''(x)\right] = f'''(x) + \frac{7}{12n}f^{(4)}(x) + \frac{x}{2}f^{(4)}(x)$$

$$+\frac{n^3}{x^2}\left(S_n h(t;x)\,(t-x)^6\right)(x) - \frac{n^2}{x^2}\left(S_n h(t;x)(t-x)^5\right)(x) - \frac{n^2}{x}\left(S_n h(t;x)(t-x)^4\right)(x).$$

Now the function $h^2(t;x)$ satisfies the hypotheses of Lemma 7. Hence

$$\lim_{n\to\infty} S_n\left(h^2(t;x)\right)(x) = h^2(x;x) = 0.$$

Further, recalling (29) with $m = 2r, r = 4,5,6$, and using the Cauchy-Schwarz inequality we obtain

$$n^{[r/2]} \left| S_n \left(h(t;x)(t-x)^r \right)(x) \right| \leq \sqrt{S_n \left(h^2(t;x) \right)(x)} \sqrt{A_{2r}(x)} \to 0,$$

as $n \to \infty$. Thus the proof of the theorem is complete. ∎

U. Abel and M. Ivan [1] established the foregoing result with a different method, one based on the complete asymptotic expansion of $(S_n f)''$. It also works for higher derivatives but it tends to be rather technical.

7. The Favard operator on $(-\infty, \infty)$.

An operator giving a genuine approximation on the whole real axis $\mathbb{R}$ is that of Jean Favard [13] of 1944, namely

$$F_n f(x) := \frac{1}{\sqrt{\pi n}} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \exp\left(-n\left(\frac{k}{n} - x\right)^2\right),$$

who proved the following result:

Let $f \in C(-\infty, \infty)$ such that $|f(x)| \leq A \exp\left(Bx^2\right)$ for certain constants A, B and $x \in \mathbb{R}$. Then

$$\lim_{n \to \infty} F_n f(x) = f(x)$$

pointwise for every $x \in \mathbb{R}$, and uniformly on any compact subinterval of $\mathbb{R}$.

Becker, Butzer and Nessel [7] studied this operator not only on the polynomial weight space

$$C_{2N}(-\infty,\infty) \;:\; = \left\{ f \in C(\mathbb{R});\ f(x) = o(1 + x^{2N}),\ |x| \to \infty \right\}$$

$$\|f\|_{C_{2N}} \;:\; = \sup_{x \in \mathbb{R}} \left| (1 + x^{2N})^{-1} f(x) \right|,$$

but also on the exponential weight space

$$C_{2\beta}(\mathbb{R}) := \left\{ f \in C(\mathbb{R});\ \sup_{x \in \mathbb{R}} \left| e^{-\beta x^2} f(x) \right| < \infty \right\}$$

for fixed $N \in \mathbb{N}$ and $\beta > 0$.

In the latter case F_n maps $C_{2\beta}(\mathbb{R})$ into $C_{2\gamma}(\mathbb{R})$ for any $\gamma > \beta$, provided n is sufficiently large.

So instead of working in the individual Banach space $C_{2\gamma}(\mathbb{R})$ it is appropriate to study F_n in the more general locally convex weight space

$$C^*(\mathbb{R}) := \bigcap_{\beta > 0} C_{2\beta}(\mathbb{R}).$$

The Voronovskaya-type theorem for the F_n themselves here reads:

If f, f' *and* f'' *belong to* $C^*(\mathbb{R})$ with $\beta > 0$, *then*

$$\lim_{n\to\infty} \left\| e^{-\beta x^2} \left\{ n\left[F_n f(x) - f(x)\right] - \frac{f''(x)}{4} \right\} \right\|_{C(\mathbb{R})} = 0.$$

The paper [7] together with Becker [6], who established direct and inverse theorems for the Favard operator in the polynomial weight space, seem to have initiated a whole variety of papers dealing with various aspects or generalizations of the Favard operator. Thus there are the papers by Gawronski and Stadtmüller [14] and Kratz and Stadtmüller [18]. They were followed up by mathematicians from the Poznań school, thus, for example by Pych-Taberska [26], Nowak and Pych-Taberska [23], [24], Nowak and Sikorska-Nowak [22] and Nowak [21]. They worked not only in the above exponential weight space but also in

$$L_{p,\beta}(\mathbb{R}) = \left\{ f;\ \left\| e^{-\beta x^2} f(x) \right\|_p < \infty \right\}$$

for $1 \le p \le \infty$, and studied not only a generalized Favard operator but also the Favard-Kantorovich operator as well as generalized Favard-Durrmeyer operators.

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