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Semi-compatibility in non-archimedean Menger PM-space

Abstract. The object of this paper is to establish fixed point theorem for six self maps and an example using the concept of semi-compatible self maps in a non-Archimedean Menger PM-space. Our result generalizes the result of Cho et. al. [2].

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1. Introduction. There have been a number of generalizations of metric space. One such generalization is Menger space initiated by Menger [11]. It is a probabilistic generalization in which we assign to any two points x and y, a distribution function $F_{x,y}$. Schweizer and Sklar [13] studied this concept and gave some fundamental results on this space

The notion of compatible mapping in a Menger space has been introduced by Mishra [12]. Using the concept of compatible mappings of type (A), Jain et. al. [5, 6] proved some interesting fixed point theorems in Menger space. Afterwards, Jain et. al. [7] proved the fixed point theorem using the concept of weak compatible maps in Menger space. Cho, Sharma and Sahu [4] introduced the concept of semi-compatibility in a d-complete topological space. In Menger space, Singh et. al. [15] defined the concept of semi-compatibility of pair of self-maps.

The notion of non-Archimedean Menger space has been established by Istratescu and Crivat [10]. The existence of fixed point of mappings on non- Archimedean Menger space has been given by Istratescu [9]. This has been the extension of the results of Sehgal and Bharucha - Reid [14] on a Menger space. Cho. et. al. [2] proved a common fixed point theorem for compatible mappings in non- Archimedean Menger PM-space.

In this paper, we generalize the result of Cho et. al. [2] by introducing the notion of semi-compatible self maps. Also, we cited an example in support of this.

2. Preliminaries. For terminologies, notations and properties of probabilistic metric spaces, refer to [1], [8] and [14].

Definition 2.1 ([2]) Let X be a non-empty set and $\mathcal{D}$ be the set of all left-continuous distribution functions. An order pair $(X, \mathcal{F})$ is called a non-Archimedean probabilistic metric space (briefly, a N.A. PM-space) if $\mathcal{F}$ is a mapping from $X \times X$ into $\mathcal{D}$ satisfying the following conditions (the distribution function $\mathcal{F}(x, y)$ is denoted by $F_{x,y}$ for all $x, y \in X$):

(PM-1) $$F_{u,v}(x) = 1, \text{ for all } x > 0, \text{ if and only if } u = v;$$

(PM-2) $$F_{u,v} = F_{v,u};$$

(PM-1) $$F_{u,v}(0) = 0;$$

(PM-4) $$\text{If } F_{u,v}(x) = 1 \text{ and } F_{v,w}(y) = 1$$
$$\text{then } F_{u,w}(max\{x, y\}) = 1, \text{ for all } u, v, w \in X \text{ and } x, y \geq 0.$$

Definition 2.2 ([2]) A t-norm is a function $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ which is associative, commutative, nondecreasing in each coordinate and $\Delta(a, 1) = a$ for every $a \in [0, 1]$.

Definition 2.3 ([2]) A N.A. Menger PM-space is an order triple $(X, \mathcal{F}, \Delta)$, where $(X, \mathcal{F})$ is a non-Archimedean PM-space and Δ is a t-norm satisfying the following condition:

(PM-5) $F_{u,w}(max\{x, y\}) \geq \Delta(F_{u,v}(x), F_{v,w}(y))$, for all $u, v, w \in X$ and $x, y \geq 0$.

Definition 2.4 ([2]) A PM-space $(X, \mathcal{F})$ is said to be of type $(C)_g$ if there exists a $g \in \Omega$ such that
$$g(F_{x,y}(t)) \leq g(F_{x,z}(t)) + g(F_{z,y}(t))$$
for all $x, y, z \in X$ and $t \geq 0$, where $\Omega = \{g : \; g : [0, 1] \to [0, \infty)$ is continuous, strictly decreasing, $g(1) = 0$ and $g(0) < \infty\}$.

Definition 2.5 ([2]) A N.A. Menger PM-space $(X, \mathcal{F}, \Delta)$ is said to be of type $(D)_g$ if there exists a $g \in \Omega$ such that
$$g(\Delta(s, t)) \leq g(s) + g(t)$$
for all $s, t \in [0, 1]$.

REMARK 2.6 ([3]) (1) If a N.A. Menger PM-space $(X, \mathcal{F}, \Delta)$ is of type $(D)_g$ then $(X, \mathcal{F}, \Delta)$ is of type $(C)_g$.

(2) If a N.A. Menger PM-space $(X, \mathcal{F}, \Delta)$ is of type $(D)_g$, then it is metrizable, where the metric d on X is defined by

$$(*) \qquad d(x, y) = \int_0^1 g(F_{x,y}(t))d(t) \text{ for all } x, y \in X.$$

Throughout this paper, suppose $(X, \mathcal{F}, \Delta)$ be a complete N.A. Menger PM-space of type $(D)_g$ with a continuous strictly increasing t-norm Δ. Let $\phi : [0, \infty) \to [0, \infty)$ be a function satisfied the condition (Φ) :

(Φ) ϕ is upper-semicontinuous from the right and $\phi(t) < t$ for all $t > 0$.

LEMMA 2.7 *If a function $\phi : [0, \infty) \to [0, \infty)$ satisfies the condition (Φ), then we have*

(1) *For all $t \geq 0$, $\lim_{n\to\infty} \phi^n(t) = 0$, where $\phi^n(t)$ is n-th iteration of $\phi(t)$.*

(2) *If $\{t_n\}$ is a non-decreasing sequence of real numbers and $t_{n+1} \leq \phi(t_n)$, $n = 1, 2, \ldots$ then $\lim_{n\to\infty} t_n = 0$. In particular, if $t \leq \phi(t)$ for all $t \geq 0$, then $t = 0$.*

DEFINITION 2.8 ([2]) Let $A, S : X \to X$ be mappings. A and S are said to be compatible if $\lim_{n\to\infty} g(F_{SA_{x_n},AS_{x_n}}(t)) = 0$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = Sx_n = z$ for some z in X.

DEFINITION 2.9 Let $A, S : X \to X$ be mappings. A and S are said to be semi-compatible if $\lim_{n\to\infty} g(F_{ASx_n,Sz}) = 0$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = Sx_n = z$ for some z in X.

PROPOSITION 2.10 *If (S, T) is a semi-compatible pair of self maps in a N.A. Menger PM-space $(X, \mathcal{F}, \Delta)$ and T is continuous then (S, T) is compatible.*

PROOF Consider a sequence $\{x_n\}$ in X such that $\{Sx_n\}\to u$ and $\{Tx_n\}\to u$ as $n\to\infty$. As T is continuous we get $TSx_n\to Tu$ as $n\to\infty$. By semi-compatibility of (S, T), we have $\lim_{n\to\infty} g(F_{ST_{x_n},Tu}(t))$ for all $t > 0$ and so

$$\lim_{n\to\infty} g(F_{ST_{x_n},TS_{x_n}}(t)) \leq g(F_{ST_{x_n},Tu}(t)) + g(F_{Tu,TS_{x_n}}(t))\to 0$$

as $n\to\infty$. Hence, the pair (S, T) is compatible. ∎

The following is an example of pair of self maps in a N.A. Menger PM-space which are semi-compatible but not compatible.

EXAMPLE 2.11 Let $(X, \mathcal{F}, \Delta)$ be the N.A. Menger PM-space, where $X = [0, 2]$ and the metric d on X is defined in condition (*) of Remark 2.6. Define self maps A and S as follows :

$$Ax = \begin{cases} 2 - x, & \text{if } 0 \le x < 1 \\ 2, & \text{if } 1 \le x \le 2 \end{cases} \text{ and } Sx = \begin{cases} x & \text{if } 0 \le x < 1 \\ 2 & \text{if } 1 \le x \le 2 \end{cases}$$

Take $x_n = 1 - 1/n$. Then $Ax_n \to 1$ as $n \to \infty$. Similarly, $Sx_n \to 1$ as $n \to \infty$. Therefore, $\lim_{n\to\infty} g(F_{ASx_n, SAx_n}(t)) \neq 0 \ \forall t \ge 0$. Hence, the pair (A, S) is not compatible.

Also, if $\lim_{n\to\infty} x_n = 1 = u$ (say), then $\lim_{n\to\infty} g(F_{ASx_n, Su}(t)) = 0 \ \forall t \ge 0$. Hence, the pair (A, S) is semi-compatible.

From the above example it is obvious that the concept of semi-compatibility is more general than that of compatibility.

PROPOSITION 2.12 *Let A and S be compatible self maps of a N.A. Menger PM-space $(X, \mathcal{F}, \Delta)$ and let $\{x_n\}$ be a sequence in X such that Ax_n, $Sx_n \to u$ for some u in X. Then $ASx_n \to Su$ provided S is continuous.*

PROOF Suppose S is continuous at u. Since Ax_n, $Sx_n \to u$ for some $u \in X$, $SSx_n \to Su$ as $n \to \infty$. Since A and S are compatible maps,

$$\lim_{n\to\infty} g(F_{ASx_n, SAx_n}(t)) = 0, \quad \forall t \ge 0.$$

Hence, we have

$$g(F_{ASx_n, Su}(t)) \le g(F_{ASx_n, SAx_n}(t)) + g(F_{SAx_n, Su}(t)) \to 0$$

for all $t > 0$, as $n \to \infty$, which implies that $ASx_n \to Su$ as $n \to \infty$. ■

PROPOSITION 2.13 *Let S and T be compatible self maps of N.A. Menger PM-space $(X, \mathcal{F}, \Delta)$ and $Su = Tu$ for some u in X then $STu = TSu = SSu = TTu$.*

PROOF Let $\{x_n\}$ be a sequence in X defined as $x_n = u$, $n = 1, 2, 3, \ldots$ and $Su = Tu$. Then we have Sx_n, $Tx_n \to Su$. Since S and T are compatible and so for $t > 0$, we have

$$g(F_{STu, TTu}(t)) = \lim_{n\to\infty} g(F_{STx_n, TSx_n}(t)) = 0.$$

Hence $STu = TTu$. Similarly $TSu = SSu$. But $Su = Tu$ implies that $TTu = TSu$. Hence $STu = TSu = SSu = TTu$. ■

LEMMA 2.14 ([2]) *Let $A, B, S, T : X \to X$ be mappings satisfying the condition (1) and (2) as follows :*

(1) *$A(X) \subset T(X)$ and $B(X) \subset S(X)$.*

(2)

$$g(F_{Ax,By}(t)) \le \phi(max\{ g(F_{Sx,Ty}(t)), g(F_{Sx,Ax}(t)), g(F_{Ty,By}(t)), 0.5(g(F_{Sx,By}(t)) + g(F_{Ty,Ax}(t))) \})$$

for all $t > 0$, where a function $\phi : [0, +\infty) \to [0, +\infty)$ satisfies the condition (Φ). Then the sequence $\{y_n\}$ in X, defined by $Ax_{2n} = Tx_{2n+1} = y_{2n}$ and $Bx_{2n+1} = Sx_{2n+2} = y_{2n+1}$ for $n = 0, 1, 2, \ldots,$ such that $\lim_{n\to\infty} g(F_{y_n,y_{n+1}}(t)) = 0$ for all $t > 0$ is a Cauchy sequence in X.

Cho et. al. [2] established the following result :

THEOREM 2.15 ([2]) *Let $A, B, S, T : X \to X$ be mappings satisfying the conditions (1), (2), (3), (4),*

(3) *S and T is continuous,*

(4) *the pairs (A, S) and (B, T) are compatible maps.*

Then A, B, S and T have a unique common fixed point in X.

3. Main Result. In the following, we extend this result to six self maps and generalize it in other respects too.

THEOREM 3.1 *Let $A, B, S, T, L, M : X \to X$ be mappings satisfying the conditions*

(3.1.1) $$L(X) \subset ST(X), \quad M(X) \subset AB(X);$$

(3.1.2) $$AB = BA, \ ST = TS, \ LB = BL, \ MT = TM;$$

(3.1.3) *either AB or L is continuous*;

(3.1.4) *(L, AB) is compatible and (M, ST) is semi-compatible*;

(3.1.5) $$g(F_{Lx,My}(t)) \le \phi(max\ \{g(F_{ABx,STy}(t)), g(F_{ABx,Lx}(t)), g(F_{STy,My}(t)), 0.5(g(F_{ABx,My}(t)) + g(F_{STy,Lx}(t)))\ \})$$

for all $t > 0$, where a function $\phi : [0, +\infty) \to [0, +\infty)$ satisfies the condition (Φ). Then A, B, S, T, L and M have a unique common fixed point in X.

PROOF Let $x_0 \in X$. From condition (3.1.1) $\exists x_1, x_2 \in X$ such that $Lx_0 = STx_1 = y_0$ and $Mx_1 = ABx_2 = y_1$. Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

(3.1.6) $$Lx_{2n} = STx_{2n+1} = y_{2n} \text{ and } Mx_{2n+1} = ABx_{2n+2} = y_{2n+1}$$

for $n = 0, 1, 2, \ldots$.

Step 1. We prove that $\lim_{n\to\infty} g(F_{y_n,y_{n+1}}(t)) = 0$ for all $t > 0$. From (3.1.5) and (3.1.6), we have

$$\begin{aligned}
g(F_{y_{2n},y_{2n+1}}(t)) &= g(F_{Lx_{2n},Mx_{2n+1}}(t)\\
&\le \phi(max\{g(F_{ABx_{2n},STx_{2n+1}}(t)), g(F_{ABx_{2n},Lx_{2n}}(t)),\\
&\quad g(F_{STx_{2n+1},Mx_{2n+1}}(t)),\\
&\quad 0.5(g(F_{ABx_{2n},Mx_{2n+1}}(t)) + g(F_{STx_{2n+1},Lx_{2n}}(t)))\ \})\\
&= \phi(max\{g(F_{y_{2n-1},y_{2n}}(t)), g(F_{y_{2n-1},y_{2n}}(t)), g(F_{y_{2n},y_{2n+1}}(t)),\\
&\quad 0.5(g(F_{y_{2n-1},y_{2n+1}}(t)) + g(1))\ \})\\
&\le \phi(max\{g(F_{y_{2n-1},y_{2n}}(t)), g(F_{y_{2n},y_{2n+1}}(t)),\\
&\quad 0.5(g(F_{y_{2n-1},y_{2n}}(t)) + g(F_{y_{2n},y_{2n+1}}(t)))\ \})
\end{aligned}$$

If $g(F_{y_{2n-1},y_{2n}}(t)) \le g(F_{y_{2n},y_{2n+1}}(t))$ for all $t > 0$, then by (3.1.5)

$$g(F_{y_{2n},y_{2n+1}}(t)) \le \phi(g(F_{y_{2n},y_{2n+1}}(t)))$$

on applying Lemma 2.7, we have $g(F_{y_{2n},y_{2n+1}}(t)) = 0$ for all $t > 0$. Similarly, we have $g(F_{y_{2n+1},y_{2n+2}}(t)) = 0$ for all $t > 0$.

Thus, we have $g(F_{y_n,y_{n+1}}(t)) = 0$ for all $t > 0$.

On the other hand, if $g(F_{y_{2n-1},y_{2n}}(t)) \ge g(F_{y_{2n},y_{2n+1}}(t))$ then by (3.1.5), we have $g(F_{y_{2n},y_{2n+1}}(t)) \le \phi(g(F_{y_{2n-1},y_{2n}}(t)))$ for all $t > 0$. Similarly, $g(F_{y_{2n+1},y_{2n+2}}(t)) \le \phi(g(F_{y_{2n},y_{2n+1}}(t)))$ for all $t > 0$.

Thus, we have $g(F_{y_n,y_{n+1}}(t)) \le \phi(g(F_{y_{n-1},y_n}(t)))$ for all $t > 0$ and $n = 1, 2, \ldots$.

Therefore, by Lemma 2.7, $\lim_{n\to\infty} g(F_{y_n,y_{n+1}}(t)) = 0$ for all $t > 0$, which implies that $\{y_n\}$ is a Cauchy sequence in X by Lemma 2.14.

Since $(X, \mathcal{F}, \Delta)$ is complete, the sequence $\{y_n\}$ converges to a point $z \in X$.

Also its subsequences converges as follows :

$$\{Mx_{2n+1}\}\to z \text{ and } \{STx_{2n+1}\}\to z, \tag{3.1.7}$$

$$\{Lx_{2n}\}\to z \text{ and } \{ABx_{2n}\}\to z. \tag{3.1.8}$$

Case I. AB is continuous.

As AB is continuous, $(AB)^2x_{2n}\to ABz$ and $(AB)Lx_{2n}\to ABz$. As (L, AB) is compatible, so by Proposition 2.12, $L(AB)x_{2n}\to ABz$.

Step 2. Putting $x = ABx_{2n}$ and $y = x_{2n+1}$ for $t > 0$ in (3.1.5), we get

$$\begin{aligned}
g(F_{LABx_{2n},Mx_{2n+1}}(t)) &\le \phi(max\{g(F_{ABABx_{2n},STx_{2n+1}}(t)), g(F_{ABABx_{2n},LABx_{2n}}(t)),\\
&\quad g(F_{STx_{2n+1},Mx_{2n+1}}(t)),\\
&\quad 0.5(g(F_{ABABx_{2n},Mx_{2n+1}}(t)) + g(F_{STx_{2n+1},LABx_{2n}}(t)))\ \}).
\end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned}
g(F_{ABz,z}(t)) &\le \phi(max\{g(F_{ABz,z}(t))), g(F_{ABz,ABz}(t)), g(F_{z,z}(t)),\\
&\quad 0.5(g(F_{ABz,z}(t)) + g(F_{z,ABz}(t)))\ \}) = \phi(g(F_{ABz,z}(t)))
\end{aligned}$$

which implies that $g(F_{ABz,z}(t)) = 0$ by Lemma 2.7 and so we have $ABz = z$.

Step 3. Putting $x = z$ and $y = x_{2n+1}$ for $t > 0$ in (3.1.5), we get

$$\begin{aligned} g(F_{Lz,Mx_{2n+1}}(t)) \leq\ & \phi(max\{g(F_{ABz,STx_{2n+1}}(t)), g(F_{ABz,Lz}(t)), \\ & g(F_{STx_{2n+1},Mx_{2n+1}}(t)), \\ & 0.5(g(F_{ABz,Mx_{2n+1}}(t)) + g(F_{STx_{2n+1},Lz}(t)))\ \}). \end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} g(F_{Lz,z}(t)) \leq\ & \phi(max\{g(F_{z,z}(t)), g(F_{z,Lz}(t)), g(F_{z,z}(t)), \\ & 0.5(g(F_{z,z}(t)) + g(F_{z,Lz}(t)))\ \}) = \phi(g(F_{Lz,z}(t))) \end{aligned}$$

which implies that $g(F_{Lz,z}(t)) = 0$ by Lemma 2.7 and so we have $Lz = z$. Therefore, $ABz = Lz = z$.

Step 4. Putting $x = Bz$ and $y = x_{2n+1}$ for $t > 0$ in (3.1.5), we get

$$\begin{aligned} g(F_{LBz,Mx_{2n+1}}(t)) \leq\ & \phi(max\{g(F_{ABBz,STx_{2n+1}}(t)), g(F_{ABBz,LBz}(t)), \\ & g(F_{STx_{2n+1},Mx_{2n+1}}(t)), \\ & 0.5(g(F_{ABBz,Mx_{2n+1}}(t)) + g(F_{STx_{2n+1},LBz}(t)))\ \}). \end{aligned}$$

As $BL = LB$, $AB = BA$, so we have $L(Bz) = B(Lz) = Bz$ and $AB(Bz) = B(ABz) = Bz$. Letting $n \to \infty$, we get

$$\begin{aligned} g(F_{Bz,z}(t)) \leq\ & \phi(max\{g(F_{Bz,z}(t)), g(F_{Bz,Bz}(t)), g(F_{z,z}(t)), \\ & 0.5(g(F_{Bz,z}(t)) + g(F_{z,Bz}(t)))\ \}) = \phi(g(F_{Bz,z}(t))) \end{aligned}$$

which implies that $g(F_{Bz,z}(t)) = 0$ by Lemma 2.7 and so we have $Bz = z$. Also, $ABz = z$ and so $Az = z$. Therefore,

$$Az = Bz = Lz = z. \tag{3.1.9}$$

Step 5. As $L(X) \subset ST(X)$, there exists $v \in X$ such that $z = Lz = STv$. Putting $x = x_{2n}$ and $y = v$ for $t > 0$ in (3.1.5), we get

$$\begin{aligned} g(F_{Lx_{2n},Mv}(t)) \leq\ & \phi(max\{g(F_{ABx_{2n},STv}(t)), g(F_{ABx_{2n},Lx_{2n}}(t)), \\ & g(F_{STv,Mv}(t)), \\ & 0.5(g(F_{ABx_{2n},Mv}(t)) + g(F_{STv,Lx_{2n}}(t)))\ \}). \end{aligned}$$

Letting $n \to \infty$ and using equation (3.1.8), we get

$$\begin{aligned} g(F_{z,Mv}(t)) \leq\ & \phi(max\{g(F_{z,z}(t)), g(F_{z,z}(t)), g(F_{z,Mv}(t)), \\ & 0.5(g(F_{z,Mv}(t)) + g(F_{z,z}(t)))\ \}) = \phi(g(F_{z,Mv}(t))) \end{aligned}$$

which implies that $g(F_{z,Mv}(t)) = 0$ by Lemma 2.7 and so we have $z = Mv$. Hence, $STv = z = Mv$. As (M, ST) semi-compatible, we have $STMv = MSTv$. Thus, $STz = Mz$.

Step 6. Putting $x = x_{2n}$, $y = z$ for $t > 0$ in (3.1.5), we get

$$\begin{aligned} g(F_{Lx_{2n},Mz}(t)) \quad \leq \quad & \phi(max\{g(F_{ABx_{2n},STz}(t)), g(F_{ABx_{2n},Lx_{2n}}(t)), \\ & g(F_{STz,Mz}(t)), \\ & 0.5(g(F_{ABx_{2n},Mz}(t)) + g(F_{STz,Lx_{2n}}(t)))\ \}). \end{aligned}$$

Letting $n \to \infty$ and using equation (3.1.8) and Step 5 we get

$$\begin{aligned} g(F_{z,Mz}(t)) \quad \leq \quad & \phi(max\{g(F_{z,Mz}(t)), g(F_{z,z}(t)), g(F_{Mz,Mz}(t)), \\ & 0.5(g(F_{z,Mz}(t)) + g(F_{z,z}(t)))\ \}) = \phi(g(F_{z,Mz}(t))) \end{aligned}$$

which implies that $g(F_{z,Mz}(t)) = 0$ by Lemma 2.7 and so we have $z = Mz$.

Step 7. Putting $x = x_{2n}$ and y= Tz for $t > 0$ in (3.1.5), we get

$$\begin{aligned} g(F_{Lx_{2n},MTz}(t)) \quad \leq \quad & \phi(max\{g(F_{ABx_{2n},STTz}(t)), g(F_{ABx_{2n},Lx_{2n}}(t)), \\ & g(F_{STTz,MTz}(t)), \\ & 0.5(g(F_{ABx_{2n},MTz}(t)) + g(F_{STTz,Lx_{2n}}(t)))\ \}). \end{aligned}$$

As $MT = TM$ and $ST = TS$ we have $MTz = TMz = Tz$ and $ST(Tz) = T(STz) = Tz$. Letting $n \to \infty$ we get

$$\begin{aligned} g(F_{z,Tz}(t)) \quad \leq \quad & \phi(max\{g(F_{z,Tz}(t)), g(F_{z,z}(t)), g(F_{Tz,Tz}(t)), \\ & 0.5(g(F_{z,Tz}(t)) + g(F_{Tz,z}(t)))\ \}) = \phi(g(F_{z,Tz}(t))) \end{aligned}$$

which implies that $g(F_{z,Tz}(t)) = 0$ by Lemma 2.7 and so we have $z = Tz$.

Now $STz = Tz = z$ implies $Sz = z$. Hence

$$Sz = Tz = Mz = z. \tag{3.1.10}$$

Combining (3.1.9) and (3.1.10), we get $Az = Bz = Lz = Mz = Tz = Sz = z$. Hence, the six self maps have a common fixed point in this case.

Case II. L is continuous.
As L is continuous, $L^2x_{2n} \to Lz$ and $L(AB)x_{2n} \to Lz$.
As (L, AB) is compatible, so by Proposition 2.12, $(AB)Lx_{2n} \to Lz$.

Step 8. Putting $x = Lx_{2n}$ and $y = x_{2n+1}$ for $t > 0$ in (3.1.5), we get

$$\begin{aligned} g(F_{LLx_{2n},Mx_{2n+1}}(t)) \quad \leq \quad & \phi(max\{g(F_{ABLx_{2n},STx_{2n+1}}(t)), g(F_{ABLx_{2n},LLx_{2n}}(t)), \\ & g(F_{STx_{2n+1},Mx_{2n+1}}(t)), \\ & 0.5(g(F_{ABLx_{2n},Mx_{2n+1}}(t)) + g(F_{STx_{2n+1},LLx_{2n}}(t)))\ \}). \end{aligned}$$

Letting $n \to \infty$ we get

$$\begin{aligned} g(F_{Lz,z}(t)) &\le \phi(max\{g(F_{Lz,z}(t)), g(F_{Lz,Lz}(t)), g(F_{z,z}(t)), \\ &\quad 0.5(g(F_{Lz,z}(t)) + g(F_{z,Lz}(t)))\ \}) = \phi(g(F_{Lz,z}(t))), \end{aligned}$$

which implies that $g(F_{Lz,z}(t)) = 0$ by Lemma 2.7 and so we have $Lz = z$. Now, using steps 5-7 gives us $Mz = STz = Sz = Tz = z$.

Step 9. As $M(X) \subset AB(X)$, there exists $w \in X$ such that $z = Mz = ABw$. Putting $x = w$ and $y = x_{2n+1}$ for $t > 0$ in (3.1.5), we get

$$\begin{aligned} g(F_{Lw,Mx_{2n+1}}(t)) &\le \phi(max\{g(F_{ABw,STx_{2n+1}}(t)), g(F_{ABw,Lw}(t)), \\ &\quad g(F_{STx_{2n+1},Mx_{2n+1}}(t)), \\ &\quad 0.5(g(F_{ABw,Mx_{2n+1}}(t)) + g(F_{STx_{2n+1},Lw}(t)))\ \}). \end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} g(F_{Lw,z}(t)) &\le \phi(max\{g(F_{z,z}(t)), g(F_{z,Lw}(t)), g(F_{z,z}(t)), \\ &\quad 0.5(g(F_{z,z}(t)) + g(F_{z,Lw}(t)))\ \}) = \phi(g(F_{Lw,z}(t))), \end{aligned}$$

which implies that $g(F_{Lw,z}(t)) = 0$ by Lemma 2.7 and so we have $Lw = z$.

Thus, we have $Lw = z = ABw$. Since (L, AB) is compatible and so by Proposition 2.13, $LABw = ABLw$ and hence, we have $Lz = ABz$. Also, $Bz = z$ follows from Step 4. Thus, $Az = Bz = Lz = z$ and we obtain that z is the common fixed point of the six maps in this case also.

Step 10. (Uniqueness) Let u be another common fixed point of A, B, S, T, L and M; then $Au = Bu = Su = Tu = Lu = Mu = u$. Putting $x = z$ and $y = u$ for $t > 0$ in (3.1.5), we get

$$\begin{aligned} g(F_{Lz,Mu}(t)) &\le \phi(max\{g(F_{ABz,STu}(t)), g(F_{ABz,Lz}(t)), \\ &\quad g(F_{STu,Mu}(t)), \\ &\quad 0.5(g(F_{ABz,Mu}(t)) + g(F_{STu,Lz}(t)))\ \}). \end{aligned}$$

Letting $n \to \infty$ we get

$$\begin{aligned} g(F_{z,u}(t)) &\le \phi(max\{g(F_{z,u}(t)), g(F_{z,z}(t)), g(F_{u,u}(t)), \\ &\quad 0.5(g(F_{z,u}(t)) + g(F_{u,z}(t)))\ \}) = \phi(g(F_{z,u}(t))), \end{aligned}$$

which implies that $g(F_{z,u}(t)) = 0$ by Lemma 2.7 and so we have $z = u$. Therefore, z is a unique common fixed point of A, B, S, T, L and M. This completes the proof. ∎

REMARK 3.2 If we take $B = T = I$, the identity map on X in Theorem 3.1, then the condition (3.1.2) is satisfied trivially and we get

COROLLARY 3.3 *Let $A, S, L, M : X \to X$ be mappings satisfying the conditions :*

(3.1.11) $$L(X) \subset S(X),\ M(X) \subset A(X);$$

(3.1.12) *Either A or L is continuous*;

(3.1.13) *(L, A) is compatible and (M, S) is semi-compatible*;

(3.1.14) $$g(F_{Lx,My}(t)) \leq \phi(max\{g(F_{Ax,Sy}(t)), g(F_{Ax,Lx}(t)), g(F_{Sy,Ly}(t)), \\ 0.5(g(F_{Ax,My}(t)) + g(F_{Sy,Lx}(t)))\})$$

for all $t > 0$, where a function $\phi : [0, +\infty) \to [0, +\infty)$ satisfies the condition (Φ). Then A, S, L and M have a unique common fixed point in X.

REMARK 3.4 In view of Remark 3.2, Corollary 3.3 is a generalization of the result of Cho et. al. [2] in the sense that condition of compatibility of the pairs of self maps has been restricted to compatible and semi-compatible self maps and only one of the mappings of the compatible pair is needed to be continuous.

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