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On some properties of Musielak-Orlicz sequence spaces

Abstract. We consider a nontrivial vector space X and a semimodular $M : X \to [0, \infty]$ with property: $(\forall x \in X)(\exists \alpha > 0)\ M(\alpha x) < \infty$ (in other words, M is normal (i.e. $(\forall x \in X \setminus \{0\})\ (\exists \alpha > 0)\ M(\alpha x) > 0$) pregenfunction). The function M generates in X a metric d with

$$d(x, y) := inf\{a > 0 :\ M(a^{-1}(x - y)) \leq a\}.$$

At the same time M generates a metric ρ in Musielak-Orlicz sequence space l_M, namely

$$\rho(\varphi, \psi) := inf\{a > 0 :\ I(a^{-1}(\varphi - \psi)) \leq a\}.$$

with $I(\varphi) = \sum_{n \geq 1} M(\varphi(n))$.

It is proved that the space (l_M, ρ) is complete if and only if the space (X, d) is complete.

We consider also the closed subspace $G_M \subset l_M$ of sequences $\varphi = \{\varphi(n)\}$ such that $(\forall \alpha > 0)\ (\exists m \in \mathbb{N})\ \sum_{n \geq m} M(\alpha\varphi(n)) < \infty$ and prove that (G_M, ρ) is separable if and only if (X, d) is the same. Several examples are considered.

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1. Preliminaries. In what follows $\alpha, \gamma, \delta, \varepsilon$ denote positive numbers, and j, k, m, n - positive integers. Let X be a vector space ($X \neq \{0\}$) over scalar field $\mathbb{K}$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition 1.1 A function $M : X \to [0, \infty]$ is called pregenfunction if the following conditions are fulfilled:

(a) $M(0) = 0$;

(b) if $\lambda \in \mathbb{K}$ with $|\lambda| = 1$, then$(\forall x \in X)$ $M(\lambda x) = M(x)$;

(c) for any $x, y \in X$ and $a \in (0, 1)$

$$M(ax + (1-a)y) \leq M(x) + M(y);$$

(d) $(\forall x \in X)$ $(\exists \alpha)$ $M(\alpha x) < \infty$.

In other words, the pregenfunction is a pseudomodular [1] that satisfies to condition (d). If moreover the condition

(e) $(\forall x \in X \setminus \{0\})$ $(\exists \alpha > 0)$ $M(\alpha x) > 0$

is fulfiled, we shall call such function M by normal pregenfunction (in this case M is semimodular [1] with property (d)).

A normal pregenfunction M generates in X a metric. Really, if

$$p(x) := inf\{a > 0 : \ M(a^{-1}(x)) \leq a\}, \quad \forall x \in X,$$

then [1] $0 \leq p(x) < \infty$; $p(x) = 0$ if and only if $x = 0$; $p(-x) = p(x)$; $p(x+y) \leq p(x) + p(y)$ (it is evidently also that $(\forall b > p(x))$ $M(b^{-1}x) < b$). Hence the formula

$$d(x, y) := p(x - y) = inf\{a > 0 : \ M(a^{-1}(x - y)) \leq a\}$$

defines a metric in X.

Remark 1.2 Suppose taht condition (d) is not fulfilled, i.e. $\exists x_0 \neq 0$ $(\forall \alpha)$ $M(\alpha x_0) = \infty$. Then $\{a > 0 : \ M(a^{-1}x_0) \leq a\} = \emptyset$, and $p(x_0)$ is not defined (or one may set that $p(x_0) = \infty$). In this situation it is possible to define p (and hence the metric d) in the subspace

$$X_0 := \{x \in X : \ \exists \alpha \ M(\alpha x) < \infty\}.$$

In particular, if $(\forall x \neq 0)$ $M(x) = \infty$, then $X_0 = \{0\}$.

Note yet that if in place of (d) to accept more strong condition: $(\forall x \in X)$ $\lim_{\alpha \to 0} M(\alpha x) = 0$, then p is a F-norm in X [1]. However, in this paper it is enough to have a metric.

Lemma 1.3 *Let M be a normal pregenfunction, and $x_k \in X$, $k = 0, 1, 2 \ldots$. Then*

(1) $(\lim_{k \to \infty} d(x_k, x_0) = 0) \Leftrightarrow ((\forall \alpha) \lim_{k \to \infty} M(\alpha(x_k - x_0)) = 0)$;

(2) $(\lim_{j,k \to \infty} d(x_j, x_k) = 0) \Leftrightarrow ((\forall \alpha) \lim_{j,k \to \infty} M(\alpha(x_j - x_k)) = 0)$.

Proof (1) Let $d(x_k, x_0) \xrightarrow{k} 0$. Take $\alpha > 1$ nad $\varepsilon < 1$. Then $(\exists m)$ $(\forall k > m)$ $d(x_k, x_0) < \varepsilon/\alpha$, hence $M(\alpha(x_k - x_0)) \leq M(\varepsilon^{-1}\alpha(x_k - x_0)) < \varepsilon/\alpha < \varepsilon$, i.e. $M(\alpha(x_k - x_0)) \xrightarrow{k} 0$.

Conversely, let $(\forall \alpha)$ $M(\alpha(x_k - x_0)) \xrightarrow{k} 0$. Then $(\forall \varepsilon)$ $(\exists m)$ $(\forall k > m)$ $M(\varepsilon^{-1}(x_k - x_0)) < \varepsilon$. Hence $d(x_k, x_0) \leq \varepsilon$.

(2) It is proved similarly to (1). ∎

Remark 1.4 Lemma 1.3 coincides in essence with Theorem 1.6 from [1], which is proved for pseudomodular ρ and subspace $\{x \in X:\ \lim_{\lambda\to 0}\rho(\lambda x)=0\}$.

2. Sequence spaces generated by pregenfunctions. . Consider the vector space $X^{\mathbb{N}}$ of all sequences $\varphi=(\varphi(1),\varphi(2),\dots,)$, where $(\forall n)\ \varphi(n)\in X$. Denote the zero-element of $X^{\mathbb{N}}$ by θ, i.e. $(\forall\ \mathrm{n})\ \theta(n)=0\in X$.

Let $M: X\to[0,\infty]$ be a pregenfunction. Put

$$I(\varphi):=\sum_{n\geq 1}M(\varphi(n)),\ \ \varphi\in X^{\mathbb{N}},$$

$$l_M:=\{\varphi\in X^{\mathbb{N}}:\ (\exists\alpha)I(\alpha\varphi)<\infty\}.$$

The conditions (a), (b), (c) imply that l_M is a vector subspace (in other words, lineal) in $X^{\mathbb{N}}$. By condition (d) $l_M\neq\{\theta\}$. Indeed, let $x_0\neq 0$ and $M(\alpha x_0)<\infty$. Put $\varphi(1)=x_0$ and $(\forall n>1)\ \varphi(n)=0$. Then $\varphi\neq\theta$ and $I(\alpha\varphi)<\infty$, i.e. $\varphi\in l_M$.

Lemma 2.1 *The functional* $I: l_M\to[0,\infty]$ *is a pregenfunction. Moreover, it is normal if and only if* M *is the same.*

Proof The properties (a), (b), (c) for I follow directly from this properties for M. The property (d) is contained in the definition of the lineal l_M.

Let M be a normal pregenfunction. If $\varphi\in l_M\setminus\{\theta\}$, then $(\exists n_0)\ \varphi(n_0)\neq 0$. Hence, by (e), $(\exists\alpha)\ M(\alpha\varphi(n_0))>0$, so that $I(\alpha\varphi)>0$, i.e. I is normal. Evidently also, if M is not normal, then I is the same. ∎

The Lemma 2.1 implies that if the pregenfunction M is normal, then the formula

$$\rho(\varphi,\psi):=inf\{a>0:\ I(a^{-1}(\varphi-\psi))\leq a\}$$

defines a metric in l_M. Moreover, according to Lemma 1.3 the following statement is valid. Let $\varphi_k\in l_M$, $k=0,1,\dots$. Then

(1) $(\lim_{k\to\infty}\rho(\varphi_k,\varphi_0)=0)\Leftrightarrow((\forall\alpha)\lim_{k\to\infty}I(\alpha(\varphi_k-\varphi_0))=0)$;

(2) $(\lim_{j,k\to\infty}\rho(\varphi_j,\varphi_k)=0)\Leftrightarrow((\forall\alpha)\lim_{j,k\to\infty}I(\alpha(\varphi_j-\varphi_k))=0)$.

Consider now some other lineals in $X^{\mathbb{N}}$. Denote by F the collection of all sequences from $X^{\mathbb{N}}$, that have only finite set of nonzero members. Further, by the given pregenfunction M put

$$G_M=\{\varphi\in X^{\mathbb{N}}:\ (\forall\alpha)\,(\exists m)\,I_m(\alpha\varphi)<\infty\},$$

with $I_m(\varphi)=\sum_{n\geq m}M(\varphi(n))$. It is easy to check that G_M is a lineal.

Lemma 2.2 $F\subset G_M\subset l_M$.

Proof Let $\varphi \in F \setminus \{\theta\}$, and $m = max\{n: \varphi(n) \neq 0\} + 1$. Then $(\forall\alpha)$ $I_m(\alpha\varphi) = 0$, so that $\varphi \in G_M$. Thus $F \subset G_M$.

Let now $\varphi \in G_M$. Then $(\exists m)$ $I_m(\varphi) < \infty$. If $m = 1$, i.e. $I(\varphi) < \infty$, then $\varphi \in l_M$. Let $m > 1$. Then, by (d), $(\forall n < m)$ $(\exists \alpha_n)$ $M(\alpha_n\varphi(n)) < \infty$. Put $\alpha = min(1, \alpha_1, \ldots, \alpha_{m-1})$. Then $I(\alpha\varphi) < \infty$, i.e. $\varphi \in l_M$. Thus $G_M \subset l_M$. ∎

The following statement is valid: $G_M = l_M$ if and only if there exist γ and δ such that if $M(x) < \delta$, then $M(2x) \leq \gamma M(x)$ (this is generalization of the known in the theory of Orlicz spaces "Δ_2-condition" by small x). We publish the proof of this statement in [2].

Let now the pregenfunction M be normal and consider the metric space (l_M, ρ).

Lemma 2.3 *The lineal G_M is closed in (l_M, ρ). Moreover, $G_M = \overline{F}$.*

Proof Let $\varphi_k \in G_M$, $k = 1, 2, \ldots$, and $\rho(\varphi_k, \varphi) \xrightarrow{k} 0$ with $\varphi \in l_M$. Take an arbitrary α. Since $I(2\alpha(\varphi_k - \varphi)) \xrightarrow{k} 0$, there exists j such that $I(2\alpha(\varphi_j - \varphi)) < \infty$. As $\varphi_j \in G_M$ $(\exists m)$ $I_m(2\alpha\varphi_j) < \infty$. Then, by condition (c),

$$I_m(\alpha\varphi) \leq I_m(2\alpha(\varphi - \varphi_j)) + I_m(2\alpha\varphi_j) < \infty,$$

so that $\varphi \in G_M$. Thus the lineal G_M is closed. From here and Lemma 2.2 it follows that $\overline{F} \subset G_M$. Prove that $G_M \subset \overline{F}$. Indeed, let $\varphi \in G_M$. Put $(\forall k)$ $\varphi_k(n) = \varphi(n)$ by $n \leq k$, and $\varphi_k(n) = 0$ by $n > k$. Then $(\forall k)$ $\varphi_k \in F$ and $(\forall\alpha)$ $I(\alpha(\varphi - \varphi_k)) = \sum_{n>k} M(\alpha\varphi(n)) \xrightarrow{k} 0$, i.e. $\rho(\varphi_k, \varphi) \xrightarrow{k} 0$. Hence $\varphi \in \overline{F}$. ∎

3. On completeness of l_M and separability of G_M. let M be a normal pregenfunction, so that (X, d), (l_M, ρ) and (G_M, ρ) are metric spaces.

Theorem 3.1 *The space (l_M, ρ) is complete if and only if (X, d) is the same.*

Proof Let (l_M, ρ) be complete. Take a Cauchy sequence $\{x_k\} \in (X, d)$ and put $(\forall k)$ $\varphi_k(1) = x_k$ and $(\forall n > 1)$ $\varphi_k(n) = 0$. Since

$$(\forall\alpha) \quad \lim_{j,k\longrightarrow\infty} I(\alpha(\varphi_j - \varphi_k)) = \lim_{j,k\longrightarrow\infty} M(\alpha(x_j - x_k)) = 0,$$

then $\{\varphi_k\}$ is a Cauchy sequence in (l_M, ρ). Hence, $(\exists\varphi \in l_M)$ $\rho(\varphi_k, \varphi) \xrightarrow{k} 0$, i.e. $(\forall\alpha)$ $I(\alpha(\varphi_k - \varphi)) \xrightarrow{k} 0$, whence $(\forall\alpha)$ $M(\alpha(x_k - \varphi(1))) \xrightarrow{k} 0$, i.e. $d(x_k, \varphi(1)) \xrightarrow{k} 0$. Thus (X, d) is complete.

Let now (X, d) be complete, and let $\{\varphi_k\}$ be a Cauchy sequence in (l_M, ρ), i.e. $(\forall\alpha)$ $\lim_{j,k\longrightarrow\infty} I(\alpha(\varphi_j - \varphi_k)) = 0$. Hence

$$\forall(n, \alpha) \quad \lim_{j,k\longrightarrow\infty} M(\alpha(\varphi_j(n) - \varphi_k(n))) = 0,$$

i.e. $(\forall n)$ $\{\varphi_k(n)\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, then $(\forall n)$ $(\exists x_n \in X)$ $d(\varphi_k(n), x_n) \xrightarrow{k} 0$. Put $(\forall n)$ $\varphi(n) = x_n$ and show taht $(\forall\alpha)$ $I(\alpha(\varphi_k - \varphi)) \xrightarrow{k} 0$. To this end fix α and ε and find k_0 such that

$$(\forall j, k > k_0) \; I(2\alpha(\varphi_k - \varphi_j)) < \varepsilon. \tag{1}$$

Further fix $k > k_0$. Then $(\forall n, j)$

$$M(\alpha(\varphi_k(n) - \varphi(n))) \leq M(2\alpha(\varphi_k(n) - \varphi_j(n))) + M(2\alpha(\varphi_j(n) - \varphi(n))).$$

Since $M(2\alpha(\varphi_j(n) - \varphi(n))) \xrightarrow{j} 0$, then $(\forall n)$

$$M(\alpha(\varphi_k(n) - \varphi(n))) \leq \liminf_{j \longrightarrow \infty} M(2\alpha(\varphi_k(n) - \varphi_j(n))).$$

Denote here the right side by $a_k(n)$. Then

$$I(\alpha(\varphi_k - \varphi)) \leq \sum_{n \geq 1} a_k(n). \tag{2}$$

Considering $\sum_{n \geq 1} a_k(n)$ as an integral over $\mathbb{N}$, we have by Fatou lemma

$$\sum_{n \geq 1} a_k(n) \leq \liminf_{j \longrightarrow \infty} I(2\alpha(\varphi_k - \varphi_j)).$$

Then, by (1) and (2), $(\forall k > k_0)$ $I(\alpha(\varphi_k - \varphi)) \leq \varepsilon$. From here it follows that $\varphi \in l_M$ and $\rho(\varphi_k, \varphi) \xrightarrow{k} 0$, i.e. the space (l_M, ρ) is complete. ∎

Corollary 3.2 *If X is a Banach space (with norm $|\cdot|$), and for pregenfunction M the following conditions are fulfilled:*

$$\lim_{|x| \longrightarrow 0} M(x) = 0, \tag{3}$$

$$\liminf_{|x| \longrightarrow \infty} M(x) > 0, \tag{4}$$

(note that (4) *implies condition (e)), then the space (l_M, ρ) is complete.*

Proof Show that the space (X, d) is complete. To this end take in (X, d) a Cauchy sequence $\{x_k\}$, i.e.

$$(\forall \alpha) \quad \lim_{j,k \longrightarrow \infty} M(\alpha(x_j - x_k)) = 0, \tag{5}$$

and show that $\{x_k\}$ is a Cauchy sequence in $(X, |\cdot|)$.

Suppose the contrary. Then there exist γ and indexes $k(1) < k(2) < \ldots$ such that $(\forall m)$ $|x_{k(m)} - x_{k(m+1)}| > \gamma$. On the other hand, by (4),

$$(\exists \delta, \xi > 0) \ \ inf\{M(x) : \ |x| > \xi\} > \delta.$$

Therefore, if $\alpha > \xi/\gamma$, then $(\forall m)$ $M(\alpha(x_{k(m)} - x_{k(m+1)})) > \delta$, that contradicts to (5). So $\{x_k\}$ is a Cauchy sequence in Banach space $(X, |\cdot|)$. Hence, $(\exists x \in X)$ $|x_k - x| \xrightarrow{k} 0$. Then, by (3), $(\exists \alpha)$ $M(\alpha(x_k - x)) \xrightarrow{k} 0$, so that (X, d) is complete. It remains to applay Theorem 3.1. ∎

Example 3.3 Let X be the space of continuous functions $x : [0,1] \to \mathbb{R}$ with the norm $|x| = \int_0^1 |x(t)|dt$, and $M(x) := \max_t |x(t)|$. Evidently M is a pregenfunction with property (4), but condition (3) is not fulfilled. In addition, the space $(X, |\cdot|)$ is not complete. Therefore, Corollary 3.2 is not applicable here. Nevertheless, the space (l_M, ρ) is complete by Theorem 3.1 since the space (X, d) is complete (it is easy to check that $d(x,y) = (max_t|x(t) - y(t)|)^{1/2}$).

Example 3.4 Let $X = L^\infty[0,1]$, so that X is a Banach space with the norm $|x| = ess\, sup_{t\in[0,1]}|x(t)|$. Let futher $M(x) = \int_0^1 |x(t)|dt$ (Lebesgue integral). Evidently M is a normal pregenfunction, for which (3) is fulfilled, but (4) is not. So that Corollary 3.2 is not applicable. But Theorem 3.1 does not help here too since (X, d) is not complete. Indeed, consider a sequence $\{x_k\} \subset L^\infty[0,1]$ that converges by the norm of the space $L[0,1]$ to a function $x \in L[0,1] \setminus L^\infty[0,1]$. Hence, $\{x_k\}$ is a Cauchy sequence in (X, d) that has not a limit in this space.

Thus in Example 3.4 the space (l_M, ρ) is not complete. We have a similar situation if $X = C[0,1]$ ($|x| = max_t|x(t)|$) and $M(x) = \int_0^1 |x(t)|dt$.

Pass to the question on separability of the space (G_M, ρ).

Theorem 3.5 *The space (G_M, ρ) is separable if and only if (X, d) is the same.*

Proof Since, by Lemma 2.3 $G_M = \overline{F}$, it is possible to consider F in place of G_M.

So, let F be separable in metric ρ, i.e. there exists a countable set $A \subset F$, such that $F \subset \overline{A}$. Show that the countable set $A_1 := \{\varphi(1) : \varphi \in A\}$ is dense in (X, d). To this end fix $x \in X$ and put $\varphi(1) = x$ and $(\forall n > 1)\ \varphi(n) = 0$. Then $\varphi \in F$, so that $(\exists \varphi_k \in A,\ k = 1, 2, \dots)\ (\forall\alpha)\ I(\alpha(\varphi_k - \varphi)) \xrightarrow{k} 0$, whence $(\forall\alpha)\ M(\alpha(\varphi_k(1) - x)) \xrightarrow{k} 0$. Hence $d(\varphi_k(1), x) \xrightarrow{k} 0$ with $\varphi_k(1) \in A$, $k = 1, 2, \dots$. Thus (X, d) is separable.

Conversely, let some countable set B be dense in (X, d) with $0 \in B$. Put $A = \{\varphi \in F :\ (\forall n)\varphi(n) \in B\}$. Evidently, A is countable. It remains to show that $F \subset \overline{A}$.

So, let $\varphi \in F$, i.e. $(\exists m)\ (\forall n > m)\ \varphi(n) = 0$. Then $(\forall n \le m)\ (\exists x_{n,k} \in B,\ k = 1, 2, \dots)\ d(x_{n,k}, \varphi(n)) \xrightarrow{k} 0$, i.e. $(\forall\alpha)\ M(\alpha(x_{n,k} - \varphi(n))) \xrightarrow{k} 0$. Put $(\forall k)\ (\forall n \le m)$ $\varphi_k(n) = x_{n,k}$ and $(\forall n > m)\ \varphi_k(n) = 0$. Then $(\forall k)\ \varphi_k \in A$ and

$$(\forall\alpha)\ \ I(\alpha(\varphi_k - \varphi)) = \sum_{n\le m} M(\alpha(x_{n,k} - \varphi(n))) \xrightarrow{k} 0,$$

i.e. $\rho(\varphi_k, \varphi) \xrightarrow{k} 0$, so that $\varphi \in \overline{A}$. ∎

Corollary 3.6 *Let X be a separable normed space (with a norm $|\cdot|$), and for M the condition (3) is valid. Then (G_M, ρ) is separable.*

Proof Let a countable set A be dense in $(X, |\cdot|)$. Then $(\forall x \in x)\ (\exists a_k \in A,\ k = 1, 2, \dots)\ |a_k - x| \xrightarrow{k} 0$, whence, by (3), $(\forall\alpha)\ M(\alpha(a_k - x)) \xrightarrow{k} 0$, i.e. $d(a_k, x) \xrightarrow{k} 0$. Thus the space (X, d) is separable. It remains to applay Theorem 3.5. ∎

Example 3.7 Let a normed space $(X, |\cdot|)$ be separable, $M(x) = 0$ by $|x| \leq 1$ and $M(x) = \infty$ by $|x| > 1$. Then

$$l_M = l^\infty(X) := \{\varphi \in X^{\mathbb{N}} : \sup_n |\varphi(n)| < \infty\},$$

$$G_M = c_0(X) := \{\varphi \in X^{\mathbb{N}} : |\varphi(n)| \xrightarrow{n} 0\},$$

and, by Corollary 3.6, (G_M, ρ) is separable. By the way, in this example $\rho(\varphi, \psi) = \sup_n |\varphi(n) - \psi(n)|$, so that the space (l_M, ρ) is not separable.

Example 3.8 Let X be the space of bounded functions $x : [0,1] \to \mathbb{R}$, $M(x) = \sup_t |x(t)|$. Since $d(x, y) = (\sup_t |x(t) - y(t)|)^{1/2}$, then (X, d) is not separable. Hence, $(G_M, \rho)(=(l_M, \rho))$ is the same.

In Example 3.3 the space $(X, |\cdot|)$ is separable, but (3) is not valid, so that Corollary 3.6 is not applicable. But since M represents a norm in separable space $C[0,1]$, then (X, d) is separable (note that $d(x, y) = (\max_t |x(t) - y(t)|)^{1/2}$). So that $(G_M, \rho)(=(l_M, \rho))$ is separable.

In Example 3.4 the space $(X, |\cdot|)$ is not separable, so that Corollary 3.6 does not work. At the same time, since M represents a norm in a subspace of the separable space $L[0,1]$, then (X, d) is separable (note that $d(x, y) = (\int_0^1 |x(t) - y(t)| dt)^{1/2}$). So that $(G_M, \rho)(=(l_M, \rho))$ is separable.

Remark 3.9 The stated results it is possible to extend to situation when on X normal pregenfunction M_n, $n = 1, 2, \ldots$ are given, so that every M_n generates in X the metric d_n. In this case lineals l_M and G_M and metric ρ are defined as before, but now

$$I(\varphi) := \sum_{n \geq 1} M_n(\varphi(n)), \quad I_m(\varphi) := \sum_{n \geq m} M_n(\varphi(n)).$$

In this situation, in Theorems 3.1 and 3.5 it ought to replace (X, d) by (X, d_n), $n = 1, 2, \ldots$. Similar modifications it is necessary to bring in Corollaries 3.2 and 3.6.

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