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# A spectral synthesis property for $C_b(X, \beta)$

**Abstract.** Let $(C_b(X), \beta)$ be the algebra of all continuous bounded real or complex valued functions defined on a completely regular Hausdorff space $X$ with the usual algebraic operations and with the strict topology $\beta$. It is proved that $(C_b(X), \beta)$ has a spectral synthesis, i.e. every of its closed ideals is an intersection of closed maximal ideals of codimension 1. We give one necessary and two sufficient conditions over $X$ in order that $(C_b(X), \beta)$ has no proper non-zero closed principal ideals. Moreover if $X$ satisfy any of these two conditions and is also a $k$-space, then any non zero element of $C_b(X)$ is invertible or a topological divisor of zero.

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**1. Introduction.** Throughout this work $X$ will be a non-empty completely regular Hausdorff space and $\mathbb{F}$ will denote the field of scalars $\mathbb{R}$ or $\mathbb{C}$.

A bounded function $f : X \to \mathbb{F}$ is said to vanish at infinity if given any $\epsilon > 0$, there is a compact subset $K$ such that $|f(x)| < \epsilon$ whenever $x \in X \setminus K$.

Let $(C_b(X), \beta)$ be the algebra over $\mathbb{F}$ of all continuous bounded $\mathbb{F}$-valued functions defined on $X$ with the usual algebraic operations and endowed with the *strict topology* $\beta$ [5], i.e. the topology defined by the following seminorms:

$$\|f\|_{\varphi} = \sup_{x \in X} |f(x)| \, |\varphi(x)|$$

for $f \in C_b(X)$ and $\varphi$ varying over the space $B_0$ of all bounded scalar-valued functions on $X$ vanishing at infinity.

A *topological algebra* over $\mathbb{F}$ is a topological Hausdorff vector space with associative jointly continuous multiplication.

A *locally convex* algebra is a topological algebra $A$ which is a locally convex space. In this case its topology can be given by a family $\{\|\cdot\|_{\alpha} : \alpha \in \Lambda\}$ of seminorms satisfying the following condition: for every $\alpha \in \Lambda$ there exists $\beta \in \Lambda$ such that

$$\|xy\|_\alpha \leq \|x\|_\beta \|y\|_\beta \tag{1}$$

for all $x$, $y \in A$.

A locally convex algebra is said to be *multiplicatively locally convex* (shortly *m-convex*) if every seminorm is submultiplicative i.e. the above inequality can be replaced by

$$\|xy\|_\alpha \leq \|x\|_\alpha \|y\|_\alpha \tag{2}$$

for all $\alpha \in \Lambda$ and all $x$, $y \in A$.

The seminorms $\|\cdot\|_\varphi$ that define the topology of $(C_b(X), \beta)$ satisfy (1) and so $(C_b(X), \beta)$ is a commutative locally convex algebra. It is also complete when $X$ is a *k-space* (i.e. $F \subset X$ is closed if and only if $F \cap K$ is closed for every compact $K \subset X$).

$Z(f)$ will denote the *zero set* $f^{-1}(0)$ for any scalar valued function $f$. When $f$ is linear, $Z(f)$ is usually called the null space of $f$. For $f$ defined on $X$ let *supp*$f = $ *cl* $(X \diagdown Z(f))$, where *cl* denotes the closure operator.

For any topological algebra $A$, an element $x \in A$ different from 0 is a *bilateral topological divisor of zero* if there exist two nets $(y_v)$ and $(z_v)$ in $A$ not convergent to zero and such that $y_v x \to 0$ and $x z_v \to 0$. In commutative topological algebras we simply say that $x$ is a topological divisor of zero. A bilateral topological divisor of zero is called *proper* whenever it is not a divisor of zero.

Suppose $A$ has a unit $e$. An element $x \in A$ is *topologically invertible* provided *cl* $(Ax) =$ *cl* $(xA) = A$. This is equivalent to the existence of a pair of nets $\widetilde{a} = (a_\lambda)$ and $\widetilde{b} = (b_\lambda)$, called *right* and *left topological inverses* respectively, such that $aa_\lambda \to e$ and $b_\lambda a \to e$. By $G(A)$ and $G_t(A)$ we denote the set of all the invertible and topologically invertible elements of $A$, respectively.

We recall the concept of *bounded net* given in [3] that has been called *almost bounded* by W. Żelazko [6]: a net $(a_\lambda)$ in a topological vector space $X$ is bounded if for every neighborhood of zero $U$ there exist $\lambda_U$ and $k_U > 0$ such that $a_\lambda \in k_U U$ if $\lambda > \lambda_U$.

A topological space $X$ is called *Fréchet-Urysohn* [4] if for every $S \subset X$, a point $x \in$ *cl* $(S)$ if and only if there exists a sequence in $S$ that converges to $x$. Every first countable space is a Fréchet-Urysohn space.

Let $A$ be a topological algebra with unit. We write $\mathfrak{M}(A)$ for the space of all non-trivial continuous linear and multiplicative functionals on $A$, endowed with the $w^*$-topology.

In any commutative unital topological algebra $A$ the null spaces of the elements in $\mathfrak{M}(A)$ are precisely the maximal closed ideals of codimension 1. For a complex commutative unital Banach algebra any maximal ideal is closed and of codimension 1, and for a complex commutative unital $m$-convex algebra every closed maximal ideal is of codimension 1.

If $A = (C_b(X), \beta)$, then $\mathfrak{M}(A) = \{\phi_x : x \in X\}$, where $\phi_x(f) = f(x)$ for every $f \in C_b(X)$. Based on this we write $\mathfrak{M}((C_b(X), \beta)) = X$ and we can set up an injective correspondence between $X$ and the set of all the maximal closed ideals of $A$ via

$$x \to \phi_x \to Z(\phi_x).$$

The range of this correspondence is the set of all maximal closed ideals of codimension 1 of $C_b(X)$.

It is said that a commutative unital Banach algebra has the spectral synthesis property if every of its closed ideals is an intersection of maximal ideals. Similarly we shall say that a commutative unital topological algebra $A$ has the *spectral synthesis property (s.s.p.)* if every of its closed ideals is an intersection of closed maximal ideals of $A$ of codimension 1, i.e. an intersection of null spaces of functionals in $\mathfrak{M}(A)$.

Let $A$ be a commutative unital topological algebra. For $E \subset \mathfrak{M}(A)$ the *kernel* $k(E)$ is defined as the closed ideal $k(E) = \bigcap_{\phi \in E} Z(\phi)$ if $E$ is not empty and $k(\varnothing) = A$. For an ideal $I$ of $A$ the *hull* $h(I)$ is defined as $h(I) = \{\phi \in \mathfrak{M}(A) : I \subset Z(\phi)\}$.

It is clear that $A$ has the spectral synthesis property if and only if $I = k(h(I))$ for every closed ideal $I$ of $A$.

Having in mind that $\mathfrak{M}(C_b(X), \beta)) = X$, it is easy to see that $k(E) = \{f \in C_b(X) : f(x) = 0 \text{ for all } x \in E\}$ if $E \subset X$ is not empty, and $h(I) = \{x \in X : f(x) = 0 \text{ for all } f \in I\}$ if $I$ is an ideal of $(C_b(X), \beta)$.

Clearly, $h(cl(I)) = h(I)$ if $I$ is an ideal of $C_b(X)$ and therefore $k(h(cl(I))) = k(h(I))$. It is also obvious that $k(h(I)) = I$ if $I = C_b(X)$.

In Section 2 we shall prove that $h(I) \neq \emptyset$ and $I = k(h(I))$ for each proper closed ideal $I$ of $(C_b(X), \beta)$. Therefore, it will follow that $C_b(X)$ has the spectral synthesis property.

We can point out that the algebra $A = C(\mathbb{R})$ of all complex continuous functions defined on $\mathbb{R}$, endowed with the compact-open topology, is a commutative complex $m$-convex algebra with unit in which every non-invertible element different from 0 is a topological divisor of zero. In particular, every non-zero element in $cl(G(A)) \setminus G(A)$ is a topological divisor of zero.

In the last section we show that connectedness of $X$ is a necessary condition for $C_b(X, \beta)$ has no proper non-zero closed principal ideal, and we prove, using the s.s.p., that $C_b(X, \beta)$ has no proper non-zero closed principal ideal if $X$ is a connected space that it is also: a) locally connected or b) Fréchet-Urysohn.

As consequence of this, each non-zero element of $A = (C_b(X), \beta)$ is invertible or a topological divisor of zero, when $X$ is a connected $k$-space satisfying a) or b); and then, this $A$ is another example of a topological algebra on which any non-zero element in $cl(G(A)) \setminus G(A)$ is a topological divisor of zero. The same is true for $G_t(A)$.

## 2. A property of spectral synthesis.

LEMMA 2.1 *Let $K \subset X$ be a non-empty compact set and $\{U_i\}_{i=1,\ldots,n}$ a finite open cover in $X$ of $K$. There exist continuous functions $\lambda_i : X \to [0,1]$, with $1 \le i \le n$, such that:*

*i)* $supp\lambda_i \subset U_i$ *for* $1 \le i \le n$.
*ii)* $\sum_{i=1}^{n} \lambda_i(x) \le 1$ *for all* $x \in X$.
*iii)* $\sum_{i=1}^{n} \lambda_i(z) = 1$ *if* $z \in K$.

Proof It is easy to see that there exist functions $\alpha_i : X \to \mathbb{R}$ for $1 \le i \le n$ such that $supp\alpha_i \subset U_i$ and $\sum_{i=1}^{n} \alpha_i(z) > 0$ for all $z \in K$. In particular, we have $m = \min_{z\in K} \sum_{i=1}^{n} \alpha_i(z) > 0$. For $1 \le i \le n$ let us define $\lambda_i : X \to \mathbb{R}$ as $\lambda_i(x) = \frac{\alpha_i(x)}{M(x)}$, where $M(x) = \max\left(\sum_{j=1}^{n} \alpha_j(x), m\right)$ for $x \in X$. ■

In what follows we suppose that $X$ is not a singleton.

Theorem 2.2 *If* $I \subset (C_b(X), \beta)$ *is a proper closed ideal, then* $h(I) \neq \emptyset$.

Proof Assume $h(I) = \emptyset$. This will lead us to the equality $I = C_b(X)$ which contradicts the hypothesis. In fact, we are going to prove that $1 \in cl(I)$, where 1 is the unit function.

According to our assumption, for each $x \in X$ the set $\{f(x) : f \in I\}$ is all of $\mathbb{F}$. Therefore for each $x \in X$ we have $f(x) = 1$ for some $f \in I$.

Take $\varphi \in B_0$ and $0 < \epsilon < 1$. There exist a compact subset $K$ of $X$ and a constant $M > 1$ such that

$$|\varphi(x)| < \epsilon \text{ if } x \in X \setminus K$$

and

$$|\varphi(x)| < M \text{ for all } x \in X.$$

For each $z \in K$ take $f_z \in I$ with $f_z(z) = 1$. By the continuity of $f_z$ there is an open neighborhood $U(z)$ of $z$ in $X$ such that

$$|f_z(x) - 1| < \frac{\epsilon}{M} \text{ if } x \in U(z).$$

The collection $\{U(z)\}_{z\in K}$ is an open cover of $K$ in $X$ . Let $\{U(z_1), ..., U(z_n)\}$ be a finite subcover. By Lemma 2.1 we can find continuous functions $\lambda_i : X \to [0,1]$ for $1 \le i \le n$ satisfying i)-iii). The function $F_{\varphi,\in}(x) = \sum_{i=1}^{n} \lambda_i(x) f_{z_i}(x)$ belongs to $I$. In order to estimate $\|F_{\varphi,\in} - 1\|_\varphi$ we consider three cases: $x \in K, x \in \bigcup_{i=1}^{n} U(z_i) \setminus K$ and $x \in X \setminus \bigcup_{i=1}^{n} U(z_i)$, obtaining $\|F_{K,\in} - 1\|_\varphi < \epsilon$. Thus, $1 \in cl(I)$. ■

Theorem 2.3 *A proper ideal* $I$ *of* $(C_b(X), \beta)$ *is closed if and only if* $I = k(h(I))$. *Therefore* $(C_b(X), \beta)$ *has the spectral synthesis property and* $cl(I) = k(h(I))$ *for any ideal* $I$, *i.e* $g \in cl(I)$ *if and only if* $h(I) \subset Z(g)$.

PROOF Let $I \subset (C_b(X), \beta)$ be a proper closed ideal. It follows from Theorem 2.2 that $E = h\ (I) \neq \emptyset$. We have only to prove $k(E) \subset I$. Let $g \in k\ (E)$ be different from the zero function, $\varphi \in B_0$ and $0 < \epsilon < 1$. There exist a compact subset $K$ of $X$ and a constant $M > 1$ such that

$$|\varphi(x)| < \frac{\epsilon}{1 + \|g\|_\infty} \text{ if } x \in X \diagdown K,$$

where $\|g\|_\infty = \sup\limits_{x \in X} |g(x)|$, and

$$|\varphi(x)| < M \text{ for all } x \in X.$$

For each $z \in K$ take $f_z \in I$ with $f_z(z) = g(z)$. By the continuity of these two functions there exists an open neighborhood $U(z)$ of $z$ in $X$ such that

$$|f_z(x) - g(x)| < \frac{\epsilon}{M} \text{ if } x \in U(z).$$

The collection $\{U(z)\}_{z \in K}$ is an open cover of $K$ in $X$ . Let $\{U(z_1), ..., U(z_n)\}$ be a finite subcover. For Lemma 2.1 there exist continuous functions $\lambda_i : X \to [0,1]$ for $1 \leq i \leq n$, satisfying i)-iii). The function $F_{K,\in}(x) = \sum\limits_{i=1}^{n} \lambda_i(x) f_{z_i}(x)$ belongs to $I$. Proceeding as in the proof of Theorem 2.2 we obtain $\|F_{K,\in} - g\|_\varphi < \epsilon$. Therefore, $g \in cl(I)$ and $I = k\ (E)$.

The converse result and the equality $cl(I) = k(h(I))$ follow from the comments after definitions of $h$ and $k$, and from what we have just proved. ∎

In [1] it is established, as Corollary 4, the following:

PROPOSITION 2.4 *Let $A$ be a complete locally convex or locally pseudoconvex algebra with a unit $e$. If $a \in A$ is topologically invertible and it is not invertible, then its lateral topological inverses $\widetilde{b} = (b_\lambda)$ and $\widetilde{c} = (c_\lambda)$ are not bounded and $a$ is a bilateral topological divisor of zero.*

Here bounded net means almost bounded net.

COROLLARY 2.5 *If $f \in C_b(X)$ does not vanish in any point $x \in X$, then $fC_b(X)$ is dense in $(C_b(X), \beta)$, i.e. $f$ is topologically invertible; if $X$ is a $k$-space and $f$ is also not invertible, then $f$ is a proper topological divisor of zero and its topological inverse is not bounded.*

PROOF Let $I = fC_b(X)$. Then, $h\ (I) = \emptyset$ and $cl(I) = C_b(X)$. Thus $f$ is topologically invertible. When $X$ is a $k$-space, then $C_b(X, \beta)$ is a complete locally convex algebra with unit and the conclusion follows from Proposition 2.4. ∎

**3. Spaces $C_b(X, \beta)$ without any non-zero proper closed principal ideal.**
The next lemma is easy to prove.

Lemma 3.1 *Let $(a_n)$ and $(b_n)$ be two sequences of positive numbers which converge to 0, with $(a_n)$ being also a strictly decreasing sequence. Then there exists a continuous function $h:\mathbb{R}\to\mathbb{R}$ satisfying: $h(a_n)=b_n$ for all $n\geq 1$, $h(0)=0$ and $h(t)\neq 0$ for all $t\neq 0$.*

Theorem 3.2 *If $C_b(X,\beta)$ has no proper non-zero closed principal ideals then $X$ is a connected space. Conversely, if $X$ is a connected space and satisfies any of the following two conditions:*

*a) $X$ is a locally connected space;*

*b) $X$ is a Fréchet-Urysohn space;*

*then, $C_b(X,\beta)$ has no proper non-zero closed principal ideals.*

Proof Suppose $X$ is a disconnected space. There exists a surjective continuous function $f:X\to\{0,1\}$. Thus, $fC_b(X,\beta)$ is a non-zero proper closed principal ideal of $C_b(X,\beta)$.

Conversely, let $X$ be a connected space and $I=fC_b(X)$ a proper non-zero principal ideal of $C_b(X,\beta)$. Then $f$ is neither the zero function nor an invertible element of $C_b(X)$.

If $f(x)\neq 0$ for all $x\in X$, then $f$ is topologically invertible and therefore the unit function belongs to $cl(I)\setminus I$, and $I$ is not a closed ideal.

Now suppose $Z(f)\neq\emptyset$. Since we also have $Z(f)\neq X$, there exists an element $z$ belonging to the boundary of $Z(f)$.

We shall prove that if the connected space $X$ satisfies condition a) or b), then $I$ is not closed.

a) Assume that $X$ is a locally connected space. The real function $g$ defined on $X$ as $g(x)=0$ if $x\in Z(f)$ and $g(x)=f(x)\sin\frac{1}{|f(x)|}$ if $x\notin Z(f)$, belongs to $C_b(X)$. Since $h(I)=Z(f)\subset Z(g)$, it follows from Theorem 2.3 that $g\in cl(I)$.

We claim $g\notin I$; in order to prove it assume the contrary, then $g=f\cdot f_1$ for some $f_1\in C_b(X)$. Since $f_1$ is continuous at $z$ there exists a connected neighborhood $V$ of $z$ such that $f_1(V)\subset B(f_1(z))$, where $B(f_1(z))$ is the open unitary ball centered at $f_1(z)$. The image $|f|(V)$ is an interval $J$ of positive length with 0 as left extreme, and all the non-zero elements of $J$ are images of points in $V\cap(X\setminus Z(f))$. Thus, $\sin\frac{1}{|f(x)|}$ takes the values 1 and $-1$ in $V\cap(X\setminus Z(f))$. Since $f_1=\sin\frac{1}{|f(x)|}$ if $x\in V\cap(X\setminus Z(f))$, then $-1,1\in B(f_1(z))$. This contradiction proves our claim and therefore $I$ is not a closed ideal.

b) Now suppose that $X$ is a Fréchet-Urysohn space. Since $z\in cl(X\setminus Z(f))$ there is a sequence $(x_i)$ in $X\setminus Z(f)$ such that $x_i\to z$ when $i\to\infty$. Then, $|f(x_i)|\to 0$ when $i\to\infty$.

We can assume, choosing a subsequence if necessary, that $|f(x_i)|\neq|f(x_j)|$ if $i\neq j$. Then, there is a strictly decreasing subsequence $(a_k)$ of $(|f(x_i)|)$. Let $b_{2n-1}=\frac{2}{(4n+1)\pi}$ and $b_{2n}=\frac{2}{(4n+3)\pi}$ for $n\geq 1$. By the Lemma 3.1, there exists a continuous function $h:\mathbb{R}\to\mathbb{R}$ with $h(a_k)=b_k$ for all $k\geq 1$, $h(0)=0$ and $h(t)\neq 0$ for all $t\neq 0$. Then, $\sin\frac{1}{h(|f(x)|)}$ takes the values 1 and $-1$ in $V\cap(X\setminus Z(f))$ for any neighborhood $V$ of $z$.

The real function $g$ defined on $X$ as $g(x) = 0$ if $x \in Z(f)$ and $g(x) = f(x) \sin \frac{1}{h(|f(x)|)}$ if $x \notin Z(f)$, belongs to $C_b(X)$. Moreover, since $h(I) = Z(f) \subset Z(g)$, it follows from Theorem 2.3 that $g \in cl(I)$. Similarly to proof in a) we can conclude that $g \notin I$. Thus, $I$ is not a closed ideal. ∎

We do not know if the hypothesis a) and b) are essential in Theorem 3.2, but with them we cover a wide class of topological spaces including the locally convex and the metric spaces.

Corollary 3.3 *If $X$ is a connected $k$-space satisfying a) or b), then each non-zero element of $(C_b(X), \beta)$ is invertible or a topological divisor of zero and therefore every non-zero element belonging to $cl(G(A)) \setminus G(A)$ or $cl(G_t(A)) \setminus G_t(A)$ is a topological divisor of zero.*

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