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Bochner representable operators on Köthe-Bochner spaces

Abstract. Let $E$ be a Banach function space and $X$ be a real Banach space. We study Bochner representable operators from a Köthe-Bochner space $E(X)$ to a Banach space $Y$. We consider the problem of compactness and weak compactness of Bochner representable operators from $E(X)$ (provided with the natural mixed topology) to $Y$.

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1. Introduction and preliminaries. The integral representation of linear operators on function spaces (in particular, Lebesgue spaces and Orlicz spaces) has been the object of much study (see [Ph], [DP], [G1], [G2], [DS], [D], [U1], [U2], [A1], [A2], [Na1], [Na2]). In particular, K. Andrews [A1, Theorems 2 and 5], [A2, Theorem 3] and V.G. Navodnov [Na2, Corollary] obtained a Dunford-Pettis-Phillips type theorem for compact operators (resp. weakly compact operators) from the space $L^1(X)$ of Bochner integrable functions to a Banach space $Y$. Moreover, V.G. Navodnov ([Na2], [Na1]) has considered Bochner representable operators from a Köthe-Bochner space $E(X)$ to a Banach space $Y$.

In Section 2 we study the problem of compactness and weak compactness of Bochner representable operators from a Köthe-Bochner space $E(X)$ (provided with the natural mixed topology $\gamma_{E(X)}$) to a Banach space $Y$. The space $(E(X), \gamma_{E(X)})$ is a generalized DF-space, so we can apply the Grothendieck’s DF techniques (see [G1], [G2], [Ru]).

Throughout the paper $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ are real Banach spaces with the Banach duals $X^*$ and $Y^*$. Let $B_X$ and $B_Y$ denote the closed unit balls in $X$ and $Y$. Let $L(X,Y)$ stand for the space of all bounded linear operators
from \(X\) to \(Y\) provided with the uniform convergence norm \(\|\cdot\|_{X\to Y}\). The strong operator topology (briefly SOT) is the topology on \(\mathcal{L}(X,Y)\) defined by the family of seminorms \(\{p_U : x \in X\}\), where \(p_U(U) := \|U(x)\|_Y\) for \(U \in \mathcal{L}(X,Y)\) (see [DS, p. 475–477]). Let \(\mathbb{N}\) stand for the set of natural numbers.

Let \((\Omega, \Sigma, \mu)\) be a complete finite measure space. Assume that \((E, \|\cdot\|_E)\) is a Banach function space and let \(E'\) stand for the Köthe dual of \(E\). Then the associated norm \(\|\cdot\|_{E'}\) on \(E'\) can be defined for \(v \in E'\) by

\[
\|v\|_{E'} = \sup \left\{ \left| \int_{\Omega} u(\omega) v(\omega) \, d\mu \right| : u \in E, \|u\|_E \leq 1 \right\}.
\]

A Banach function space \((E, \|\cdot\|_E)\) is said to be perfect, if \(E = E''\) and \(\|u\|_{E''} = \|u\|_E\) for \(u \in E\). It is well known that \((E, \|\cdot\|_E)\) is perfect if and only if \(\|\cdot\|_E\) satisfies both the \(\sigma\)-Fatou property and the \(\sigma\)-Levy property (see [KA, Theorem 6.1.7]).

From now we will assume that \(L^\infty \subset E \subset L^1\), where the inclusion maps are continuous. Moreover, we will assume that \(\|\cdot\|_{E'}\) on \(E'\) is order continuous. Since \((L^1)^\prime = L^\infty\), the space \(L^1\) is excluded.

By \(L^0(X)\) we denote the set of \(\mu\)-equivalence classes of all strongly \(\Sigma\)-measurable functions \(f : \Omega \to X\). Let \(T_0(X)\) stand for the topology on \(L^0(X)\) of the \(F\)-norm \(\|\cdot\|_{L^0(X)}\) that generates convergence in measure on sets of finite measure.

For \(f \in L^0(X)\) let \(\tilde{f}(\omega) = \|f(\omega)\|_X\) for \(\omega \in \Omega\). Then the space

\[
E(X) = \{ f \in L^0(X) : \tilde{f} \in E \}
\]

equipped with the norm \(\|f\|_{E(X)} := \|\tilde{f}\|_E\) is a Banach space and is usually called a Köthe-Bochner space (see [CM], [L] for more details). We will denote by \(T_E(X)\) the topology of the norm \(\|\cdot\|_{E(X)}\). Recall that the algebraic tensor product \(E \otimes X\) is the subspace of \(E(X)\) spanned by the functions of the form \(u \otimes x, (u \otimes x)(\omega) = u(\omega)x\), where \(u \in E, x \in X\) and \(\omega \in \Omega\). For \(r > 0\) denote

\[
B_{E(X)}(r) = \{ f \in E(X) : \|f\|_{E(X)} \leq r \}.
\]

Let \(\tau\) be a linear topology on \(E(X)\). A linear operator \(T : E(X) \to Y\) is said to be \((\tau, \|\cdot\|_Y)\)-compact (resp. \((\tau, \|\cdot\|_Y)\)-weakly compact) if there exists a neighbourhood \(U\) of \(0\) for \(\tau\) such that \(T(U)\) is relatively norm compact (resp. relatively weakly compact) in \(Y\). By \(Bd(E(X), \tau)\) we will denote the collection of all \(\tau\)-bounded subsets of \(E(X)\).

2. Bochner representable operators on Köthe-Bochner spaces. We start by recalling terminology concerning Bochner representable operators \(T : E(X) \to Y\) (see [A1], [A2], [Na1], [Na2] for more details).

A function \(K : \Omega \to \mathcal{L}(X,Y)\) is said to be SOT-measurable if for every \(x \in X\) the function \(K_x : \Omega \ni \omega \mapsto K(\omega)(x) \in Y\) is strongly \(\Sigma\)-measurable. We
say that two SOT-measurable functions $K_1$ and $K_2$ are SOT-equivalent (briefly, $K_1 \approx K_2$) if $K_1(\omega)(x) = K_2(\omega)(x)$ for all $x \in X$ and $\mu$-almost all $\omega \in \Omega$ (see [Na1], [Na2]).

Recall that a bounded linear operator $T : E(X) \to Y$ is Bochner representable if there exists a SOT-measurable function $\mathcal{K} : \Omega \to L(X, Y)$ (called the representing kernel for $T$) such that for each $f \in E(X)$ the function $\langle f, \mathcal{K} \rangle : \Omega \ni \omega \Rightarrow \langle f(\omega), \mathcal{K}(\omega) \rangle \in Y$ is Bochner integrable and

$$T(f) = \int_\Omega \langle f(\omega), \mathcal{K}(\omega) \rangle \, d\mu \quad \text{for all } f \in E(X).$$

The following theorem will be of importance (see [Na1, Theorem 1]).

**Theorem 2.1** For a SOT-measurable function $\mathcal{K} : \Omega \to L(X, Y)$ the following statements are equivalent:

(i) $\mathcal{K}$ is the representing kernel for a Bochner representable operator $T : E(X) \to Y$.

(ii) There exists a SOT-measurable function $\mathcal{K}_0 : \Omega \to L(X, Y)$ such that $\mathcal{K}_0 \approx \mathcal{K}$ and the $\mu$-equivalence class of the function $\|\mathcal{K}_0(\cdot)\|_Y$ belongs to $E'$.

Recall that the mixed topology $\gamma[T_\mathcal{E}(X), T_0(X) \mid E(X)]$ (briefly $\gamma_E(X)$) on $E(X)$ is the finest Hausdorff locally convex topology on $E(X)$ which agrees with $T_0(X)$ on $\|\cdot\|_{E(X)}$-bounded subsets of $E(X)$ (see [B, Chap. III], [R2], [F, §3] for more details). Then

$$T_0(X) \mid E(X) \subset \gamma_E(X) \subset T_E(X).$$

Since $B_{E(X)}(1)$ is closed in $(E(X), T_0(X) \mid E(X))$ (see [KA, Lemma 4.3.4]), by [W, Theorem 2.4.1] we get

$$(1) \quad Bd(E(X), \gamma_E(X)) = Bd(E(X), \|\cdot\|_{E(X)}).$$

This means that $(E(X), \gamma_E(X))$ is a generalized DF-space (see [Ru]). Hence using the Grothendieck classical results (see [Ru, p. 429], [G1, Corollary 1 of Theorem 11], [G2, Chap. IV, 4.3, Corollary 1 of Theorem 2]) we get:

**Theorem 2.2** Let $T : E(X) \to Y$ be a $(\gamma_E(X), \|\cdot\|_Y)$-continuous linear operator which transforms $\gamma_E(X)$-bounded sets into relatively norm compact (resp. relatively weakly compact) sets in $Y$. Then $T$ is $(\gamma_E(X), \|\cdot\|_Y)$-compact (resp. $(\gamma_E(X), \|\cdot\|_Y)$-weakly compact).

A linear operator $T : E(X) \to Y$ is called $(\gamma, \|\cdot\|_Y)$-linear if $\|T(f_n)\|_Y \to 0$ whenever $\|f_n\|_{E(X)} \to 0$ and $\sup_n \|f_n\|_{E(X)} < \infty$ (see [W]). It is known that a linear operator $T : E(X) \to Y$ is $(\gamma, \|\cdot\|_Y)$-linear if and only if $T$ is $(\gamma_E(X), \|\cdot\|_Y)$-continuous (see [W, Theorem 2.6.1(iii)]).
Proposition 2.3 Assume that \((E, \|\cdot\|_E)\) is a perfect Banach function space such that the associated norm \(\|\cdot\|_{E'}\) on \(E'\) is order continuous. Then every Bochner representable operator \(T : E(X) \to Y\) is \((\gamma_{E(X)}, \|\cdot\|_Y)\)-continuous.

Proof Assume that \(T : E(X) \to Y\) is a Bochner representable operator. Then, by Theorem 2.1, there exists a SOT-measurable function \(K_0 : \Omega \to \mathcal{L}(X,Y)\) such that the \(\mu\)-equivalence class \(v_0 = [\|K_0(\cdot)\|_{X \to Y}]\) belongs to \(E'\) and

\[
T(f) = \int_\Omega \langle f(\omega), K_0(\omega) \rangle \, d\mu \quad \text{for all } f \in E(X).
\]

Hence for \(f \in E(X)\) we get

\[
\|T(f)\|_Y = \left\| \int_\Omega \langle f(\omega), K_0(\omega) \rangle \, d\mu \right\|_Y \\
\leq \int_\Omega \|\langle f(\omega), K_0(\omega) \rangle\|_Y \, d\mu \\
\leq \int_\Omega \|f(\omega)\|_X \cdot \|K_0(\omega)\|_{X \to Y} \, d\mu.
\]

Putting

\[
\varphi_{v_0}(u) = \int_\Omega u(\omega) \, v_0(\omega) \, d\mu \quad \text{for } u \in E,
\]

we see that \(\varphi_{v_0}\) is a \(\gamma\)-linear functional on \(E\) (see [N1, Theorem 3.1]). It follows that \(T\) is a \((\gamma, \|\cdot\|_Y)\)-linear operator, i.e., \(T\) is \((\gamma_{E(X)}, \|\cdot\|_Y)\)-continuous.

Before stating our main result we require a preliminary definition (see [A1, p. 258]). A SOT-measurable function \(K : \Omega \to \mathcal{L}(X,Y)\) is said to have its essential range in the uniformly norm compact operators (resp. uniformly weakly compact operators) if there exists a relatively norm compact (resp. relatively weakly compact) set \(C\) in \(Y\) such that \(\{x, K(\omega)\} \in C\) for \(\mu\)-almost all \(\omega \in \Omega\) and all \(x \in B_X\).

First we recall the well known Dunford-Pettis-Phillips type theorem for compact operators (resp. weakly compact operators) from the space \(L^1(X)\) to a Banach space \(Y\) (see [Na2, Corollary]).

Theorem 2.4 For a bounded linear operator \(T : L^1(X) \to Y\) the following statements are equivalent:

(i) \(T\) is \((\|\cdot\|_{L^1(X)}, \|\cdot\|_Y)\)-compact (resp. \((\|\cdot\|_{L^1(X)}, \|\cdot\|_Y)\)-weakly compact).

(ii) \(T\) is Bochner representable operator with the representing kernel \(K\) having its essential range in the uniformly norm compact operators (resp. uniformly weakly compact operators).

Now we are in position to extend and strengthen the implication (ii)\(\Rightarrow\)(i) of Theorem 2.4 for the case of linear operators from a Köthe-Bochner space \(E(X)\) to a Banach space \(Y\).
Theorem 2.5 Assume that $(E, \| \cdot \|_E)$ is a perfect Banach function space such that the associated norm $\| \cdot \|_{E'}$ on $E'$ is order continuous. Let $T : E(X) \to Y$ be a Bochner representable operator with the representing kernel $K$ having its essential range in the uniformly norm compact operators (resp. uniformly weakly compact operators). Then $T$ is $(\gamma_{E(X)}, \| \cdot \|_Y)$-compact (resp. $(\gamma_{E(X)}, \| \cdot \|_Y)$-weakly compact).

Proof In view of Proposition 2.3 $T$ is $(\gamma_{E(X)}, \| \cdot \|_Y)$-continuous. Since $Bd(E(X), \gamma_{E(X)}) = Bd(E(X), \| \cdot \|_{E(X)})$ (see (1)), by Theorem 2.2 it is enough to show that for every $r > 0$, the set $T(B_{E(X)}(r))$ is relatively norm compact (resp. relatively weakly compact) in $Y$. Indeed, let $r > 0$. Since the inclusion map $(E, \| \cdot \|_E) \hookrightarrow (L^1, \| \cdot \|_{L^1})$ is supposed to be continuous, there exists $r_0 > 0$ such that $B_{E(X)}(r_0) \subset B_{L^1}(X)(\frac{r}{2})$.

Moreover, by our assumption there exists a relatively norm compact (resp. relatively weakly compact) set $C$ in $Y$ such that $(x, K(\omega)) \in C$ for $\mu$-almost all $\omega \in \Omega$ and all $x \in B_X$. Now we shall show that $T(f) \in \overline{\text{conv}} C$ for all $f \in B_{E(X)}(r_0)$, where $\overline{\text{conv}} C$ stands for the norm closed (=weakly closed) convex hull of $C$ in $Y$. Indeed, let $f \in B_{E(X)}(r_0)$. Then by [DS, p. 117] there exists a sequence $(s_n)$ of $X$-valued $\Sigma$-simple functions such that $\| s_n(\omega) \|_X \leq 2 \| f(\omega) \|_X$ $\mu$-a.e. on $\Omega$ and $s_n \to f$ in $\mu$-measure. Then $\| s_n - f \|_{E(X)} \to 0$ and $\sup_n \| s_n \|_{E(X)} \leq 2 \| f \|_{E(X)} \leq 2 r_0$. Hence $\| T(s_n) - T(f) \|_Y \to 0$, because $T$ is $(\gamma, \| \cdot \|_Y)$-linear.

Since $s_n = \sum_{i=1}^{k_n} 1_{A_{n,i}} \otimes x_{n,i}$ and $s_n \in B_{E(X)}(2r_0)$ for $n \in \mathbb{N}$, we get $s_n \in B_{L^1(X)}(1)$, i.e., $\sum_{i=1}^{k_n} \| x_{n,i} \| \mu(A_{n,i}) \leq 1$ for $n \in \mathbb{N}$. Hence, using [DU, Corollary 2.2.8, p. 48] we get

$$T(s_n) = \sum_{i=1}^{k_n} T(1_{A_{n,i}} \otimes x_{n,i})$$

$$= \sum_{i=1}^{k_n} \int_{A_{n,i}} \langle x_{n,i}, K(\omega) \rangle d\mu$$

$$= \sum_{i=1}^{k_n} \| x_{n,i} \|_X \int_{A_{n,i}} \langle \frac{x_{n,i}}{\| x_{n,i} \|_X}, K(\omega) \rangle d\mu$$

$$\in \left( \sum_{i=1}^{k_n} \| x_{n,i} \|_X \mu(A_{n,i}) \right) \overline{\text{conv}} C \subset \overline{\text{conv}} C.$$

Hence $T(f) \in \overline{\text{conv}} C$, that is, $T(B_{E(X)}(r_0))$ is relatively norm compact (resp. relatively weakly compact) in $Y$, because $\overline{\text{conv}} C$ is norm compact in $Y$ by Mazur’s theorem (resp. $\overline{\text{conv}} C$ is weakly compact in $Y$ by Krein-Smulian’s theorem). Since $T(B_{E(X)}(r)) \subset \frac{r}{r_0} \overline{\text{conv}} C$, we obtain that $T(B_{E(X)}(r))$ is relatively norm compact (resp. relatively weakly compact) in $Y$, as desired.

Now we will consider Bochner representable operators on the space $L^\infty(X)$. By a Young function we mean here a continuous convex mapping $\Phi : [0, \infty) \to [0, \infty)$ that vanishes only at 0 and $\Phi(t)/t \to 0$ as $t \to 0$ and $\Phi(t)/t \to \infty$ as $t \to \infty$. 
The Orlicz-Bochner space \( L^\Phi(X) := \{ f \in L^0(X) : \int \Omega \Phi(\lambda \| f(\omega) \|_X) d\mu < \infty \text{ for some } \lambda > 0 \} \) is a Banach space under the norm \( \| f \|_{L^\Phi(X)} := \inf \{ \lambda > 0 : \int \Omega \Phi(\| f(\omega) \|_X / \lambda) d\mu \leq 1 \} \) (see [RR] for more details).

We will need the following characterization of the mixed topology \( \gamma_{L^\infty(X)} \) on \( L^\infty(X) \) (see [N_2, Theorem 4.5]).

**Theorem 2.6** Then mixed topology \( \gamma_{L^\infty(X)} \) on \( L^\infty(X) \) is generated by the family of norms \( \| \cdot \|_{L^\Phi(X)} \), where \( \Phi \) runs over the family of all Young functions.

As a consequence of Theorems 2.5 and 2.6 we get:

**Corollary 2.7** Let \( T : L^\infty(X) \to Y \) be a Bochner representable operator and assume that the representing kernel \( K \) for \( T \) has its range in the uniformly norm compact operators (resp. uniformly weakly compact operators). Then there exists a Young function \( \Phi \) such that the set

\[
\left\{ \int \Omega (f(\omega), K(\omega)) d\mu : f \in L^\infty(X), \| f \|_{L^\Phi(X)} \leq 1 \right\}
\]

is relatively norm compact (resp. relatively weakly compact) in \( Y \).

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