Best constant approximants in Orlicz-Lorentz spaces

Abstract. The best constant approximant operator is extended from an Orlicz-Lorentz space $\Lambda_{w,\phi}$ to the space $\Lambda_{w,\phi'}$, where $\phi'$ is the derivative of $\phi$. Monotonicity property of its extension is established.

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1. Introduction.

Let $\mathcal{M}_0$ be the class of all real extended $\mu$-measurable functions on $[0, \alpha)$, $0 < \alpha \leq \infty$, where $\mu$ is the Lebesgue measure. As usual, for $f \in \mathcal{M}_0$ we denote its distribution function by

$$
\mu_f(\lambda) = \mu(\{x \in [0, \alpha) : |f(x)| > \lambda\}), \quad \lambda \geq 0,
$$

and its decreasing rearrangement by

$$
f^*(t) = \inf\{\lambda : \mu_f(\lambda) \leq t\}, \quad t \geq 0.
$$

For properties of $\mu_f$ and $f^*$, the reader can see in ([1], pp. 36-42).

Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable and convex function, $\phi(0) = 0$, $\phi(t) > 0$ for $t > 0$ and let $w : (0, \alpha) \to (0, \infty)$ be a weight function, non-increasing and locally integrable. If $\alpha = \infty$, we also assume $\lim_{t \to \infty} w(t) = 0$ and $\int_0^\infty w(t)d\mu(t) = \infty$. For $f \in \mathcal{M}_0$, let

$$
\Psi_{w,\phi}(f) = \int_{\mu_f(0)}^\infty \phi(f^*(t))w(t)d\mu(t).
$$
In [3]-[9] and [11]-[13], several authors studied geometric properties of the regular Orlicz-Lorentz space \( \{ f \in \mathcal{M}_0 : \Psi_{w,\varphi}(\lambda f) < \infty \text{ for some } \lambda > 0 \} \). We consider the following subspace

\[
\Lambda_{w,\varphi} := \{ f \in \mathcal{M}_0 : \Psi_{w,\varphi}(\lambda f) < \infty \text{ for all } \lambda > 0 \}.
\]

With the Luxemburg norm, the Orlicz - Lorentz space is a Banach space. If \( w \) is constant, it coincide with the Orlicz space \( L_{\varphi} \) (see [19]). On the other hand, setting \( \varphi(t) = t^p, 1 \leq p < \infty \), we obtain the Lorentz space \( L_{w,p} \) and \( \Psi_{w,\varphi}(f) = \| f \|_{w,p}^p \).

These Lorentz spaces have been studied in [2]. If \( w(t) = \frac{t^{q-1}}{q}, 1 \leq q \leq p < \infty \), a good reference for a description of these spaces is [10].

If \( \phi' \) is the derivative of the function \( \phi \), the space \( \Lambda_{w,\phi'} \) is defined analogously.

For \( g \in \mathcal{M}_0 \) we denote

\[
N(g) = \{ x \in [0, \alpha) : g(x) \neq 0 \} \quad \text{and} \quad Z(g) = \{ x \in [0, \alpha) : g(x) = 0 \}.
\]

Observe that the inequalities \( \phi(t) \leq t\phi'(t) \leq \phi(2t), t \geq 0, \) hold. Therefore

\[
\{ f \in \Lambda_{w,\phi} : \mu(N(f)) < \infty \} \subset \Lambda_{w,\phi'}.
\]

In fact, for \( \lambda > 0 \) we have

\[
\Psi_{w,\phi'}(\lambda f) \leq \phi'(1) \int_{0<\lambda f^* \leq 1} w + \int_{\lambda f^* > 1} \phi(2\lambda f^*)w < \infty.
\]

Let \( A \subset [0, \alpha) \) be a finite measure set and we call \( \chi_A \) the characteristic function of \( A \). We define \( C(f, A) \) as the set of all constants \( c \) minimizing the expression

\[
\mathcal{E}(c) := \Psi_{w,\phi}((f - c)\chi_A).
\]

Since \( \Psi_{w,\phi} \) is a convex functional (see [11]), \( \mathcal{E}(c) \) is a continuous function. Moreover, \( \lim_{c \to \pm \infty} \mathcal{E}(c) = +\infty \). Now, it is easy to see that \( C(f, A) \) is a nonempty compact interval for every \( f \in \Lambda_{w,\phi} \). We call

\[
f = \min C(f, A) \quad \text{and} \quad \bar{f} = \max C(f, A).
\]

We denote by \( T_A \) the mapping which assigns to each \( f \in \Lambda_{w,\phi} \) the set \( C(f, A) = [f, \bar{f}] \). \( T_A \) is called the best constant approximant operator. In [14], the authors introduced the following monotonicity concept.

**Definition 1.1** \( T_A \) is said to be monotone if and only if the conditions \( f, g \in \Lambda_{w,\phi}, f \leq g \) a.e. on \( A \), \( c \in T_A(f) \) and \( d \in T_A(g) \), imply that \( c \land d \in T_A(f) \) and \( c \lor d \in T_A(g) \), where \( c \land d = \min\{c,d\} \) and \( c \lor d = \max\{c,d\} \). If \( T_A(f) \) is a singleton, this is the usual definition of monotonicity.

In [15], the monotonicity of \( T_A \) has been studied when \( f, g \) are simple functions, \( \phi(t) = t^p, 1 \leq p < \infty \) and \( A = [0, \alpha) \). In [17] the authors extended the best constant approximant operator from \( L_p \) to \( L_{p^{-1}}, p \geq 1 \), for the case that \( A \) is a finite measure.
set. $L_0$ means there the set of measurable functions which are finite a.e.. Moreover, they showed that $T_A$ is monotone. Similar results for a suitable subspace of the Orlicz space have recently appeared in [20].

In this paper we study an analogous problem in the setting of Orlicz-Lorentz spaces. More precisely, in Section 2, we extend $T_A$ from an Orlicz-Lorentz space $Λ_{w,φ}$ to the space $Λ_{w,φ'}$, while in Section 3, monotonicity property of the extended operator is established.

2. Extension of the best constant approximant operator.

In this section we extend the best constant approximant operator from an Orlicz-Lorentz space $Λ_{w,φ}$ to the space $Λ_{w,φ'}$. Throughout of this paper $A ⊂ [0,α)$ will denote a set of finite measure.

Let $f, h \in M_0$. For $x \in [0,α)$ we write

$\sigma_{h_{N(h)\cap Z(f)}}(x) = \mu_{h_{N(h)\cap Z(f)}}(|h(x)|)$

and

$\tau_{f,h}(x) = \mu_f(|f(x)|)$

$\mu \{ y : |f(y)| = |f(x)| \text{ and } h(y)sg(f(y)) > h(x)sg(f(x)) \}$

$\mu \{ y : |f(y)| = |f(x)|, h(y)sg(f(y)) = h(x)sg(f(x)) \text{ and } y \leq x \}$.

For $f, h \in Λ_{w,φ}$, we shall use in this work the one-sided Gateaux derivative

$\gamma^+(f, h) = \lim_{s \to 0^+} \frac{\Psi_{w,φ}(f + sh) - \Psi_{w,φ}(f)}{s}$.

In [16], it was proved that

$\gamma^+(f, h) = \int_{N(f)} w(\rho_{f,h})\phi'(|f|)sg(f)hd\mu + \phi'_+(0) \int_{N(h)\cap Z(f)} w(\rho_{f,h})h|d\mu|$, (1)

where $\phi'_+(0)$ is the left derivative of $\phi$ at 0, and

$\rho_{f,h}(x) = \begin{cases} \tau_{f,h}(x) & \text{if } x \in N(f) \\ \mu_f(0) + \sigma_{h_{N(h)\cap Z(f)}}(x) & \text{if } x \in N(h) \cap Z(f) \end{cases}$.

In (1), we write $w(∞) = 0$.

Remark 2.1 If $f \in Λ_{w,φ'}$ and $h = χ_A$, the right side of (1) is finite.

Now, we can give the following definition.

Definition 2.2 Let $f \in Λ_{w,φ'}$. We define the function $S_{f,A} : \mathbb{R} \to \mathbb{R}$ by

$S_{f,A}(u) = \gamma^+(f - u)\chi_A, \chi_A)$. 
Remark 2.3 Let \( f \in \Lambda_{w,\phi} \). As a direct consequence of ([18], Theorem 1.6), we have that \( c \in C(f, A) \) if and only if
\[
S_{f, A}(c) \geq 0 \quad \text{and} \quad S_{-f, A}(-c) \geq 0.
\]
Replacing in ([16], Lemmas 4.8-4.10), \( f \in [0, \alpha) \), \( f - c \), \( f - d \) and 1 by \( x \in A \), \((f - c)\chi_A, (f - d)\chi_A \) and \( \chi_A \), respectively, we obtain the following lemma.

Lemma 2.4 Let \( f \in \mathcal{M}_0 \) and \( c < d \). Then for all \( x \in A \) we have,
(a) \( \mu_{(f-c)\chi_A}(f(x) - c) \leq \mu_{(f-d)\chi_A}(f(x) - d) \) if \( f(x) \geq d \);
(b) \( \mu_{(f-d)\chi_A}(d - f(x)) \leq \mu_{(f-c)\chi_A}(c - f(x)) \) if \( f(x) < c \);
(c) \( \rho_{(f-c)\chi_A,\chi_A}(x) \leq \rho_{(f-d)\chi_A,\chi_A}(x) \) if \( f(x) \geq d \);
(d) \( \rho_{(f-d)\chi_A,\chi_A}(x) \leq \rho_{(f-c)\chi_A,\chi_A}(x) \) if \( f(x) < c \).
In addition, if \( f \in \Lambda_{w,\phi'} \) then \( S_{f, A}(d) \leq S_{f, A}(c) \).

Lemma 2.5 Let \( f \in \mathcal{M}_0 \) and \( u_0 \in \mathbb{R} \). Then
\[
\lim_{u \to u_0} \rho_{(f-u)\chi_A,\chi_A}(x) = \rho_{(f-u_0)\chi_A,\chi_A}(x), \quad x \in A.
\]
Proof Let \( x \in A \) and let \( u_n \uparrow u_0 \). First, we suppose \( f(x) \geq u_0 \). Then \( f(x) > u_n \), \( n \in \mathbb{N} \). Let
\[
E_n = \{ y \in A : |f(y) - u_n| \leq f(x) - u_n \}.
\]
Clearly, \( E_{n+1} \subset E_n \), \( n \in \mathbb{N} \), \( \mu(E_1) < \infty \), and \( \bigcap_{n=1}^{\infty} E_n = E_0 \). Then
\[
\lim_{n \to \infty} \mu(E_n) = \mu(E_0).
\]
Since
\[
\rho_{(f-u_n)\chi_A,\chi_A}(x) = \mu(A - E_n) + \mu(\{y \in A : f(y) = f(x) \text{ and } y \leq x \}),
\]
we have
\[
\lim_{n \to \infty} \rho_{(f-u_n)\chi_A,\chi_A}(x) = \mu(A - E_0) + \mu(\{y \in A : f(y) = f(x) \text{ and } y \leq x \}) = \rho_{(f-u_0)\chi_A,\chi_A}(x).
\]
Now, suppose \( f(x) < u_0 \). Let \( N \in \mathbb{N} \) be such that \( f(x) < u_n \), \( n \geq N \). For \( n \geq N \) we consider the following sets:
\[
F_n = \{ y \in A : |f(y) - u_0| \neq u_0 - f(x) \text{ and } |f(y) - u_n| > u_n - f(x) \},
\]
\[
G_n = \{ y \in A : f(y) = 2u_0 - f(x) \text{ and } |f(y) - u_n| > u_n - f(x) \}, \quad \text{and}
\]
\[
H_n = \{ y \in A : f(y) \neq 2u_0 - f(x) \text{ and } f(y) = 2u_n - f(x) \}.
\]
Clearly $F_{n+1} \subset F_n$ and $G_{n+1} \subset G_n$, $n \geq N$, $\mu(F_N) < \infty$ and $\mu(G_N) < \infty$. Moreover, $\bigcap_{n=N}^{\infty} F_n = \{y \in A : |f(y) - u_0| > u_0 - f(x)\}$ and $\bigcap_{n=N}^{\infty} G_n = \{y \in A : f(y) = 2u_0 - f(x)\}$. Therefore

$$\lim_{n \to \infty} \mu_{(f-u_n)\chi_{\Delta A}}(u_n - f(x)) = \mu(F_0) + \mu(\{y \in A : f(y) = 2u_0 - f(x)\}).$$

In addition, $\lim_{n \to \infty} \mu(H_n) = 0$. Since

$$\rho_{(f-u_n)\chi_{\Delta A}}(x) = \mu_{(f-u_n)\chi_{\Delta A}}(u_n - f(x)) + \mu(H_n) + \mu(\{y \in A : f(y) = f(x) \text{ and } y \leq x\}),$$

we have

$$\lim_{n \to \infty} \rho_{(f-u_n)\chi_{\Delta A}}(x) = \mu(F_0) + \mu(\{y \in A : f(y) = 2u_0 - f(x)\}) + \mu(\{y \in A : f(y) = f(x) \text{ and } y \leq x\}) - \rho_{(f-u_0)\chi_{\Delta A}}(x).$$

**Lemma 2.6** Let $f \in \Lambda_{w, \phi'}$ and $u_0 \in \mathbb{R}$. Then $\lim_{u \to u_0} S_{f,A}(u) = S_{f,A}(u_0)$.

**Proof** Let $x \in A$ and let $u_n \uparrow u_0$. Let $B_u = Z(f-u) \cap A$, $C_u = N(f-u) \cap A$, $D = \liminf_n C_{u_n}$, $E = [0, \mu(A)]$, $H = \{t \in E : w \text{ is continuous at } t\}$, and $F = \rho^{-1}_{(f-u_0)\chi_{\Delta A}}(H)$. As $\rho_{(f-u_0)\chi_{\Delta A}} : A \to E$ is a measure preserving transformation (see [16]),

$$\mu(A - F) = \mu\left(\rho^{-1}_{(f-u_0)\chi_{\Delta A}}(E - H)\right) = \mu(E - H) = 0.$$

From (3) and the equality $C_{u_0} = C_{u_0} \cap D$, we get

$$\mu(C_{u_0}) = \mu(C_{u_0} \cap D \cap F).$$

We consider the functions $g_u, h_u : A \to \mathbb{R}$ defined by

$$g_u = w\left(\rho_{(f-u)\chi_{\Delta A}}\right) \text{ and } h_u = w\left(\rho_{(f-u)\chi_{\Delta A}}\right)_{|A \cap C_u}.$$

From (4) and Lemma 2.5,

$$\lim_{n \to \infty} h_u \phi'(|f-u_n|)sg(f-u_n) = h_u \phi'(|f-u_0|)sg(f-u_0), \quad \text{a.e. on } C_{u_0}.$$

According to Lemma 2.4 (c) and (d), for all $n \in \mathbb{N}$ we get

$$|h_u \phi'(|f-u_n|)sg(f-u_n)| \leq g_u \phi'(|f-u_1|) + g_{u_0} \phi'(|f-u_0|), \quad \text{on } C_{u_0}.$$

Therefore, the Lebesgue Dominated Convergence Theorem implies

$$\lim_{n \to \infty} \int_{C_{u_0} \cap C_{u_n}} h_u \phi'(|f-u_n|)sg(f-u_n)d\mu = \int_{C_{u_0}} h_u \phi'(|f-u_0|)sg(f-u_0)d\mu.$$
Clearly, we have
\[ \int_{B_{u_0} \cap C_u} h_u \phi'(|f - u_n|)sg(f - u_n) d\mu = \phi'_+ (u_0 - u_n) \int_{B_{u_0}} h_u d\mu, \]
and by Lemma 2.4, (c), \( h_u \leq h_{u_1} \) on \( B_{u_0} \). From Lemma 2.5 and the Lebesgue Dominated Convergence Theorem, it follows immediately that
\[ \lim_{n \to \infty} \int_{B_{u_0} \cap C_u} h_u \phi'(|f - u_n|)sg(f - u_n) d\mu = \phi'_+ + (u_0 - u_n) \int_{B_{u_0}} h_u d\mu. \]

Finally, we observe that
\[ \int_{B_u} g_u d\mu = \frac{w}{\mu(C_u)} \int_{\mu(C_u)} w, \text{ and } \lim_{n \to \infty} \mu(C_{u_n}) = \mu(A). \]
Then
\[ \lim_{n \to \infty} \int_{B_{u_n}} g_u d\mu = 0. \]

Thus, from (5)-(7), we get \( \lim_{n \to \infty} S_{f,A}(u_n) = S_{f,A}(u_0). \)

The proof of the next theorem of a characterization of the best constant approximant maximum and minimum in \( \Lambda_{w,\phi'} \) follows in the same way as in ([16], Lemma 4.11).}

**Theorem 2.7** If \( f \in \Lambda_{w,\phi}, \)

\[ \overline{f} = \max\{u : S_{f,A}(u) \geq 0\} \text{ and } \underline{f} = \min\{u : S_{-f,A}(-u) \geq 0\}. \]

**Proof** Suppose that there exists \( u, u > \overline{f}, \) such that
\[ S_{f,A}(u) \geq 0. \]
By Lemma 2.4,
\[ S_{-f,A}(-u) \geq S_{-f,A}(-\overline{f}) \geq 0. \]
Then, (2), (9) and (10) imply that \( u \in C(f,A), \) a contradiction.
Similarly, we can see that \( \underline{f} = \min\{u : S_{-f,A}(-u) \geq 0\}. \)

The following Lemma shows us the way to extend the best constant approximant operator from \( \Lambda_{w,\phi} \) to \( \Lambda_{w,\phi'}. \)

**Lemma 2.8** If \( f \in \Lambda_{w,\phi'}, \) then \( S_{f,A} \) is a non-increasing left-continuous function. Moreover, \( \lim_{u \to +\infty} S_{f,A}(u) < 0 \) and \( \lim_{u \to -\infty} S_{f,A}(u) > 0. \) Here the limits can be infinity.
Proof From Lemmas 2.4 and 2.6, \( S_{f,A} \) is a non-increasing left-continuous function. Now, we define
\[
P(u) = \int_{\{ f \geq u \} \cap A} w(\rho_{f-u}) \phi'(f-u) d\mu, \quad u \in \mathbb{R}
\]
and
\[
Q(u) = \int_{\{ f < u \} \cap A} w(\rho_{f-u}) \phi'(u-f) d\mu, \quad u \in \mathbb{R}.
\]
Clearly \( S_{f,A}(u) = P(u) - Q(u) \). Let \( c \in \mathbb{R} \) be such that \( \mu(\{ f < c \} \cap A) > 0 \). As consequence of Lemma 2.4, \( P \) is a non-increasing function. Further, \( \rho_{f-u}(x) \leq \mu(A) \) for \( x \in A \), whence
\[
S_{f,A}(u) \leq P(u) - w(\mu(A)) \int_{\{ f < c \} \cap A} \phi'(u-f) d\mu \leq P(c) - w(\mu(A)) \int_{\{ f < c \} \cap A} \phi'(u-f) d\mu,
\]
for all \( u > c \). If \( \phi'(+\infty) = +\infty \), \( \lim_{u \to +\infty} S_{f,A}(u) = -\infty \). If \( \phi'(+\infty) < +\infty \), \( P(u) \leq \mu(\{ f \geq u \} \cap A) \int_0^\infty w, \) consequently
\[
\lim_{u \to +\infty} S_{f,A}(u) \leq -w(\mu(A)) \mu(\{ f < c \} \cap A) \phi'(+\infty) < 0.
\]
On the other hand, let \( d \in \mathbb{R} \) be such that \( \mu(\{ f \geq d \} \cap A) > 0 \). As \( Q \) is a non-decreasing function, in a similar way we obtain
\[
w(\mu(A)) \int_{\{ f \geq d \} \cap A} \phi'(f-u) d\mu - Q(d) \leq w(\mu(A)) \int_{\{ f \geq d \} \cap A} \phi'(f-u) d\mu - Q(u) \leq S_{f,A}(u),
\]
for all \( u < d \). If \( \phi'(+\infty) = +\infty \), \( \lim_{u \to -\infty} S_{f,A}(u) = +\infty \). If \( \phi'(+\infty) < +\infty \), \( Q(u) \leq \mu(\{ f < u \} \cap A) \int_0^\infty w, \) so
\[
\lim_{u \to -\infty} S_{f,A}(u) \geq w(\mu(A)) \mu(\{ f \geq d \} \cap A) \phi'(+\infty) > 0.
\]

Theorem 2.9 If \( f \in \Lambda_{w,\phi'} \), there are constants \( \overline{f} \) and \( \underline{f} \) satisfying (8). In addition, \( -\overline{f} = -f \).

Proof From Lemma 2.8 there are constant \( \overline{f} \) and \( f \) satisfying (8). Since \( S_{f,A}(-f) \geq 0 \) and \( S_{f,A}(-\overline{f}) \geq 0 \), by definition of \( -\overline{f} \) and \( \overline{f} \) we have \( -\overline{f} = -f \). □

So, Theorem 2.9 aware us to define the extended best constant approximant operator by
\[
T_A(f) = C(f, A) := [\underline{f}, \overline{f}], \quad f \in \Lambda_{w,\phi'}.
\]
3. Monotonicity of the best constant approximant operator.

In this section we show that the extension of the best constant approximant operator $T_A$ is monotone.

Given a non-constant simple function $h = \sum_{k=1}^{n} a_k \chi_{I_k}$, $I_i \cap I_j = \emptyset$, $i \neq j$, defined on $[0, \alpha)$, we write

$$\delta_h = \min\{|h_i| - |h_j| : i \neq j, \gamma_h = \min\{|h_i| : |h_i| > 0\}, \quad \beta_h = \begin{cases} \frac{\min\{|\delta_h, \gamma_h\|}{4} & \text{if } \delta_h > 0 \vspace{1ex} \\
\frac{\gamma_h}{4} & \text{if } \delta_h = 0 \end{cases}.$$ 

**Lemma 3.1** Let $a \geq 0$, $f = \sum_{k=1}^{n} a_k \chi_{I_k}$, and $h = \chi_{A_j}$, $1 \leq j \leq n$. If $0 < |a - \epsilon| < \beta_{(f + uh)\chi_A}$ and $\epsilon > 0$ then

$$S_{f + eh, A}(0)s g(a - \epsilon) \leq S_{f + ah, A}(0)s g(a - \epsilon).$$

**Proof** Without loss of generality we can assume $j = 1$. Let $z = \mu(I_1)$. For $u \in \mathbb{R}$, we denote the function $f_u : A \to \mathbb{R}$ by

$$f_u = (f + uh)\chi_A.$$ 

Assume $0 < \epsilon - a < \beta_{f_u}$. Replacing in ([15], Lemma 2.2), $[0, 1]$, $2\epsilon$, $h$, $u$, and $1$, by $A$, $\epsilon - a$, $f_u$, $(\epsilon - a)\chi_{I_1\cap A}$ and $\chi_A$ respectively, we get a measure preserving transformation $\sigma : A \to [0, \mu(A)]$ such that for all $t \in \{0, 1\}$, $s \in [0, \epsilon)$, we have

$$s g(t - t_0 + t_0 + t e + s \chi_A), \quad \text{on } A.$$ 

By the convexity of the function $\phi$, we also have for all $s \geq 0$

$$\phi(|f_s|) - \phi(|f|) \leq \phi(|f_s + s \chi_A|) - \phi(|f_s + s \chi_A|), \quad \text{on } A.$$ 

From (12) and (13), we have

$$\Phi_{w, \phi}(f_s) - \Phi_{w, \phi}(f_a) \leq \Phi_{w, \phi}(f_s + s \chi_A) - \Phi_{w, \phi}(f_s + s \chi_A), \quad s \in [0, \epsilon].$$ 

Therefore, (11) holds.

Now assume $0 < a - \epsilon < \beta_{f_s}$.

Let

$$R_a := \{|a_1 + a|, |a_2|, \ldots, |a_n|\} \quad \text{and} \quad E_a := R_a - \{0, |a_1 + a|\}.$$ 

Suppose $a_1 + a \neq 0$. If $\lambda \in R_a$, by hypotheses we obtain

$$s g(a_1 + a) = s g(a_1 + \epsilon) \quad \text{and} \quad |a_1 + \epsilon| \neq \lambda.$$ 

If $\lambda \in E_a$, then

$$\lambda > \max\{|a_1 + a|, |a_1 + \epsilon|\} \quad \text{or} \quad \lambda < \min\{|a_1 + a|, |a_1 + \epsilon|\}.$$
(15) \( \mu_{f_\lambda}(\lambda) = \mu_{f_\lambda}(\lambda), \quad \lambda \in E_a. \)

Moreover, if \( |a_k| \neq |a_1 + a|, k \geq 2, \)

(16) \( \mu(\{|f_\lambda| = |a_1 + a|\}) = 0. \)

From (14) we get

(17) \( \mu_{f_\lambda}(0) = \mu_{f_\lambda}(0), \)

(18) \( \mu(\{|f_\lambda| = \lambda\}) = \mu(\{|f_\lambda| = \lambda\}), \) and \( \mu(\{|f_\lambda| = -\lambda\}) = \mu(\{|f_\lambda| = -\lambda\}), \)

for all \( \lambda \in E_a. \)

If \( a_1 + a < 0, \) then \( s := \mu(\{|f_\lambda| = |a_1 + a|\}) = \mu(\{|f_\lambda| = |a_1 + a|\}), \)

\( r := \mu_{f_\lambda}(|a_1 + a|) = \mu_{f_\lambda}(|a_1 + a|) - z = \mu_{f_\lambda}(|a_1 + \epsilon|) = \mu_{f_\lambda}(|a_1 + \epsilon|), \)

\( t := \mu(\{|f_\lambda| = a_1 + a\}) = \mu(\{|f_\lambda| = a_1 + a\}) + z, \)

\( \mu(\{|f_\lambda| = a_1 + \epsilon\}) = \mu(\{|f_\lambda| = a_1 + \epsilon\}) = 0, \)

and \( \mu(\{|f_\lambda| = a_1 + \epsilon\}) = \mu(\{|f_\lambda| = a_1 + \epsilon\}) - z = 0. \)

From (1), (15)-(18) it follows that

\[
S_{f_{\lambda}, A}(0) = \sum_{\lambda \in E_a} \phi'(\lambda) \left( \frac{\mu_{f_\lambda}(\lambda) + \mu(\{|f_\lambda| = \lambda\})}{\mu_{f_\lambda}(\lambda)} \int_{r+z} w - \frac{\mu_{f_\lambda}(\lambda) + \mu(\{|f_\lambda| = \lambda\})}{\mu_{f_\lambda}(\lambda)} \int_{r+z} w \right) 
- \phi'(|a_1 + \epsilon|) \int r w + \phi'(|a_1 + a|) \left( \int_{r+z} w - \int_{r+z} w \right) 
+ \phi'_+(0) \int_{\mu_{f_\lambda}(0)} w.
\]

So,

\[
S_{f_{\lambda}, A}(0) - S_{f_{\lambda}, A}(0) = \phi'(|a_1 + a|) \left( \int r w - \int r w - \int_{r+z} w \right) + \phi'(|a_1 + \epsilon|) \int_{r+z} w.
\]

Since \( \phi'(|a_1 + \epsilon|) \geq \phi'(|a_1 + a|), \) we get (11).

If \( a_1 + a > 0 \) we have \( s := \mu(\{|f_\lambda| = |a_1 + a|\}) = \mu(\{|f_\lambda| = |a_1 + a|\}) + z, \)

\( r := \mu_{f_\lambda}(|a_1 + a|) = \mu_{f_\lambda}(|a_1 + a|), \)

\( t := \mu(\{|f_\lambda| = -(a_1 + a)\}) = \mu(\{|f_\lambda| = -(a_1 + a)\}), \)

\( \mu_{f_\lambda}(|a_1 + \epsilon|) = \mu_{f_\lambda}(|a_1 + \epsilon|) + z = r + s + t, \)

\( \mu(\{|f_\lambda| = a_1 + \epsilon\}) = \mu(\{|f_\lambda| = a_1 + \epsilon\}) - z = 0, \)
and \( \mu(\{f_a = -(a_1 + \epsilon)\}) = \mu(\{f_\epsilon = -(a_1 + \epsilon)\}) = 0 \).

Therefore, (1), (15)-(18) imply
\[
S_{f_a,A}(0) = \sum_{\lambda \in E_a} \phi'(\lambda) \left( \frac{\mu_{f_a}(\lambda) + \mu(\{f_a = \lambda\})}{\mu_{f_a}(\lambda)} \int w - \frac{\mu_{f_a}(\lambda) + \mu(\{f_a = \lambda\})}{\mu_{f_a}(\lambda)} \int w \right) \\
+ \phi'(|a_1 + \epsilon|) \int w + \phi'(|a_1 + \epsilon|) \left( \int w - \int w \right)
\]

In consequence,
\[
S_{f_a,A}(0) - S_{f_a,A}(0) = \phi'(|a_1 + \epsilon|) \int w + \phi'(|a_1 + \epsilon|) \int w \geq 0.
\]

The proof is complete.

\[\blacklozenge\]

**Remark 3.2** Let \( f \) be a constant function and \( h = \chi_I \), where \( I \) is a measurable subset of \( A \). Then, a straightforward computation leads to
\[
S_{f,A}(0)sg(a) \leq S_{f + ah,A}(0)sg(a), \ \ a \in \mathbb{R}.
\]

**Lemma 3.3** Let \( f \) and \( g \) be simple functions such that \( f \leq g \), a.e. on \( A \). Then
\[
S_{f,A}(0) \leq S_{g,A}(0).
\]

**Proof** We can assume \( f = \sum_{k=1}^{n} a_k \chi_{I_k} \) and \( g = \sum_{k=1}^{n} b_k \chi_{I_k} \). We define \( f_0 = f \) and \( f_m = \sum_{k=1}^{m} b_k \chi_{I_k} + \sum_{k=m+1}^{n} a_k \chi_{I_k}, 1 \leq m \leq n \). Clearly, \( f_n = g \) and
\[
f_{m+1} = f_m + (b_{m+1} - a_{m+1})h_{m+1}, 0 \leq m \leq n - 1, \ \ \text{a.e. on} \ A,
\]
where \( h_{m+1} = \chi_{I_{m+1}} \). We shall prove that

\[
S_{f_m,A}(0) \leq S_{f_{m+1},A}(0), \quad 0 \leq m \leq n-1.
\]

If \( b_{m+1} = a_{m+1} \), (19) is obvious. Assume \( a := b_{m+1} - a_{m+1} > 0 \). We consider the following set

\[
C := \{ u \in [0,a] : S_{f_m,A}(0) \leq S_{f_{m+u},A}(0) \}.
\]

We shall show that \( a \in C \). If \( f_m \) is a constant function on \( A \), from Remark 3.2, we have \( a \in C \). In otherwise, Lemma 3.1 implies \( S_{f_m,A}(0) \leq S_{f_{m+sh},A}(0) \), for all \( 0 < \epsilon < \beta_{f_m,A} \). So, \( s := \sup(C) > 0 \). If \( f_m + sh_{m+1} \) is a constant function on \( A \), as a consequence of Remark 3.2 we get \( a \in C \). Let for the contrary \( t > 0 \) be such that \( t \in C \) and \( 0 < s - t < \beta_{f_m+sh_{m+1},A} \). By Lemma 3.1,

\[
S_{f_m,A}(0) \leq S_{f_{m+sh_{m+1}},A}(0) \leq S_{f_{m+sh_{m+1}},A}(0).
\]

Therefore, \( s \in C \). Now suppose \( s < a \) and let \( r > 0 \) be such that \( r < a \) and \( 0 < r - s < \beta_{f_m+sh_{m+1},A} \). According to Lemma 3.1,

\[
S_{f_m,A}(0) \leq S_{f_{m+sh_{m+1}},A}(0) \leq S_{f_{m+rh_{m+1}},A}(0).
\]

Then \( r \in C \), a contradiction. So \( s = a \). 

**Definition 3.4** Let \( f \in \mathcal{M}_0 \). For each \( n \in \mathbb{N} \), let \( \tilde{f}_n : [0, \alpha) \to \mathbb{R} \) be the simple function defined by

\[
\tilde{f}_n(x) = \begin{cases} 
sgn(f(x)) \frac{k}{2^n} & \text{if } \frac{k}{2^n} < |f(x)| \leq \frac{k+1}{2^n}, \quad k \in \mathbb{N} \cap [1, n2^n - 1] \\
sgn(f(x)) n & \text{if } |f(x)| > n \\
0 & \text{if } |f(x)| \leq \frac{1}{2^n}
\end{cases}
\]

**Remark 3.5** If \( f, g \in \mathcal{M}_0 \) are such that \( f \leq g \), then \( \tilde{f}_n \leq \tilde{g}_n \) for all \( n \in \mathbb{N} \).

**Lemma 3.6** Let \( f \in \mathcal{M}_0 \). Then

\[
\lim_{n \to \infty} \tilde{f}_n(x) = f(x) \quad \text{and} \quad \lim_{n \to \infty} \rho_{\tilde{f}_n,\chi_A,\chi_A}(x) = \rho_{f,\chi_A,\chi_A}(x), \quad x \in A.
\]

**Proof** Let \( x \in A \). If \( f(x) = 0 \), it is obvious. Suppose \( f(x) \neq 0 \). Then there exist a sequence \( (k_n)_n \subseteq \mathbb{N} \) and \( n_0 \in \mathbb{N} \) such that \( 0 < |f(x)| - \frac{k_n}{2^n} \leq \frac{1}{2^n} \), for all \( n \geq n_0 \). Therefore

\[
\lim_{n \to \infty} \tilde{f}_n(x) = f(x).
\]

Further, if \( f(x) > 0 \) then

\[
\rho_{\tilde{f}_n,\chi_A,\chi_A}(x) = \mu(\{ y \in A : \frac{k_n}{2^n} < f(y) \leq \frac{k_n+1}{2^n} \text{ and } y \leq x \}) + \rho_{f,\chi_A}(\frac{k_n+1}{2^n}), \quad n \geq n_0.
\]
Since
\[ \lim_{n \to \infty} \mu \left( \left\{ y \in A : \frac{k_n}{2^n} < f(y) \leq \frac{k_n+1}{2^n} \text{ and } y \leq x \right\} \right) = \mu \left( \left\{ y \in A : f(y) = f(x) \text{ and } y \leq x \right\} \right) \]
and \( \mu_{f \chi A} \) is a right-continuous function, from (20), we get
\[ (21) \lim_{n \to \infty} \rho_{f \chi A, \chi A}(x) = \rho_{f \chi A, \chi A}(x). \]

In a similar way if \( f(x) < 0 \), we have
\[ (22) \rho_{f \chi A, \chi A}(x) = \mu_{f \chi A} \left( \frac{k_n+1}{2^n} \right) + \mu \left( \left\{ y \in A : \frac{k_n}{2^n} < f(y) \leq \frac{k_n+1}{2^n} \right\} \right) + \mu \left( \left\{ y \in A : \frac{k_n}{2^n} < -f(y) \leq \frac{k_n+1}{2^n} \text{ and } y \leq x \right\} \right), \quad n \geq n_0. \]

In consequence, a straightforward computation also leads to (21).

Finally, if \( f(x) = 0 \) we have
\[ (23) \rho_{f \chi A, \chi A}(x) = \mu_{f \chi A}(0) + \mu \left( \left\{ y \in A : |f(y)| \leq \frac{1}{2^n} \text{ and } y \leq x \right\} \right), \quad n \in \mathbb{N}. \]

Since
\[ (24) \lim_{n \to \infty} \mu \left( \left\{ y \in A : |f(y)| \leq \frac{1}{2^n} \right\} \right) = \mu(\left\{ y \in A : f(y) = 0 \right\}) \]
and
\[ (25) \mu_{f \chi A}(0) = \mu_{f \chi A} \left( \frac{1}{2^n} \right) = \mu(A) - \mu \left( \left\{ y \in A : |f(y)| \leq \frac{1}{2^n} \right\} \right), \quad n \in \mathbb{N}, \]
(23)-(25) imply (21).

**Lemma 3.7** Let \( f \in \Lambda_{w, \phi'} \). Then
\[ (26) \lim_{n \to \infty} S_{f \chi A}(0) = S_{f \chi A}(0). \]

**Proof** Let \( n \in \mathbb{N}, E_n = N(f_n) \cap A \) and \( F_n = Z(f_n) \cap A \). We denote \( g_n, h_n : A \to \mathbb{R} \) be the functions defined by
\[ g_n = w(\rho_{f \chi A, \chi A})\phi'(f_n) \text{sgn}(f_n) \chi_{E_n} \quad \text{and} \quad h_n = w(\rho_{f \chi A, \chi A})\phi'(f) \chi_{E_n}, \]
respectively. Since \( \phi'(f \chi A) \chi_{E_n}^* \leq \phi'(f \chi A^*) \quad \text{on } [0, \alpha) \), the Hardy - Littlewood inequality implies
\[ \int_{N(f \chi A)} h_n d\mu \leq \int_{0}^{\mu(E_n)} \phi'(f \chi A^*) w. \]
By (24) and (25), we have
\[
\liminf_{n \to \infty} \int_{N(f_{\chi A})} h_n d\mu \leq \int_{N(f_{\chi A})} \phi'(\langle f \rangle) w = \int_{N(f_{\chi A})} w(\rho_{f_{\chi A}, A}) \phi'(\langle f \rangle) d\mu.
\]

Therefore, Lemma 3.6 and Fatou’s Lemma imply
\[
\lim_{n \to \infty} \int_{N(f_{\chi A})} h_n d\mu = \int_{N(f_{\chi A})} w(\rho_{f_{\chi A}, A}) \phi'(\langle f \rangle) d\mu.
\]

Since, \( |\tilde{f}_n| \leq |f| \) on \( A \), \( |g_n| \leq h_n \) on \( N(f_{\chi A}) \). So, from Lemma 3.6, (27) and generalized Lebesgue Dominated Convergence Theorem, we get
\[
\lim_{n \to \infty} \int_{E_n} g_n d\mu = \int_{N(f_{\chi A})} w(\rho_{f_{\chi A}, A}) \phi'(\langle f \rangle) s(g(f)) d\mu.
\]

Moreover, according to (24) and (25) we get
\[
\lim_{n \to \infty} \int_{F_n} w(\rho_{f_{\chi A}, A}) d\mu = \lim_{n \to \infty} \int_{\mu(A)} w = \int_{\mu(N(f_{\chi A}))} w(\rho_{f_{\chi A}, A}) d\mu.
\]

In consequence, (28) and (29) imply (26).

**Proposition 3.8** Let \( f, g \in \Lambda_{w, \phi'} \) be such that \( f \leq g \), a.e. on \( A \), and let \( u \in \mathbb{R} \). Then
\[
S_{f, A}(u) \leq S_{g, A}(u).
\]

**Proof** We consider the functions \( h = f - u \) and \( k = g - u \). From Remark 3.5, we have
\[
\tilde{h}_n \leq \tilde{k}_n, \quad \text{a.e. on} \ A, \quad n \in \mathbb{N}.
\]

By Lemma 3.3, \( S_{h_{\chi A}, A}(0) \leq S_{k_{\chi A}, A}(0) \), \( n \in \mathbb{N} \). Therefore, Lemma 3.7 implies
\[
S_{f, A}(u) = S_{h, A}(0) \leq S_{k, A}(0) = S_{g, A}(u).
\]

The proof is complete.

**Theorem 3.9** The extended best constant approximant operator is monotone on \( \Lambda_{w, \phi'} \).
Best constant approximants in $\Lambda_{w,\phi'}$

Proof Let $f$ and $g$ be functions in $\Lambda_{w,\phi'}$ such that $f \leq g$, a.e. on $A$. Since $C(h, A)$ is a nonempty compact interval for every $h \in \Lambda_{w,\phi'}$, we can restate the definition of monotonicity as follows. $T_A$ is monotone if and only if

$$f \leq g, \text{ a.e. on } A \text{ implies } f \leq g \text{ and } \overline{f} \leq \overline{g}.$$  

From Theorem 2.9, $-h = -\overline{h}$, for each $h \in \Lambda_{w,\phi'}$. So, it will be sufficient to prove that $\overline{f} \leq \overline{g}$. According to Theorem 2.9 and Proposition 3.8, we get $0 \leq S_{f,A}(\overline{f}) \leq S_{g,A}(\overline{g})$. Thus, the definition of $\overline{f}$ implies $\overline{f} \leq \overline{g}$. ■

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References


F.E. Levis  
**UNIVERSIDAD NACIONAL DE RÍO CUARTE, CONICET, ARGENTINA**  
RUTA 36, KM. 601, RÍO CUARTE, 5800, ARGENTINA  
E-mail: flevis@exa.unrc.edu.ar  

H.H. Cuenya  
**UNIVERSIDAD NACIONAL DE RÍO CUARTE, CONICET, ARGENTINA**  
RUTA 36, KM. 601, RÍO CUARTE, 5800, ARGENTINA  
E-mail: hcuenya@exa.unrc.edu.ar  

A.N. Priori  
**UNIVERSIDAD NACIONAL DE RÍO CUARTE, ARGENTINA**  
RUTA 36, KM. 601, RÍO CUARTE, 5800, ARGENTINA  
E-mail: priorialbina@hotmail.com

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