COMMENTATIONES MATHEMATICAE Vol. 48, No. 1 (2008), 45-58

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Existence and Topological Properties of Solution Sets for Differential Inclusions with Delay

Abstract. We consider the problem $\dot{x}(t) \in A(t)x(t) + F(t, \theta_t x))$ a.e. on $[0, b]$, $x = \kappa$ on $[-d, 0]$ in a Banach space E, where κ belongs to the Banach space, $C_E([-d, 0])$, of all continuous functions from $[-d, 0]$ into E. A multifunction F from $[0, b] \times C_E([-d, 0])$ into the set, $P_{fc}(E)$, of all nonempty closed convex subsets of E is weakly sequentially hemi-continuous, $\theta_t x(s) = x(t + s)$ for all $s \in [-d, 0]$ and $\{A(t) : 0 \leqslant t \leqslant b\}$ is a family of densely defined closed linear operators generating a continuous evolution operator $\mathcal{S}(t, s)$. Under a generalization of the compactness assumptions, we prove an existence result and give some topological properties of our solution sets that generalizes earlier theorems by Papageorgiou, Rolewicz, Deimling, Frankowska and Cichon..

2000 Mathematics Subject Classification: 46E30, 28A20, 60B12, 34K25, 34K20.

Key words and phrases: Differential inclusions, mutifunctions, measures of noncompactness, delay.

1. Introduction. Differential inclusions appear in the study of nonsmooth Hamiltonian systems and nonsmooth optimal control problems as in Clark [7]. In addition, the works of de Korvin-Kleyle [27], Papageorgiou ([26], [27]), Salinetti-Wets ([35], [36]) and Yovits-Foulk-Rosi [38] illustrate the importance of the multifunctions in various applied fields, like optimization ([35], [36]), mathematical economics ([26], [27]) and in the analysis of uncertain information system ([27], [38]). In this paper we will deal with the differential inclusion

$$(P) \begin{cases} \dot{x}(t) \in A(t)x(t) + F(t, \theta_t x), & \text{t} \in \ [0, b] \\ x = \mathcal{K} & \text{on } [-d, 0], \end{cases}$$

where $b, d \geqslant 0$ and $\mathcal{K}$ belongs to the Banach space, $C_E([-d, 0])$, of all continuous functions from $[-d, 0]$ into E. $F : [0, b] \times C_E([-d, 0]) \to E$ is weakly sequentially hemi-continuous, $\theta_t x(s) = x(t + s)$ for all $s \in [-d, 0]$ and $\{A(t) : 0 \leqslant t \leqslant b\}$ is a family of densely defined closed linear operators generating a continuous evolution operator $\mathcal{S}(t,s)$.

Our purpose is to prove an existence theorem for integral solution of the problem (P). Moreover, we give some topological properties for our solution set, $S(\mathcal{K})$, of all integral solutions for (P). The problem (P) was investigated, without delay, by many authors ([10], [11], [30], [34], [6], [13] for instance). In this case when $A(t)=0$, $\mathcal{S}(t,s)=id$ and a mild solution is a Carathéodory one, we have a generalization to the existence theorems of Deimling [10], Ibrahim-Gomaa [18], Kisielewicz [20], Papageorgiou [30] and [32]. As $A(t)\neq 0$ our results extend that of [11], [30], [34] and [6]. Also in [14] we give a generalization to recent results, to the Cauchy problem

$$\begin{cases} \dot{x}(t)=f\big(t,x(t)\big), & t\in[0,T] \\ x(0)=x_0, \end{cases}$$

where $f:[0,T]\times E\longrightarrow E$ and E is a Banach space, by using weak and strong measures of noncompactness, and in [12] we study the solution set for the inclusion $\ddot{x}\ (t)\in F(t,x(t),\dot{x}(t))$ under three boundary conditions $x(0)=x_0,\ x(\eta)=x(T)$ where $0<\eta<T$. Moreover much work has been done to study the topological properties of the solution set for the differential inclusions see ([1], [2], [5], [19], [17], and [16] for instance). Papageorgiou, [31] , consider the problem $x(t)\in p(t)+\int_0^t k(t,s)extF(s,x(s))\,ds$ where $extF(t,s)$ denotes the set of extremal points of $F(s,x(s))$, this problem arise in the study of control system, also Papageorgiou, [28] , study semilinear evolution inclusions and their use in optimal problems. Volterra integral inclusions of the type $x(t)\in p(t)+\int_0^t k(t,s)F(s,x(s))ds$ has been studied by Papageorgiou, [29], in a separable Banach space E with $p(.),x(.)\in C([0,b],E)$, $K(t,s)$ be a function from $[0,b]\times[0,b]$ into the set of all continuous, linear operators from E into E $(\mathcal{L}(E))$ for all $0\leq s\leq t\leq b$, $\|k(t,s)\|\leq C<\infty$ and F is a multifunction on $[0,b]\times E$ with nonempty closed valued in E. Under certain conditions on F and K existence theorems are proved and continuity properties of the solution set are studied. Moreover, S. Hu - V. Lakshmikantham and N. S. Papageorgiou, [15], study semilinear evolution inclusions in which the linear, closed and densely defined operator which generates the strongly continuous semigroup depends on parameter.

2. Preliminaries. A multivalued function F on a Banach space E into the set of nonempty closed subsets, $P_f(E)$, of E is said to be upper semicontinuous if and only if for all closed subset A of E $F^-(A)=\{x\in E: F(x)\cap A\neq\emptyset\}$ is closed in E and F is called $w-w$ sequentially upper semicontinuous if every weakly closed subset A of E $F^-(A)$ is weakly sequentially closed. Such a multivalued function is called upper hemi-continuous (resp. weakly upper hemi-continuous) if and only if for any $x^*\in E^*$, $c\in I\!R$ $\{x\in E:\sup_{x\in F(x)}(x^*,x)<c\}$ is open in E (resp. in E_w), where E_w is the Banach space E with the weak topology and $E^*$is the dual space. Furthermore F is called weakly sequentially upper hemi-continuous if and only if for any $x^*\in E^*$ the function $h:E_w\longrightarrow I\!R$ defined by $h(x)=\sup_{x\in F(x)}(x^*,x)$ is sequentially upper hemi-continuous. For other properties of the multivalued function we refer to [21], [8], and [4] for instance.
The following lemmas is necessary in the proof our main results.

LEMMA 2.1 *([24], [22]). If γ is a measure of weak (strong) noncompactness and $A \subset C_w(I,E)$ is a family of strongly equicontinuous functions, then* $\gamma(A(I)) = \sup\{\gamma(A(t)) : t \in I\}$.

LEMMA 2.2 *([6]) Let Y and E be Banach spaces and let $F : E \to P_{fc}(Y)$ be weakly sequentially upper hemi-continuous. If there exist $a \in L^1(I, I\!R), (x_n)_{n\in I\!N} \subset C(I,E)$ and $(y_n)_{n\in I\!N\cup\{0\}} \subset L^1(I,E)$ such that $\|F(x)\| \le a(t)$ for all $x \in C(I,E)$, $x_n(t) \to x_0(t)$ weakly a.e. on I, $y_n \to y_0$ weakly in $L^1(I,E)$ and $y_n(t) \in F(x_n(t))$ a.e. on I, then $y_0(t) \in F(x_0(t))$ a.e. on I.*

LEMMA 2.3 [6] *Let $F : I \times E \longrightarrow P_{fc}(E)$ and, for each $t \in I$, let $F(t,.)$ be weakly sequentially upper hemi-continuous. If there exists $a \in L^1(I, I\!R)$ such that, for each $(t,x) \in I \times E$, $\|F(t,x)\| \le a(t)$ and, for each $x \in E$, $F(.,x)$ has a measurable selection, then for each $y \in C(I,E)$ there exists at least one measurable (and integrable) selection $\sigma(.)$ of $F(.,y(.))$.*

In this paper we consider E is a Banach space, $I = [0,b]$ and $P_{fc}(E)$ is the family of nonempty closed convex subsets of E and we shall use some steps that used by Szufla [37] . For any nonempty bounded subset Z of E, the Kuratowski measure of noncompactness, α, and the Hausdorff measure of noncompactness, α^*, are defined as:
$\alpha(Z) = \inf\{\varepsilon > 0 : Z$ admits a finite number of sets with diameter $< \varepsilon\}$.
$\alpha^*(Z) = \inf\{\varepsilon > 0 : Z$ admits a finite number of balls with radius $< \varepsilon\}$.
For the properties of α and α^* we refer to [3] and [9] for instance.
By a Kamke function we mean a function $w : I \times I\!R^+ \longrightarrow I\!R^+$ such that:
(i) w satisfies the Carathéodory conditions,
(ii) for all $t \in I$; $w(t,0) = 0$,
(iii) for any $c \in (0,b]$, $u \equiv 0$ is the only absolutely continuous function on $[0,c]$ which satisfies $\dot{u}(t) \le w(t,u(t))$ a.e. on $[0,c]$ and such that $u(0) = 0$.
Let $\mathcal{F} : I \to 2^E - \{\emptyset\}$ be measurable and integrable bounded with weakly compact values. The set of all integrable selections of $\mathcal{F}$, $S^1_{\mathcal{F}}$, is weakly compact in the Banach space, $L^1(I,E)$, of Lebesque Bochner integrable functions $f : I \to E$ endowed with the usual norm [4]. Let $\mathcal{L}(E)$ be the algebra of all continuous, linear operators from E to E. If $\mathcal{S} : I \times I \to \mathcal{L}(E)$ is such that $\mathcal{S}(t,0)x_0$ is a solution of the problem

$$(i) \begin{cases} \dot{x}(t) = A(t)x \\ x(0) = x_0 \end{cases}$$

where $\{A(t) : t \in I\}$ is a family of densely defined, closed, linear operators on E. A continuous function $x : [-d,b] \to E$ is called an integral solution of the problem (P) if

$$x = \mathcal{K} \text{ on } [-d,0] \text{ and } x(t) = \mathcal{S}(t,0)\mathcal{K}(0) + \int_0^t \mathcal{S}(t,s)f(s)ds \text{ for all } t \in I,$$

since $f(s) \in F(s,\theta_s x)$ and $f \in L^1(I,E)$. $\mathcal{S}$ (.,.) is called a fundamental solution of (i) or the family $\{A(t) : t \in I\}$ is a generator of $\mathcal{S}$ (.,.) see [34] and [33].

3. Main Results.

THEOREM 3.1 *Let β be either a Kuratowski measure of noncompactness or a Hausdorff measure of noncompactness; $0 \leq M < \infty$: w a Kamke function and let F be a multifunction from $[0,b] \times C_E([-d,0])$ into the set, $P_{fc}(E)$, of all nonempty closed convex subsets of E such that*

(F_1) for each $\varepsilon > 0$, there exists a closed subset I_ε of I with $\lambda(I - I_\varepsilon) < \varepsilon$ such that for any nonempty bounded subset A of $C_E([-d,0])$ and for each closed subset $J \subseteq I_\varepsilon$, one has

$$\beta(F(J \times A)) \leq \sup_{t \in J} w(t, \beta(A(0)))$$

(F_2) $\|F(t,\mathcal{K})\| \leq c(t)(1+ \| \mathcal{K}(0)\|$ for each $\mathcal{K} \in C_E([-d,0])$ and for some $c \in L^1(I, I\!R)$ a.e. on I,

(F_3) F(., $\mathcal{K}$) has a measurable selection, for each $\mathcal{K} \in C_E([-d,0])$

(F_4) for each $t \in I$, $F(t,.)$ is weakly sequentially upper hemi-continuous. Further, let $\{A(t) : t \in I\}$ is a generator of a fundamental solution $\mathcal{S} : I \times I \longrightarrow \mathcal{L}(E)$ such that

a) $\mathcal{S}(t,t) = id$, $t \in I$, id is the identity function on E;

b) $\mathcal{S}(t,s)\mathcal{S}(s,r) = \mathcal{S}(t,r), \quad t,s,r \in I$;

c) $\mathcal{S}$ is continuous;

d) $\|\mathcal{S}(t,s)\| \leq M$, $t,s \in I$;

e) for each $s \in I$, $\mathcal{S}(.,s)$ is uniformly continuous.

Then, for each $\mathcal{K} \in C_E([-d,0])$, problem (P) has an integral solution. and the solution set of all integral solutions of (P), $S(\mathcal{K})$, is compact.

PROOF First we drive a priori bound for the integral solutions of problem (P) on I. If x is such a solution, then we have x equal to $\mathcal{K}$ on $[-d,0]$ and $x(t) = \mathcal{S}(t,0)\mathcal{K}(0) + \int_0^t \mathcal{S}(t,s)f(s)ds$ for all $t \in I$ with $f(s) \in F(s, \theta_s x)$ and $f \in L^1(I,E)$. So, for each $t \in I$,

$$\begin{aligned}
\|x(t)\| &\leq \|\mathcal{S}(t,0)\|\|\mathcal{K}(0)\| + \int_0^t \|\mathcal{S}(t,s)\|\|f(s)\|\,ds \\
&\leq M\|\mathcal{K}(0)\| + \int_0^t Mc(t)(1 + \|x(s)\|)\,ds \\
&\leq M\|\mathcal{K}(0)\| + M\|c\| + \int_0^t Mc(t)\|x(s)\|\,ds.
\end{aligned}$$

if $M_1 = (M\|\mathcal{K}(0)\| + \|c\|)e^{M\|c\|}$, then $\|x(t)\| \leq M_1$. Put $\varphi(t) = c(t)(1 + M_1)$. So we may assume without any loss of generality $\|F_1(t,x(t))\| \leq \varphi(t)$ a.e. on I since, otherwise, with $B_{M_1} = \{x \in E : \|x(t)\| \leq M_1\}$, we can replace F by F' which is defined by

$$F'(t,x(t)) = \begin{cases} F_1(t,x(t)) & \text{if } x \in B_{M_1} \\ F_1(t, \frac{M_1.x(t)}{\|x\|}) & \text{if } x \notin B_{M_1}. \end{cases}$$

For arbitrary $n \in I\!N$, define $\Phi_1 : [-d, \frac{b}{n}] \times E \to E$ by

$$\Phi_1(t,x) = \begin{cases} \mathcal{K}(\mathrm{t}) & \text{if } t \in [-d, 0] \\ \mathcal{K}(0) + nt(x - \mathcal{K}(0)) & \text{if t} \in [0, \frac{b}{n}] \end{cases}$$

and define $F_1 : [0, \frac{b}{n}] \times E \to P_{fc}(E)$ by $F_1(t,x) = F(t, \theta_{\frac{b}{n}}(\Phi_1(.,x)))$.Thus, from Lemma 2.3, for each $v \in C([0, \frac{b}{n}], E)$ we can find at least one integrable selection σ of $F_1(., v(.))$. Consequently we can define a multivalued function $\mathcal{G} : B_{M_1} \subseteq C([0, \frac{b}{n}], E) \longrightarrow 2^{C([0,\frac{b}{n}],E)}$ by

$$(\mathcal{G}x)(t) = \mathcal{S}(t,0)\mathcal{K}(0) + \int_0^t \mathcal{S}(t,s)F(t, \theta_{\frac{b}{n}}(\Phi_1(., x(s))))ds,$$

for each $x \in B_{M_1}$, $\mathcal{G}x \neq \emptyset$. Since $\mathcal{S}$ is continuous we can define a function $\xi : L^1([0, \frac{b}{n}], E) \longrightarrow C([0, \frac{b}{n}], E)$ by $\xi(f)(t) = \mathcal{S}(t,0)\mathcal{K}(0) + \int_0^t \mathcal{S}(t,s)f(s)\,ds$. If we set $V = \{f \in L^1([0, \frac{b}{n}], E) : \|f\| \leq \varphi(t)$ a. e. on $[0, \frac{b}{n}]\}$, then V is uniformly integrable in $L^1([0, \frac{b}{n}], E)$ and, since $\mathcal{S}(., s)$ is uniformly continuous, $\xi(V) = \{x \in C([0, \frac{b}{n}], E) : x(t) = \mathcal{S}(0,t)\mathcal{K}(0) + \int_0^t \mathcal{S}(t,s)f(s)ds\ ,\ f \in V\}$ is nonempty equicontinuous subset of $C([0, \frac{b}{n}], E)$ and so, $\overline{conv}\xi(V)$ is nonempty convex closed equibounded and equicontinuous subset of $C([0, \frac{b}{n}], E)$.

Let $(x_m, y_m) \in$ Graph $\mathcal{G}$ such that $x_m \to x$, $y_m \to y$ in C$([0, \frac{b}{n}], E)$ $y_m : I \to C([0, \frac{b}{n}], E)$ is given by $y_m(t) = \mathcal{S}(t,0)\mathcal{K}(0) + \int_0^t \mathcal{S}(t,s)f_m(s)\,ds$, $f_m \in L^1([0, \frac{b}{n}], E)$, $f_m(s) \in F_1(s, x_m(s))$ and

$$f_m(t) = \begin{cases} \mathcal{S}(t,0)\mathcal{K}(0) & \text{if } 0 \leq t \leq \frac{b}{nm} \\ \mathcal{S}(t,0)\mathcal{K}(0) + \int_0^{t-\frac{b}{nm}} \mathcal{S}(t,s)f_m\,(s)ds & \text{if } \frac{b}{nm} \leq t \leq \frac{b}{n}. \end{cases}$$

Thus

$$\begin{aligned} \lim_{m\to\infty} \|\xi(f_m) - f_m\| &= \lim_{m\to\infty} \sup_{t\in[0,\frac{b}{n}]} \|\xi(f_m)(t) - f_m(t)\| \\ &\leq \lim_{m\to\infty} (\sup_{t\in[0,\frac{b}{nm}]} \|\xi(f_m)(t) - f_m(t)\| + \sup_{t\in[\frac{b}{nm},\frac{b}{n}]} \|\xi(x_n)(t) - f_m(t)\|) \\ &\leq \lim_{m\to\infty} (\sup_{t\in[0,\frac{b}{nm}]} \int_0^t \|\mathcal{S}(t,s)f_m\|ds + \sup_{t\in[\frac{b}{nm},\frac{b}{n}]} \| \int_0^t \mathcal{S}(t,s)f_m\,ds \\ &\qquad - \int_0^{t-\frac{b}{nm}} \mathcal{S}(t,s)f_m\,ds\|) \\ &\leq \lim_{m\to\infty} (\sup_{t\in[0,\frac{b}{nm}]} \int_0^t M\varphi(s)\,ds + \sup_{t\in[\frac{b}{nm},\frac{b}{n}]} \int_{t-\frac{b}{nm}}^t \|\mathcal{S}(t,s)f_m\|ds) \\ &\leq \lim_{m\to\infty} (\sup_{t\in[0,\frac{b}{nm}]} \int_0^t M\varphi(s)\,ds + \sup_{t\in[\frac{b}{nm},\frac{b}{n}]} \int_{t-\frac{b}{nm}}^t M\varphi(s)) = 0. \end{aligned}$$

Obviously the sets $H := \{f_m : m \in I\!N\}$ and $G := \{\xi(f_m) : m \in I\!N\}$ are equicontinuous. Let $\rho(t) := \beta(H(t))$, $t \in [0, \frac{b}{n}]$. Then $\rho(0) = 0$. We claim that ρ is differentiable a.e. on $[0, \frac{b}{n}]$. Since $\|f_m - \xi(f_m)\| \to 0$ as $m \to \infty$ so, from Lemma 2.1, $\beta((Id - \xi)H) = 0$ which given that

$$\beta(\{f_m : m \in I\!N\}) = \beta(\{\xi(f_m) : m \in I\!N\}).$$

Since for all t, $\tau \in [0, \frac{b}{n}]$,

$$\beta\{\xi(f_m)(\tau) : m \in I\!N\} \leq \beta\{\xi(f_m)(t) : m \in I\!N\} + \beta\{\xi(f_m)(\tau) - \xi(f_m)(t) : m \in I\!N\}$$

and

$$\beta\{\xi(f_m)(t) : m \in I\!N\} \leq \beta\{\xi(f_m)(\tau) : m \in I\!N\} + \beta\{\xi(f_m)(t) - \xi(f_m)(\tau) : m \in I\!N\},$$

then$|\ \rho(\tau) - \rho(t)| \leq 2\beta(B(0,1)) \int_t^\tau \mathrm{M}\varphi(s)\, ds$. therefore ρ is absolutely continuous function and thus it is differentiable a.e. on $[0, \frac{b}{n}]$. Let $(t, \tau) \in [0, \frac{b}{n}] \times [0, \frac{b}{n}]$ such that $t \leq \tau$. Since ρ is continuous and w is Caratheodory we can find a closed subset I_ε of $[0, \frac{b}{n}]$, $\delta > 0$, $\eta > 0\,(\eta < \delta)$ and for $s_1, s_2 \in I_\varepsilon$; $r_1, r_2 \in [0, \frac{2b}{n}]$ such that if $|s_1 - s_2| < \delta$, $|r_1 - r_2| < \delta$, then $|w(s_1, r_1) - w(s_2, r_2)| < \varepsilon$ and if $|s_1 - s_2| < \eta$, then $|\rho(s_1) - \rho(s_2)| < \frac{\delta}{2}$. Consider the following partition, to $[t, \tau]$, $t = t_0 < t_1 < \cdots < t_r = \tau$ such that $t_i - t_{i-1} < \eta$ for $i = 1, \cdots, r$. Let $A_i = \{x(s) : x \in H,\, s \in [t_{i-1}, t_i] \cap I_\varepsilon\}$. Let Z be a bounded subset of E and $A = \{\theta_{\frac{b}{n}}(\Phi_1(., x)) : x \in Z\}$. Thus, for each t$\in [0, \frac{b}{n}]$, $\beta(F_1(\{t\} \times Z)) = \beta(F(\{t\} \times A))$. From Condition (1) we can find a closed subset J_ε of $[0, \frac{b}{n}]$ such that $\lambda(I - J_\varepsilon) < \varepsilon$ and that for any compact subset C of J_ε $\beta(F_1(C \times Z)) = \beta(F(C \times A)) \leq \sup_{s \in C} w(s, \beta(Z))$. Let $T_i = J_\varepsilon \cap [t_{i-1}, t_i] \cap I_\varepsilon$, $P = \sum_{i=1}^m T_i = [t, \tau] \cap J_\varepsilon \cap I_\varepsilon$ and $Q = [t, \tau] - P$. Thus $\int_t^\tau F_1(s, H(s))\, ds \subset \int_P F_1(s, H(s))\, ds + \int_Q F_1(s, H(s))$. In virtue of Lemma 2.1 and from the continuity of ρ, we have

$$\beta(A_i) = \sup\{\beta(H(s)) : s \in [t_{i-1}, t_i] \cap I_\varepsilon\},$$

and by the mean value theorem we obtain

$$\int_P F_1(s, H(s))\, ds \subset \sum_{i=1}^m \int_{T_i} F_1(s, H(s))\, ds \subset \sum_{i=1}^n \lambda(T_i) \overline{conv} F_1(T_i \times A_i)$$

Now, we have

$$\begin{aligned}
\beta(\int_P F_1(s,H(s))\,ds) &\le \sum_{i=1}^{m} \lambda(T_i)\beta(F_1(T_i\times A_i))\\
&\le \sum_{i=1}^{m} \lambda(T_i)\sup_{s_i\in T_i} w(s_i,\beta(A_i))\\
&= \sum_{i=1}^{m} \lambda(T_i) w(q_i,\rho(p_i)); \quad (q_i,\, p_i\in T_i)\\
&\le \sum_{i=1}^{m} \int_{T_i} w(s,\rho(s))\,ds + \varepsilon\lambda(T_i)\\
&= \int_P w(s,\rho(s))\,ds + \varepsilon\lambda(P)\\
&\le \int_t^{\tau} w(s,\rho(s))\,ds + \varepsilon(\tau - t).
\end{aligned}$$

Moreover, we get $\beta(\int_Q F_1(s,H(s))\,ds) \le 2\int_Q \varphi(s)\,ds$.As $\lambda(Q) < 2\varepsilon$ and since ε is arbitrary, then

$$\beta(\int_t^{\tau} F_1(s,H(s))\,ds) \le \int_t^{\tau} w(s,\rho(s))\,ds. \tag{1}$$

On the other hand, we have

$$\beta(\xi(H)(\tau)) \le \beta(\xi(H)(t)) + \beta(\int_t^{\tau} F_1(s,H(s))\,ds). \tag{2}$$

By relations (1) and (2) we get

$$\rho(\tau) - \rho(t) \le \beta(\int_t^{\tau} F_1(s,H(s))\,ds) \le \int_t^{\tau} w(s,\rho(s))\,ds.$$

Therefore $\dot\rho(t) \le w(t,\rho(t))$ a.e. on $[0,\frac{b}{n}]$. Since $\rho(0)=0$ and w is a Kamke function, then $\rho\equiv 0$. Thus the weak closure of $(f_m)_{m\in I\!N}$ is weakly compact and so we can suppose that the sequence $(f_m)_{m\in I\!N}$ converges to a continuous function x_1 such that $x_1 = \mathcal{K}$ on $[-d,0]$ and for each $t\in[0,\frac{b}{n}]$

$$x_1(t) = \mathcal{S}(t,0)\mathcal{K}(0) + \int_0^t \mathcal{S}(t,s) l_1(s)ds$$

where $l_1(s) \in F(s,\theta_{\frac{b}{n}}(\Phi_1(.,x_1(t))))$ a.e. on $[0,\frac{b}{n}]$.

Now, by the mathematical induction for some $k\in\{2,3,...n\}$, we can assume that there exists the function x_{k-1} such that $x_{k-1} = \mathcal{K}$ on $[-d,0]$ and for each $t\in[0,\frac{(k-1)b}{n}]$

$$x_{k-1}(t) = \mathcal{S}(t,0)\mathcal{K}(0) + \int_0^t \mathcal{S}(t,s) l_{k-1}(s)ds$$

$$l_{k-1}(s) \in F(t, \theta_{\frac{(k-1)b}{n}} \Phi_{k-1}(., x_{k-1}(s))) \text{ a.e. and } l_{k-1} \in L^1([0, \frac{(k-1)b}{n}], E)$$

also let $\Phi_k : [-d, \frac{kb}{n}] \times E \to E$ be such that

$$\Phi_k(t,x) = \begin{cases} x_{k-1}(\mathrm{t}) & \text{if } t \in [-d, \frac{(k-1)b}{n}] \\ x_{k-1}(\frac{(k-1)b}{n}) + n(t - \frac{(k-1)b}{n})(x - x_{k-1}(\frac{(k-1)b}{n})) & \text{if t} \in [\frac{(k-1)b}{n}, \frac{kb}{n}]. \end{cases}$$

Arguing as in above, for the multifunction $F_k : [\frac{(k-1)b}{n}, \frac{kb}{n}] \times E \to P_{fc}(E)$ which is defined by $F_k(t,x) = F(t, \theta_{\frac{kb}{n}}(\Phi_k(., x)))$, we have a continuous function x_k defined on $[\frac{(k-1)b}{n}, \frac{kb}{n}]$ by

$$x_k(t) = \mathcal{S}(t, \frac{(k-1)b}{n}) x_{k-1}(\frac{(k-1)b}{n}) + \int_{\frac{(k-1)b}{n}}^{t} \mathcal{S}(t,s) l_k(s) ds$$

where $l_k(s) \in F(s, \theta_{\frac{kb}{n}}(\Phi_k(., x_k(s))))$ a.e on $[\frac{(k-1)b}{n}, \frac{kb}{n}]$ and $l_k \in L^1([\frac{(k-1)b}{n}, \frac{kb}{n}], E)$. Moreover, if we put $k' = k-1$, then for each $t \in [\frac{k'b}{n}, \frac{kb}{n}]$ we have

$$x_{k'}(\frac{k'b}{n}) = \mathcal{S}(\frac{k'b}{n}, 0)\mathcal{K}(0) + \int_0^{\frac{k'b}{n}} \mathcal{S}(\frac{k'b}{n}, s) l_{k'}(s) ds$$

hence

$$\begin{aligned} x_k(t) &= \mathcal{S}(t, \frac{k'b}{n})\mathcal{S}(\frac{k'b}{n}, 0)\mathcal{K}(0) + \int_0^{\frac{k'b}{n}} \mathcal{S}(t, \frac{k'b}{n})\mathcal{S}(\frac{k'b}{n}, s) l_{k'}(s) ds \\ &\quad + \int_{\frac{k'b}{n}}^{t} \mathcal{S}(t,s) l_k(s) ds \\ &= \mathcal{S}(t,0)\mathcal{K}(0) + \int_0^{\frac{k'b}{n}} \mathcal{S}(t,s) l_{k'}(s) ds + \int_{\frac{k'b}{n}}^{t} \mathcal{S}(t,s) l_k(s) ds \\ &= \mathcal{S}(t,0)\mathcal{K}(0) + \int_0^t \mathcal{S}(t,s) g_k(s) ds, \end{aligned}$$

where

$$g_k(t) = \begin{cases} l_{k'}(\mathrm{t}) & \text{if } t \in [0, \frac{k'b}{n}] \\ l_k(t) & \text{if t} \in [\frac{k'b}{n}, \frac{kb}{n}]. \end{cases}$$

Consequently, for all $n \in I\!N$, we have a continuous function v_n such that $v_n = \mathcal{K}$ 0n [-d,0] and for each $t \in I$ is defined by

$$v_n(t) = \mathcal{S}(t,0)\mathcal{K}(0) + \int_0^t \mathcal{S}(t,s) h_n(s) ds,$$

with

$$h_n(t) \in F(t, \theta_{\frac{kb}{n}} \Phi_k(., v_n(t))) \text{ a.e. on } I.$$

where $t \in [\frac{(k-1)b}{n}, \frac{kb}{n}] \subset I$, for $k \in \{1,2,3,...n\}$. Now we claim that the set $L = \{v_n : n \in I\!N\}$ is an equicontinuous set . So let $t_1, t_2 \in I$ with $t_1 < t_2$. Then

$$\begin{aligned}
&\| \quad v_n(t_1) - v_n(t_2) \| \leq \| \mathcal{S}(t_1,0) - \mathcal{S}(t_2,0) \| \| \mathcal{K}(0) \| \\
+\int_0^{t_1} &\| \quad \mathcal{S}(t_1,s) - \mathcal{S}(t_2,s) \| \| h_n(s) \| \, ds + \int_{t_1}^{t_2} \| \mathcal{S}(t_2,s) \| \| h_n(s) \| \, ds \\
&\leq \quad \| \mathcal{S}(t_1,0) - \mathcal{S}(t_2,0) \| \| \mathcal{K}(0) \| \\
+\int_0^{t_1} &\| \quad \mathcal{S}(t_1,s) - \mathcal{S}(t_2,s) \| \| \varphi(s) \| \, ds + M \int_{t_1}^{t_2} \| \varphi(s) \| \, ds
\end{aligned}$$

and, since $v_n = \mathcal{K}$ on [-d,0], this shows that L is equicontinuous in $C_E[-d,b]$. Moreover the set $\beta(L(t)) = \beta(\{v_n(t) : n \in I\!N\})$ is such that $\beta(L(0)) = 0$ and, by the same as above, we get $\beta(L(t)) = 0$ for all $t \in I$. Thus by Ascoli's theorem we may the sequence $\{v_n : n \in I\!N\}$ converges uniformly to a function $v \in C_E([-d,b])$ such that $y = \mathcal{K}$ on [-d,0]. But $\beta(\{h_n(t) : n \in I\!N\}) = 0$ and so $\{h_n(t) : n \in I\!N\}$ is relatively compact. Create a new multivalued function $\mathcal{F}(t) = \overline{conv}\{h_n(t) : n \in I\!N\}$. Thus $\mathcal{F}(t)$ is nonempty convex and compact. Now we can say that the set $\delta^1_{\mathcal{F}} = \{l \in L^1(I,E) : l(t) \in \mathcal{F}(t)\}$ is nonempty convex and weakly compact. By Eberlein-Šmulian Theorem there exists a subsequence (h_{n_k}) of (h_n) such that $h_{n_k} \to l$ weakly, $l \in \delta^1_{\mathcal{F}}$. Thus v_n tends weakly to $\mathcal{S}(t,0)\mathcal{K}(0) + \int_0^t \mathcal{S}(t,s)l(s)ds$. Moreover since, for each $n \in I\!N$, $v_n \in C_E([-d,b])$, v_n converges uniformly to v on each compact subset of $[-d,b]$ and v is uniformly continuous on [-d,0]; also for each $t \in I$, there exists $n > \frac{b}{d}$ with $t \in [\frac{(k-1)b}{n}, \frac{kb}{n}]$ for $k \in \{1,2,...n-1\}$ so, as $k' = k-1$,

$$\begin{aligned}
&\| \quad \theta_{\frac{kb}{n}} \Phi_k(.,v_n(t)) - \theta_t v \| \\
&\leq \quad \sup_{s \in [-d, -\frac{b}{n}]} [\| \Phi_k(\frac{kb}{n}+s, v_n(t)) - v(\frac{kb}{n}+s) \| \\
&+ \quad \| \quad v(\frac{kb}{n}+s) - v(t+s) \|] \\
+ \sup_{s \in [-\frac{b}{n},0]} [\Big(&\| \quad v_n(\frac{k'b}{n}) + n(\frac{b}{n}+s)(v_n(t) - v_n(\frac{k'b}{n})) - v(\frac{kb}{n}+s) \| \Big) \\
&+ \quad \| \quad v(\frac{kb}{n}+s) - v(t+s) \|] \\
&\leq \quad \sup_{s \in [-d, -\frac{b}{n}]} [\| v_n(\frac{kb}{n}+s) - v(\frac{kb}{n}+s) \| \\
&+ \quad \| \quad v(\frac{kb}{n}+s) - v(t+s) \|] \\
+ \sup_{s \in [-\frac{b}{n},0]} [\Big(b &\| \quad (v_n(t) - v_n(\frac{k'b}{n})) \| + \| v_n(\frac{k'b}{n}) - v(\frac{kb}{n}+s) \| \\
&+ \quad \| \quad v(\frac{kb}{n}+s) - v(t+s) \| \Big)] \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus, from Lemma 2.2, we conclude that the solution set , $S(\mathcal{K})$, of integral solutions of (P) is nonempty. Next if $\{v_n : n \in I\!N\}$ is a sequence of $S(\mathcal{K})$, then arguing as in the proof above we can show that, for each $t \in I$, $\beta(\{v_n(t) : n \in I\!N\}) = 0$. Thus this sequence has a convergent subsequence, so $S(\mathcal{K})$ is compact. ■

If we replace in Theorem 3.1 β by a measure of weak noncompactness, then for each $\mathcal{K} \in C_E([-d, 0])$ the solution set of all integral solutions of Problem (P), $S(\mathcal{K})$, is nonempty weakly compact subset of $C([-d, b], E)$. Moreover we can define the multifunction $S : C_E([-d, 0]) \to 2^{C([-d,b],E)}$ such that, for each $\mathcal{K} \in C_E([-d, 0])$, $S(\mathcal{K})$ is the solution set of problem (P).

In the following theorem we assume:

C(H) $\{H_n : n \in I\!N\}$ be a sequence of multifunctions from $I \times C_E([-d, 0])$ into the set, $P_{fc}(E)$, of nonempty closed convex subsets of E such that

(1) $H(t, \mathcal{K}) = \cap_{n=1}^{n=\infty} H_n(t, \mathcal{K})$,
(2) $H_{n+1}(t, \mathcal{K}) \subset H_n(t, \mathcal{K})$, for all $n \in I\!N$,
(3) if h is the Hausdorff distance, then $\lim_{n\to\infty} h(H_n(t, \mathcal{K}), H(t, \mathcal{K})) = 0$,
(4) for some $C > 0$ $\| H_n(t, \mathcal{K}) \| \leq C$,
(5) for $n \in I\!N$, H_n satisfies conditions F_1, F_3 and F_4 of Theorem 3.1.

THEOREM 3.2 *If E is a separable Banach space, $H(t, .)$ is weakly sequentially upper hemi-continuous, $H(., \mathcal{K})$ has a measurable selection and hypotheses C(H) hold, then for each $\mathcal{K} \in C_E([-d, 0])$ $S_H(\mathcal{K}) = \cap_{n=1}^{n=\infty} S_{H_n}(\mathcal{K})$.*

PROOF Thanks to our assumptions, we obtain the solution set $S_H(\mathcal{K})$ is nonempty and for any $n \in I\!N$, $S_H(\mathcal{K}) \subseteq S_{H_n}(\mathcal{K})$, so $S_H(\mathcal{K}) \subseteq \cap_{n=1}^{n=\infty} S_{H_n}(\mathcal{K})$. Conversely, let $v \in \cap_{n=1}^{n=\infty} S_{H_n}(\mathcal{K})$. Thus there exists h_n such that, for each $t \in I$, $h_n(t) \in H_n(t, \theta_t v)$ and $v(t) = \mathcal{S}(t, 0)\mathcal{K}(0) + \int_0^t \mathcal{S}(t, s)h_n(s)ds$. From condition C(H)(3) we have $h_n(t) \in H(t, \theta_t v) + \varepsilon_n(t)\overline{B_1}$ a.e. on I, where $\varepsilon_n(t) = h(H_n(t, \theta_t v), H(t, \theta_t v)) \to 0$ as $n \to \infty$ and $\overline{B_1}$ is the closed unit ball in E. By condition C(H)(4), the sequence $\{h_n : n \in I\!N\}$ is uniformly bounded. We consider a subsequence $\{h_{n_k}(t) : n_k \in I\!N\}$ and we can passing to convex combination of $h_{n_k}(t)$ denoted by $\tilde{h}_{n_k}(t)$. Thus $\tilde{h}_{n_k}(t) \to l(t) \in E$ and moreover $\tilde{h}_{n_k}(t) \in \sum_{m\geq n} \gamma_m(H(t, \theta_t v) + \varepsilon_m(t)\overline{B_1})$ a.e. on I, where $\gamma_m(t) \geq 0$ and also $\sum_{m\geq n} \gamma_m = 1$. At this point, we let $n_k \to \infty$ and since H has convex values, so $l(t) \in H(t, \theta_t v)$ and hence the result. ■

THEOREM 3.3 *The multifunction S is upper semicontinuous and both the multifunctions $S_t : C_E([-d, 0]) \to 2^E$, defined by $S_t(\mathcal{K}) = \{v(t) : v \in S(\mathcal{K})\}$ and that $S_{\mathcal{K}} : I \to 2^E$ which is defined by $S_{\mathcal{K}}(t) = \{v(t) : v \in S(\mathcal{K})\}$ is upper semicontinuous and has compact values. Further, the set $\cup_{t\in I} S_{\mathcal{K}}(t)$ is compact in E.*

PROOF For each closed subset Z of $C_E([-d, b])$, to show that S is upper semicontinuous, we claim that $\mathcal{A}=\{\mathcal{K} \in C_E([-d, 0]) : S_{\mathcal{K}} \cap Z \neq \emptyset\}$ is sequentially closed in $C_E([-d, 0])$. Let $\{\mathcal{K}_n : n \in I\!N\} \subset \mathcal{A}$ such that $\mathcal{K}_n \to \mathcal{K}$. Then $S_{\mathcal{K}_{\rm n}} \cap Z \neq \emptyset$ and hence there exists $v_n \in S_{\mathcal{K}_{\rm n}} \cap Z$, where $v_n(t) = \mathcal{S}(t, 0)\mathcal{K}_{\rm n}(0) + \int_0^t \mathcal{S}(t, s)g_n(s)ds$,with $g_n(s) \in F(s, \theta_s v_n)$ a.e. on I and $g_n(.) \in L^1(I, E)$. Now, for each $t \in I$, we have

$$\beta(\{v_n(t) : n \in I\!N\}) \leq M\beta(\{\mathcal{K}_{\rm n}(0) : n \in I\!N\}) + M\beta(\{\int_0^t g_n(s)ds : n \in I\!N\}).$$

But $\beta(\{\mathcal{K}\text{n}(0) : n \in I\!\!N\}) = 0$, where $\mathcal{K}_n \to \mathcal{K}$. Thus

$$\beta(\{v_n(t) : n \in I\!\!N\}) \leq M\beta(\{\int_0^t g_n(s)ds : n \in I\!\!N\}).$$

Arguing as in the proof of Theorem 3.1 we have $\beta(\{v_n(t) : n \in I\!\!N\}) = 0$. Now since the sequence $\{v_n(t) : n \in I\!\!N\}$ is equicontinuous, so from Arzela-Ascoli theorem we can find a subsequence (v_{n_k}) converges to v_0 in $C_E([-d, b])$. Let $v_{n_k}(t) = \mathcal{S}(t, 0)\mathcal{K}\text{n}_\text{k}(0) + \int_0^t \mathcal{S}(t, s)g_{n_k}(s)\text{ds}$, where $g_{n_k}(s) \in F(s, \theta_s v_{n_k})$ a.e. on I and $g_{n_k}(.) \in L^1(I, E)$. Then we can write $g_{n_k} = \mathcal{K}$ on $[-d, 0]$ and

$$g_{n_k}(t) = \begin{cases} \mathcal{S}(t, 0)\mathcal{K}(0) & \text{if } 0 \leq t \leq \frac{b}{n_k} \\ \mathcal{S}(t, 0)\mathcal{K}(0) + \int_0^{t-\frac{b}{nm}} \mathcal{S}(t, s)g_{n_k}(s)ds & \text{if } \frac{b}{n_k} \leq t \leq b. \end{cases}$$

As in the proof of Theorem 3.1 we obtain $\beta(\{g_{n_k}(t) : n_k \in I\!\!N\}) = 0$ for $t \in I$, so $g_{n_k} \to g_0 \in L^1(I, E)$ and from Lemma 2.2 $g_0(t) \in F(t, \theta_t v_0)$. Thus

$$v_0(t) = \mathcal{S}(t, 0)\mathcal{K}(0) + \int_0^t \mathcal{S}(t, s)g_0(s)ds$$

and consequently $\mathcal{A}=\{\mathcal{K} \in C_E([-d, 0]) : S_{\mathcal{K}} \cap Z \neq \emptyset\}$ is sequentially closed in $C_E([-d, 0])$ thus S is upper semicontinuous. Further, by the same arguments we can show that $\mathcal{P} = \{\mathcal{K} \in C_E([-d, 0]) : S_t(\mathcal{K}) \cap Z \neq \emptyset\}$is closed so, $S_t(\mathcal{K})$ is upper semicontinuous. Since $S(\mathcal{K})$ is compact, then both $S_{\mathcal{K}}$ and S_t has compact values. Lastly the set $\mathcal{Q}$={t$\in I : S_{\mathcal{K}}(t) \cap Z\}$ is closed, then from Berge's Theorem [4] $\cup_{t\in I}$ $S_{\mathcal{K}}(t)$ is compact in E. ■

Now we consider the following control problem

$$(Q)\begin{cases} \dot{x}(t) \in A(t)x(t) + F(t, \theta_t x) \\ x = \mathcal{K} \in Z \\ \quad minimise\, \omega(x(b)) \end{cases}$$

where Z is a compact subset of $C_E([-d, 0])$ and $\omega : E \longrightarrow I\!\!R$ is lower semicontinuous. We say that Problem (P) has an optimal solution if there exist $\mathcal{K}_0 \in Z$ and $v \in S(\mathcal{K}_0)$ such that $\omega(v(b)) = \inf\{\omega(x(b)) : x \in S(\mathcal{K}_0)\}$.

Theorem 3.4 *Under the assumptions of Theorem 3.1, Problem* (Q) *has an optimal solution.*

Proof If $\mathcal{K}_0 \in Z \subseteq C_E([-d, 0])$, then there exists a continuous function $v \in S(\mathcal{K}_0)$ and so, $v(b) \in S_b(\mathcal{K}_0)$. But S_b is upper semicontinuous and has compact values thus, from Berge's Theorem [4], $S_b(Z)$ is compact and so, ω has its minimum b_0 on $S_b(Z)$. Thus there exists $\mathcal{K}_1 \in Z$ such that $v_0 \in S_b(\mathcal{K}_1)$, where $\omega(v_0) = b_0$ $v_0 \in S_b(Z)$, thus $v_0 \in S_{\mathcal{K}_1}(b)$ which means that $v_0 = v(b)$ for some $v \in S(\mathcal{K}_1)$. Therefore $\omega(v(b)) = \inf\{\omega(x(b)) : x \in S(\mathcal{K}_1)\}$. ■

References

[1] A. Anguraj and K. Balachandran,*On the solution sets of differential inclusion in Banach spaces*, Tam. (**23(1)**) (1992), 59-65.

[2] E. P. Avgerinos and N. S. Papageorgiou, *On the solution set of maximal monotone differential inclusions in* $I\!R^m$, Math. Japonica (**38(1)**) (1993), 91-110.

[3] J. Banas and K. Goebel, *Measure of Noncompactness in Banach Spaces, Lecture Notes in Pure and Applied Mathematics v. 60*, Marcel Dekker Inc. New Yourk, 1980.

[4] C. Castaing and M. Valadier, *Convex analysis and measurable multifunctions, Lecture notes in mathematics v. 580*, Springer Verlage 1977.

[5] A. Cellina and A. Ornelas, *Convexity and the closure of the solution set to differential inclusions*, Bollettino U.M.I. (**4(7)-B**) (1993), 255-263.

[6] M. Cichon, *Differential inclusions and abstract control problems*, Bull. Austral Math. Soc. (**53**) (1996), 109-122.

[7] F. Clark, *Optimization and Nonsmooth Analysis v. 1*, Wiley, New York 1983.

[8] F. S. De Blasi and J. Myjak, *On continuous approximations for multifunctions*, Pacific J. Math. (**123**) (1986), 9-13.

[9] K. Deimling, *Multivalued differential equations, Walter de Gruyter, Berlin*, New York, 1992.

[10] K. Deimling,*Multivalued differential equations with usc right-hand side*, Proc. Int. Conf. Theory and Applic. of Diff. Equations (**1**) (1989), 217-222.

[11] H. Frankowska, *A priori estimates for operational differential inclusions*, J. Differential Equations (**84**) (1990), 100-128.

[12] A. M. Gomaa, *On the solution sets of three-point boundary value problems for nonconvex differential inclusions*, J. Egypt. Math., (**12(2)**) (2004), 97-107.

[13] A. M. Gomaa, *On the topological properties of the solution set of differential inclusions and abstract control problems*, J. Frac. Calcu. (**24**) (2003), 87-99.

[14] A. M. Gomaa, *Weak and strong solutions for differential equations in Banach spaces*, Chaos, Solitons & Fractals (**18(4)**) (2003), 687-692.

[15] S. Hu and V. Lakshmikantham and N. S. Papageorgiou, *On the properties of the solution set of semilinear evolution inclusions*, Nonlinear Anal. (**24(12)**) (1995), 1683-1712.

[16] A. G. Ibrahim and A. M. Gomaa, *Existence theorems for a functional multivalued three-point boundary value problem of second order*, J. Egypt. Math. Soc. (**8(2)**) (2000), 155-168.

[17] A. G. Ibrahim and A. M. Gomaa, *Extremal solutions of classes of multivalued differential equations*, Appl. Math. Comput. (**136 no. 2-3**) (2003),297-314.

[18] A. G. Ibrahim and A. M. Gomaa, *Strong and weak solutions for differential inclusions with moving constraints in Banach spaces*, PU. M. A. (**8(1)**) (1997), 53-65.

[19] A. G. Ibrahim and A. M. Gomaa, *Topological properties of the solution sets of some differential inclusions*, PU.M.A. (**10(2)**) (1999), 197-223.

[20] M. Kisielewicz, *Differential Inclusions and Optimal Control*, PWN-Polish Publishers, Kluwer Academic Publishers, Warsaw, London 1991.

[21] E. Klein and A. Thompson, *Theory of Correspondences*, Wiley, New York 1984.

[22] I. Kubiaczyka, *On the existence of solutions of differential equations in Banach spaces*, Bull. Polish Acad. Sci. Math. (**33**) (1985), 607-614.

[23] R. Kubiaczyka, *Zastosowanic miar niezwartośći w teorii ŕównań ŕozniczkowych i calkowych*, Oh. D. thesis, Lublin 1978.

[24] A. R. Mitchell and C. Smith, *An existence theorem for week solutions of differential equations in Banach spaces*, Nonlinear Equations in Abstract Spaces (1978) 387-404.

[25] A. de k Korvin and R. Kleyle, *A convergence theorem for convex set-valued supermartingalas*, Stoch. Anal. Appl. (**3**) (1985), 433-445.

[26] N. S. Papageorgiou, *A convergence theorem for set-valued submartingales with values in a separable Banach space*, Stoch. Anal. Appl. (**5**) (1987), 510-520.

[27] N. S. Papageorgiou, *Convergence theorems for Banach space-valued integral multifunctions*, Inter. J. Math. and Math. Sci. (**10**) (1987), 433-442.

[28] N. S. Papageorgiou, *Mild solutions of semilinear evolutions inclusions and optimal control*, Indian Jour. Pure Appl. Math. (**26(3)**) (1995), 189-216.

[29] N. S. Papageorgiou, *On integral inclusions of Volterra type in Banach spaces*, Czechoslovak Math. Jour. (**42(4)**) (1992), 693-714.

[30] N. S. Papageorgiou, *On multivalued evolution equation and differential inclusions in Banach space*, Comment Math. Univ. Sancti. Pauli. (**36**) (1987), 21-39.

[31] N. S. Papageorgiou, *On nonconvex valued Volterra integral inclusions in Banach Spaces*, Czechoslovak Math. Jour. (**44(4)**) (1994), 631-648.

[32] N. S. Papageorgiou, *On the attainable set of differential inclusions with control system*, J. Math. Anal. (**125**) (1987), 305-322.

[33] A. Pazy, *Semigroups of Linear operators and Applications to Partial Differential Equations*, Springer Verlage, New York, Berlin Heidelberg 1983.

[34] S. Rolewicz, *Functional Analysis and Control Theory*, PWN-Polish Scientific Publishers 1987.

[35] G. Salinetti and R. Wets, *On the convergence in distribution of measurable multifunctions (random sets), normal integrands, stochastic processes and stochastic*, Math. Oper Res. (**11**) (1986), 385-419.

[36] G. Salinetti and R. Wets, *On the convergence of closed valued measurable multifunctions*, Trans. Amer. Mat. (**266**) (1981), 275-289.

[37] S. Szufla, *On existence of solutions of differential equations in Banach spaces*, Bull. Acad. Polon. Sci. Math. (**30**) (1982), 507-514.

[38] M. C. Yovitz, A. Foulk and L. Rose, *Information flow and analysis: Theory simulation and experiment. Part I: Asic theoretical and conceptual development*, Amer. Soc. Inf. Sci. (**32**) (1981), 187-202.

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(Received: 14.05.2008)