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Duality and some topological properties of vector-valued function spaces

Abstract. Let E be an ideal of L^0 over σ -finite measure space (Ω, Σ, μ) and let $(X, \|\cdot\|_X)$ be a real Banach space. Let $E(X)$ be a subspace of the space $L^0(X)$ of μ -equivalence classes of all strongly Σ -measurable functions $f : \Omega \rightarrow X$ and consisting of all those $f \in L^0(X)$, for which the scalar function $\tilde{f} = \|f(\cdot)\|_X$ belongs to E . Let E be equipped with a Hausdorff locally convex-solid topology ξ and let $\bar{\xi}$ stand for the topology on $E(X)$ associated with ξ . We examine the relationship between the properties of the space $(E(X), \bar{\xi})$ and the properties of both the spaces (E, ξ) and $(X, \|\cdot\|_X)$. In particular, it is proved that $E(X)$ (embedded in a natural way) is an order closed ideal of its bidual iff E is an order closed ideal of its bidual and X is reflexive. As an application, we obtain that $E(X)$ is perfect iff E is perfect and X is reflexive.

2000 *Mathematics Subject Classification:* 46E30, 46E40, 46A20.

Key words and phrases: vector-valued function spaces, locally solid topologies, KB-spaces, Levy topologies, Lebesgue topologies, order dual, order continuous dual, perfectness..

1. Introduction and preliminaries. Let $(E, \|\cdot\|_E)$ be a Banach function space (over a finite measure space) and let $(X, \|\cdot\|_X)$ be a real Banach space. The mutual relationship between the properties of the Köthe-Bochner space $(E(X), \|\cdot\|_{E(X)})$ and the properties of both the spaces $(E, \|\cdot\|_E)$ and $(X, \|\cdot\|_X)$ has been examined by many authors (see [3], [4], [5], [6], [7], [8], [9], [11], [12], [14]).

Let E be an ideal of L^0 (over σ -finite measure space) with a Hausdorff locally convex-solid topology ξ and let X be a real Banach space. The duality theory of vector-valued function spaces $E(X)$ endowed with locally solid topologies was developed by M. Nowak in the series of papers [15], [16], [17]. Our aim is to characterize some properties of the space $(E(X), \bar{\xi})$ endowed with the so-called associated with ξ topology, in the terms of appropriate properties of (E, ξ) and X . It is proved that $E(X)$ (embedded in a natural way) is an order closed ideal

of its bidual iff E has an analogous property and X is reflexive (see Corollary 3.3 below). Moreover, in case X is a reflexive Banach space the general form of a linear, continuous functional over the space $(E(X)_{\tilde{n}}, |\sigma|(E(X)_{\tilde{n}}, E(X)))$ is obtained (Theorem 3.4).

It is well known that (E, ξ) (embedded in a natural way) is an order closed ideal of its bidual if and only if ξ is a Lebesgue and Levi topology (see [1, Theorem 6.63]). We extend this theorem to the vector-valued setting (see Theorem 4.5 below). As an application, we obtain that the space $E(X)$ is perfect if and only if E is perfect and X is a reflexive Banach space (see Theorem 4.7 below).

For terminology concerning Riesz spaces and function spaces we refer to [1], [13] and [21]. Given a topological vector space (L, τ) by $(L, \tau)^*$ or L_{τ}^* we will denote its topological dual.

Throughout the paper we assume that (Ω, Σ, μ) is a complete σ -finite measure space. Let L^0 denote the corresponding space of μ -equivalence classes of all Σ -measurable real valued functions. Let E be an ideal of L^0 with $\text{supp } E = \Omega$ and let E' stand for the Köthe dual of E , i.e.,

$$E' = \left\{ v \in L^0 : \int_{\Omega} |u(\omega) v(\omega)| d\mu < \infty \text{ for all } u \in E \right\}.$$

In the paper we assume that $\text{supp } E' = \Omega$. Let E^{\sim} , E_n^{\sim} stand for the order dual and the order continuous dual of E resp. Then E_n^{\sim} separates points of E and it can be identified with E' through the mapping: $E' \ni v \longrightarrow \varphi_v \in E_n^{\sim}$, where

$$\varphi_v(u) = \int_{\Omega} u(\omega) v(\omega) d\mu \text{ for } u \in E$$

(see [13, Theorem 6.1.1]).

By a locally solid (resp. locally convex-solid) function space (E, ξ) we mean an ideal E provided with a locally solid (resp. locally convex-solid) topology ξ . Recall that a Hausdorff locally convex-solid topology ξ on E is a *Lebesgue topology* iff $E_{\xi}^* \subset E_n^{\sim}$ (see [1, Theorem 3.12]).

Now we establish terminology and some basic results concerning vector-valued spaces $E(X)$ and locally solid topologies on $E(X)$ as set out in [4], [5], [6], [10], [15], [16], [17], [18].

Let $(X, \|\cdot\|_X)$ be a real Banach space and let X^* stand for the Banach dual of X . Let S_X , B_X stand for the unit sphere and the unit ball of X . By $L^0(X)$ we denote the set of μ -equivalence classes of all strongly Σ -measurable functions $f : \Omega \longrightarrow X$. For $f \in L^0(X)$ let us set $\tilde{f}(\omega) = \|f(\omega)\|_X$ for $\omega \in \Omega$. Let $E(X) = \{f \in L^0(X) : \tilde{f} \in E\}$. Recall that the algebraic tensor product $E \otimes X$ is the subspace of $E(X)$ spanned by the functions of the form $u \otimes x$, $(u \otimes x)(\omega) = u(\omega)x$, where $u \in E$, $x \in X$.

A subset H of $E(X)$ is said to be solid whenever $\tilde{f}_1 \leq \tilde{f}_2$ and $f_1 \in E(X)$, $f_2 \in H$ imply $f_1 \in H$. A linear topology τ on $E(X)$ is said to be locally solid if it has a local base at zero consisting of solid sets. A linear topology τ on $E(X)$ that is at the same time locally solid and locally convex will be called a locally convex-solid topology on $E(X)$. A seminorm ϱ on $E(X)$ is called solid if $\varrho(f_1) \leq \varrho(f_2)$

whenever $f_1, f_2 \in E(X)$ and $\tilde{f}_1 \leq \tilde{f}_2$. It is known that a locally convex topology τ on $E(X)$ is locally convex-solid iff it is generated by some family of solid seminorms defined on $E(X)$ (see [10]).

A locally solid topology τ on $E(X)$ is said to be a Lebesgue topology, whenever for a net (f_α) in $E(X)$, $\tilde{f}_\alpha \xrightarrow{(o)} 0$ in E implies $f_\alpha \xrightarrow{\tau} 0$ (see [10], [18]).

Let (E, ξ) be a Hausdorff locally convex-solid function space. Then one can topologize the space $E(X)$ as follows (see [10]). Let $\{p_\alpha : \alpha \in A\}$ be a family of Riesz seminorms on E that generates ξ . By putting

$$\bar{p}_\alpha(f) = p_\alpha(\tilde{f}) \quad \text{for } f \in E(X) \quad (\alpha \in A),$$

we obtain a family $\{\bar{p}_\alpha : \alpha \in A\}$ of solid seminorms on $E(X)$ that defines a Hausdorff locally convex-solid topology $\bar{\xi}$ on $E(X)$ (called the topology associated with ξ). Then $\bar{\xi}$ is a Lebesgue topology whenever ξ is a Lebesgue topology (see [10]).

Conversely, let τ be a Hausdorff locally convex-solid topology on $E(X)$ and let $\{\varrho_\alpha : \alpha \in A\}$ be a family of solid seminorms on $E(X)$ that generates τ . By putting for a fixed $x \in S_X$

$$\tilde{\varrho}_\alpha(u) = \varrho_\alpha(u \otimes x) \quad \text{for } u \in E \quad (\alpha \in A),$$

we obtain a family $\{\tilde{\varrho}_\alpha : \alpha \in A\}$ of Riesz seminorms on E that defines a Hausdorff locally convex-solid topology $\tilde{\tau}$ on E .

One can show that $\tilde{\xi} = \xi$ and $\tilde{\tau} = \tau$ (see [10]). Thus every Hausdorff locally convex-solid topology τ on $E(X)$ can be represented as the topology associated with some Hausdorff locally convex-solid topology ξ ($= \tilde{\tau}$) on E .

For a Banach function space $(E, \|\cdot\|_E)$ the space $E(X)$ provided with the norm $\|f\|_{E(X)} = \|\tilde{f}\|_E$ is usually called a *Köthe-Bochner space*.

For a linear functional F on $E(X)$ let us put

$$|F|(f) = \sup \{|F(h)| : h \in E(X), \tilde{h} \leq \tilde{f}\} \quad \text{for } f \in E(X).$$

The set

$$E(X)^\sim = \{F \in E(X)^\# : |F|(f) < \infty \text{ for all } f \in E(X)\}$$

will be called the order dual of $E(X)$ (here $E(X)^\#$ denotes the algebraic dual of $E(X)$).

For $F_1, F_2 \in E(X)^\sim$ we will write $|F_1| \leq |F_2|$ whenever $|F_1|(f) \leq |F_2|(f)$ for all $f \in E(X)$. A subset A of $E(X)^\sim$ is said to be solid whenever $|F_1| \leq |F_2|$ with $F_1 \in E(X)^\sim$, $F_2 \in A$ imply $F_1 \in A$. A linear subspace I of $E(X)^\sim$ will be called an ideal of $E(X)$ whenever I is solid. It is known that if τ is a locally solid topology on $E(X)$, then $(E(X), \tau)^*$ is an ideal of $E(X)^\sim$ (see [15, Theorem 3.2]).

From now on let I be an ideal of $E(X)^\sim$. Now we recall some definitions (see [16, Definition 1.2, Definition 1.3]).

DEFINITION 1.1 A net (F_σ) in I is said to be *order convergent* to $F \in I$, in symbols $F_\sigma \xrightarrow{(o)} F$ whenever there exists a net (G_σ) in I such that $|F_\sigma - F|(f) \leq |G_\sigma|(f) \downarrow_\sigma 0$ for all $f \in E(X)$.

DEFINITION 1.2 A linear functional V on I is said to be *order continuous* whenever $F_\sigma \xrightarrow{(o)} 0$ in I implies $V(F_\sigma) \rightarrow 0$. The set consisting of all order continuous linear functionals on I will be denoted by I_n^\sim and called the order continuous dual of I .

Let $L^0(X^*, X)$ be the set of weak*-equivalence classes of all weak*-measurable functions $g : \Omega \rightarrow X^*$. One can define the so-called abstract norm $\vartheta : L^0(X^*, X) \rightarrow L^0$ by $\vartheta(g) = \sup\{|g_x| : x \in B_X\}$, where $g_x(\omega) = g(\omega)(x)$ for $\omega \in \Omega$, $x \in X$. Then for $f \in L^0(X)$ and $g \in L^0(X^*, X)$ the function $\langle f, g \rangle : \Omega \rightarrow \mathbb{R}$ defined by $\langle f, g \rangle(\omega) = \langle f(\omega), g(\omega) \rangle$ is measurable and $|\langle f, g \rangle| \leq f\vartheta(g)$. Moreover, $\vartheta(g) = \tilde{g}$ for $g \in L^0(X^*)$.

For an ideal M of E' let

$$M(X^*, X) = \{g \in L^0(X^*, X) : \vartheta(g) \in M\}.$$

Then $M(X^*, X)$ is an ideal of $E'(X^*, X)$, i.e., $M(X^*, X)$ is a linear subspace of $E'(X^*, X)$ and $\vartheta(g_1) \leq \vartheta(g_2)$ with $g_1 \in E'(X^*, X)$, $g_2 \in M(X^*, X)$ imply $g_1 \in M(X^*, X)$ (see [15, Definition 1.2]). Clearly $M(X^*) \subset M(X^*, X)$.

The space $E(X)_n^\sim$ can be identified with $E'(X^*, X)$ through the mapping $E'(X^*, X) \ni g \rightarrow F_g \in E(X)_n^\sim$, where

$$F_g(f) = \int_\Omega \langle f(\omega), g(\omega) \rangle d\mu \quad \text{for all } f \in E(X),$$

and moreover

$$|F_g|(f) = \int_\Omega \tilde{f}(\omega) \vartheta(g)(\omega) d\mu \quad \text{for all } f \in E(X),$$

(see [4, Theorem 4.1]). It is well known that if X^* has the Radon-Nikodym property with respect to μ (in particular, if X is reflexive), then $E'(X^*, X) = E'(X^*)$.

Since $\text{supp } E' = \Omega$, $E(X)_n^\sim$ separates points of $E(X)$. Moreover, a Hausdorff locally convex-solid topology τ on $E(X)$ has the Lebesgue property iff $E(X)_\tau^* \subset E(X)_n^\sim$ (see [18, Theorem 2.4]).

For $F \in E(X)^\sim$ and a fixed $x \in S_X$ let us set

$$\varphi_F(u) = |F|(u \otimes x) = \sup\{|F(h)| : h \in E(X), \tilde{h} \leq u\} \quad \text{for } u \in E^+.$$

Then $\varphi_F : E^+ \rightarrow \mathbb{R}^+$ is an additive mapping and φ_F has a unique positive extension to a linear mapping from E to \mathbb{R} (denoted by φ_F again) and given by

$$\varphi_F(u) = \varphi_F(u^+) - \varphi_F(u^-) \quad \text{for all } u \in E.$$

For a linear functional V on I let us put

$$|V|(F) = \sup \{|V(G)| : G \in I, |G| \leq |F|\} \quad \text{for } F \in I.$$

The set

$$I^\sim = \{V \in I^\# : |V|(F) < \infty \quad \text{for all } F \in I\}$$

will be called the order dual of I (here $I^\#$ denotes the algebraic dual of I). For $V_1, V_2 \in I^\sim$ we will write $|V_1| \leq |V_2|$ whenever $|V_1|(F) \leq |V_2|(F)$ for all $F \in I$.

A subset K of I^\sim is said to be solid whenever $|V_1| \leq |V_2|$ with $V_1 \in I^\sim$, $V_2 \in K$ imply $V_1 \in K$. A linear subspace L of I^\sim is called an ideal of I^\sim if L is a solid subset of I^\sim . It is known that I_n^\sim is an ideal of I^\sim (see [16, Theorem 1.2]).

Now we introduce the notion of order convergence in I^\sim .

DEFINITION 1.3 A net (V_σ) in I^\sim will be called *order convergent* to V in I^\sim (in symbols $V_\sigma \xrightarrow{(o)} V$) whenever there exists a net (W_σ) in I^\sim such that $|V_\sigma - V|(F) \leq |W_\sigma|(F) \downarrow_\sigma 0$ for all $F \in I$.

DEFINITION 1.4 A subset K of I^\sim is called *order closed* whenever for (V_σ) in K , $V \in I^\sim$ such that $V_\sigma \xrightarrow{(o)} V$ we have $V \in K$.

PROPOSITION 1.1 I_n^\sim is an order closed ideal of I^\sim .

PROOF It is enough to show that I_n^\sim is an order closed set in I^\sim . Fix (V_σ) in I_n^\sim and $V \in I^\sim$ such that $V_\sigma \xrightarrow{(o)} V$ in I^\sim . Then there exists a net (W_σ) in I^\sim such that $|V_\sigma - V|(F) \leq |W_\sigma|(F) \downarrow_\sigma 0$ for all $F \in I$. We will show that $V \in I_n^\sim$.

Choose a net (F_α) in I such that $F_\alpha \xrightarrow{(o)} 0$. Then we can find a net (G_α) in I such that $|F_\alpha|(f) \leq |G_\alpha|(f) \downarrow_\alpha 0$ for all $f \in E(X)$. Let $\varepsilon > 0$ be given. Fix α_1 . Thus there exists σ_0 such that $|W_\sigma|(G_{\alpha_1}) \leq \frac{\varepsilon}{2}$ for all $\sigma \geq \sigma_0$. In particular we have $|W_{\sigma_0}|(G_{\alpha_1}) \leq \frac{\varepsilon}{2}$. Moreover, there exists α_2 such that $|V_{\sigma_0}|(F_\alpha) \leq \frac{\varepsilon}{2}$ for all $\alpha \geq \alpha_2$, because V_{σ_0} is order continuous. Take α_0 such that $\alpha_0 \geq \alpha_1$ and $\alpha_0 \geq \alpha_2$. Hence for all $\alpha \geq \alpha_0$ we get $|G_\alpha|(f) \leq |G_{\alpha_1}|(f)$ for all $f \in E(X)$, i.e., $|G_\alpha| \leq |G_{\alpha_1}|$, so $|W_{\sigma_0}|(G_\alpha) \leq |W_{\sigma_0}|(G_{\alpha_1}) \leq \frac{\varepsilon}{2}$ and $|V_{\sigma_0}|(F_\alpha) \leq \frac{\varepsilon}{2}$. Thus for all $\alpha \geq \alpha_0$ we have

$$\begin{aligned} |V(F_\alpha)| &\leq |V_{\sigma_0}(F_\alpha)| + |(V - V_{\sigma_0})(F_\alpha)| \leq |V_{\sigma_0}(F_\alpha)| + |V - V_{\sigma_0}|(F_\alpha) \\ &\leq |V_{\sigma_0}(F_\alpha)| + |W_{\sigma_0}|(F_\alpha) \leq |V_{\sigma_0}(F_\alpha)| + |W_{\sigma_0}|(G_\alpha) \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This means that $V(F_\alpha) \rightarrow 0$, so $V \in I_n^\sim$. ■

2. Basic representations. In this section we establish the basic relationships between various spaces representing the order continuous functionals over the spaces $(E, \xi), (E, \xi)^*, (E(X), \bar{\xi}), (E(X), \bar{\xi})^*$.

Let (E, ξ) be a Hausdorff locally convex-solid function space with the Lebesgue property. In view of [19, Proposition 1.1] there exists an ideal M_ξ of E' with $\text{supp } M_\xi = \Omega$ such that $E_\xi^* = \{\varphi_v : v \in M_\xi\}$ and $E(X)_{\bar{\xi}}^* = \{F_g : g \in M_\xi(X^*, X)\}$. Let us consider a natural mapping

$$(1) \quad i_\xi : M_\xi \ni v \mapsto \varphi_v \in E_\xi^*.$$

Clearly the operator i_ξ is linear and onto. Since $\text{supp } E = \Omega$ it is also one-to-one. Hence the operator i_ξ is a linear isomorphism onto. From [13, Theorem 6.1.1], it follows that both operators i_ξ and i_ξ^{-1} are positive. In view of [2, Theorems 7.3 and 7.9], i_ξ and i_ξ^{-1} are also order continuous Riesz isomorphisms.

Moreover, for every $\psi \in (M_\xi)_+^\sim$ the linear functional $\psi \circ i_\xi^{-1}$ on E_ξ^* is positive, so $\psi \circ i_\xi^{-1} \in (E_\xi^*)^\sim$.

Similarly, for every $\Phi \in (E_\xi^*)_+^\sim$ the linear functional $\Phi \circ i_\xi$ on M_ξ is positive too and consequently $\Phi \circ i_\xi \in (M_\xi)^\sim$. Hence the mappings $j_\xi : (M_\xi)^\sim \rightarrow (E_\xi^*)^\sim$ and $j'_\xi : (E_\xi^*)^\sim \rightarrow (M_\xi)^\sim$, where

$$(2) \quad j_\xi(\psi) = \psi \circ i_\xi^{-1} \quad \text{for } \psi \in (M_\xi)^\sim \quad \text{and} \quad j'_\xi(\Phi) = \Phi \circ i_\xi \quad \text{for } \Phi \in (E_\xi^*)^\sim$$

are well defined, linear and the identities

$$(3) \quad j'_\xi \circ j_\xi = id_{(M_\xi)^\sim} \quad \text{and} \quad j_\xi \circ j'_\xi = id_{(E_\xi^*)^\sim} \quad \text{hold.}$$

Thus j_ξ and j'_ξ are bijections and $j'_\xi = j_\xi^{-1}$. Both operators i_ξ and i_ξ^{-1} are positive, and so operators j_ξ and j_ξ^{-1} are. It follows by [2, Theorem 7.3], that j_ξ and j_ξ^{-1} are Riesz isomorphisms. In view of [2, Theorem 7.9], the operators j_ξ and j_ξ^{-1} are also order continuous.

From order continuity of i_ξ^{-1} and i_ξ , we obtain the inclusions $j_\xi((M_\xi)_n^\sim) \subset (E_\xi^*)_n^\sim$ and $j_\xi^{-1}((E_\xi^*)_n^\sim) \subset (M_\xi)_n^\sim$, and by (1), we actually have

$$(4) \quad j_\xi((M_\xi)_n^\sim) = (E_\xi^*)_n^\sim \quad \text{and} \quad j_\xi^{-1}((E_\xi^*)_n^\sim) = (M_\xi)_n^\sim.$$

For $u \in E$ let us put

$$(5) \quad p_u(\varphi) = \varphi(u) \quad \text{for } \varphi \in E_\xi^*.$$

Then the linear functional p_u on E_ξ^* is order continuous, i.e., $p_u \in (E_\xi^*)_n^\sim$ (see [2, p. 58–59]).

Let us consider the mapping

$$(6) \quad p : E \ni u \rightarrow p_u \in (E_\xi^*)_n^\sim.$$

Then by [2, Theorems 5.4 and 5.5] p is an order continuous, positive linear isomorphism (into).

Now, let us define the mapping $P : E \longrightarrow (M_\xi)_n^\sim$ by

$$(7) \quad P = j_\xi^{-1} \circ p.$$

It is clear that P is an order continuous linear isomorphism (into).

Finally, let $j : M'_\xi \longrightarrow (M_\xi)_n^\sim$ be the mapping defined by

$$(8) \quad j(w) = \psi_w \quad \text{for } w \in M'_\xi, \quad \text{where } \psi_w(v) = \int_\Omega v(\omega) w(\omega) d\mu \quad \text{for } v \in M_\xi.$$

In view of [13, Theorem 6.1.1], j is a linear isomorphism onto, and both operators j and j^{-1} are positive. Thus by [2, Theorems 7.3 and 7.9], we get that j and j^{-1} are order continuous Riesz isomorphisms.

Note that for $u \in E$ we have $P(u) = j(u)$ ($E \subset E'' \subset M'_\xi$). Indeed, $P(u) = (j_\xi^{-1} \circ p)(u) = j_\xi^{-1}(p(u)) = p_u \circ i_\xi$ for $u \in E$, so for every $v \in M_\xi$ we get

$$\begin{aligned} (P(u))(v) &= (p_u \circ i_\xi)(v) = p_u(i_\xi(v)) = p_u(\varphi_v) = \varphi_v(u) = \\ &= \int_\Omega u(\omega) v(\omega) d\mu = \int_\Omega v(\omega) u(\omega) d\mu = \psi_u(v) = (j(u))(v), \quad \text{as desired.} \end{aligned}$$

Thus $j|_E = P$.

$$\begin{array}{ccc} E & \xrightarrow{p} & (E_\xi^*)_n^\sim \\ & \searrow^{j|_E} & \downarrow j_\xi^{-1} \\ & & (M_\xi)_n^\sim \end{array}$$

Hence

$$(9) \quad (j^{-1} \circ P)(u) = u = id_E(u) \quad \text{for } u \in E,$$

so $j^{-1} \circ P : E \longrightarrow M'_\xi$ is an inclusion map.

Now we pay our attention to the identities between vector-valued function spaces. Throughout the rest of this section we will assume that X is a reflexive Banach space.

Then $E(X)_\xi^* = \{F_g : g \in M_\xi(X^*)\}$ (see [19, Proposition 1.2]).

Moreover, we can define a natural mapping

$$(10) \quad I_\xi : M_\xi(X^*) \ni g \mapsto F_g \in E(X)_\xi^*.$$

It is easily seen that I_ξ is a linear isomorphism onto. In view of [16, Theorem 1.1] the linear operators I_ξ and I_ξ^{-1} are order continuous (i.e., they map order convergent nets into order convergent nets).

Note that for every $G \in M_\xi(X^*)^\sim$, $V \in (E(X)_{\bar{\xi}}^*)^\sim$ and $g_0 \in M_\xi(X^*)$ we have

$$\begin{aligned}
 |G \circ I_\xi^{-1}|(F_{g_0}) &= \sup \{|(G \circ I_\xi^{-1})(F_g)| : g \in M_\xi(X^*), |F_g| \leq |F_{g_0}|\} \\
 (11) \qquad \qquad \qquad &= \sup \{|G(g)| : g \in M_\xi(X^*), \tilde{g} \leq \tilde{g}_0\} \\
 &= |G|(g_0) = |G| \left(I_\xi^{-1}(F_{g_0}) \right)
 \end{aligned}$$

and

$$\begin{aligned}
 |V \circ I_\xi|(g_0) &= \sup \{|(V \circ I_\xi)(g)| : g \in M_\xi(X^*), \tilde{g} \leq \tilde{g}_0\} \\
 (12) \qquad \qquad \qquad &= \sup \{|V(F_g)| : g \in M_\xi(X^*), |F_g| \leq |F_{g_0}|\} \\
 &= |V|(F_{g_0}) = |V|(I_\xi(g_0)).
 \end{aligned}$$

Let us define the mappings $J_\xi : M_\xi(X^*)^\sim \longrightarrow (E(X)_{\bar{\xi}}^*)^\sim$ and $J'_\xi : (E(X)_{\bar{\xi}}^*)^\sim \longrightarrow M_\xi(X^*)^\sim$ by

$$\begin{aligned}
 (13) \qquad \qquad \qquad J_\xi(G) &= G \circ I_\xi^{-1} \quad \text{for } G \in M_\xi(X^*)^\sim \quad \text{and} \\
 J'_\xi(V) &= V \circ I_\xi \quad \text{for } V \in (E(X)_{\bar{\xi}}^*)^\sim.
 \end{aligned}$$

By (11) and (12) both the mappings are well defined, linear and the following identities hold:

$$(14) \qquad \qquad \qquad J'_\xi \circ J_\xi = id_{M_\xi(X^*)^\sim} \quad \text{and} \quad J_\xi \circ J'_\xi = id_{(E(X)_{\bar{\xi}}^*)^\sim}.$$

Hence J_ξ and J'_ξ are bijections and $J'_\xi = J_\xi^{-1}$. From order continuity of operators I_ξ^{-1} and I_ξ , we easily get the inclusions: $J_\xi(M_\xi(X^*)^\sim_n) \subset (E(X)_{\bar{\xi}}^*)^\sim_n$ and $J_\xi^{-1}((E(X)_{\bar{\xi}}^*)^\sim_n) \subset M_\xi(X^*)^\sim_n$. As before, using the identities (14), those inclusions are actually identities, i.e.,

$$(15) \qquad J_\xi(M_\xi(X^*)^\sim_n) = (E(X)_{\bar{\xi}}^*)^\sim_n \quad \text{and} \quad J_\xi^{-1}((E(X)_{\bar{\xi}}^*)^\sim_n) = M_\xi(X^*)^\sim_n.$$

To get order continuity of operators J_ξ and J_ξ^{-1} it is enough to use (11) and (12) again.

For $f \in E(X)$ let us put

$$(16) \qquad \qquad \qquad \pi_f(F) = F(f) \quad \text{for } F \in E(X)_{\bar{\xi}}^*.$$

Then $\pi_f \in (E(X)_{\bar{\xi}}^*)^\sim_n$ and $|\pi_f|(F) = |F|(f)$ for $F \in E(X)_{\bar{\xi}}^*$ (see [16, p. 220]).

Let us consider the mapping

$$(17) \qquad \qquad \qquad \pi : E(X) \ni f \mapsto \pi_f \in (E(X)_{\bar{\xi}}^*)^\sim_n.$$

Since $E(X)_{\xi}^*$ separates points of $E(X)$, the mapping π is a linear isomorphism (into). In order to prove that π is order continuous, let us take a net (f_{α}) in $E(X)$ such that $\tilde{f}_{\alpha} \xrightarrow{(o)} 0$ in E . Then there exists a net (h_{α}) in $E(X)$ such that $\tilde{f}_{\alpha} \leq \tilde{h}_{\alpha} \downarrow_{\alpha} 0$. Since E is super Dedekind complete one can pick a sequence $(h_{\alpha_n}) \subset (h_{\alpha})$ such that $\tilde{h}_{\alpha_n} \downarrow 0$. By [16, Theorem 1.3], we have $|\pi_{f_{\alpha}}| \leq |\pi_{h_{\alpha}}|$, i.e., for every $F \in E(X)_{\xi}^*$ we have: $|F|(f_{\alpha}) = |\pi_{f_{\alpha}}|(F) \leq |\pi_{h_{\alpha}}|(F) = |F|(h_{\alpha}) \downarrow$. Since $I_{\xi}(M_{\xi}(X^*)) = E(X)_{\xi}^*$, there exists $g \in M_{\xi}(M^*) \subset E'(X^*)$ such that $F = I_{\xi}(g) = F_g$. But for $f \in E(X)$ and $g \in E'(X^*)$ we have $|F_g|(f) = \int_{\Omega} \tilde{f}(\omega) \tilde{g}(\omega) d\mu$ (see [4, Theorem 4.1]). By the Lebesgue Dominated Convergence theorem, we get

$$\lim_{n \rightarrow \infty} |F|(h_{\alpha_n}) = \lim_{n \rightarrow \infty} |F_g|(h_{\alpha_n}) = \lim_{n \rightarrow \infty} \int_{\Omega} \tilde{h}_{\alpha_n}(\omega) \tilde{g}(\omega) d\mu = 0.$$

Hence $|F|(h_{\alpha}) \downarrow_{\alpha} 0$, which means that $\pi(f_{\alpha}) \xrightarrow{(o)} 0$ in $(E(X)_{\xi}^*)^{\sim}$, so π is order continuous.

Now let us define the mapping $\Pi : E(X) \rightarrow M_{\xi}(X^*)_{\tilde{n}}$ by

$$(18) \quad \Pi = J_{\xi}^{-1} \circ \pi.$$

Then Π is an order continuous linear isomorphism (into).

Let $\bar{J} : M'_{\xi}(X^{**}) \rightarrow M_{\xi}(X^*)_{\tilde{n}}$ be the mapping defined by

$$(19) \quad \bar{J}(k) = \bar{F}_k \quad \text{for } k \in M'_{\xi}(X^{**}),$$

where $\bar{F}_k(g) = \int_{\Omega} \langle g(\omega), k(\omega) \rangle d\mu$ for $g \in M_{\xi}(X^*)$.

Thus \bar{J} is a linear operator onto (see [4, Theorem 4.1], [5, Theorem 3.5], [15, Theorem 2.4, Remark 2.1]). Since $\text{supp } M_{\xi} = \Omega$, \bar{J} is also one-to-one. Hence it is a linear isomorphism onto.

Let $\kappa : X \rightarrow X^{**}$ stand for the canonical isometry. For $h \in M'_{\xi}(X)$ let us put $\kappa(h) = \kappa \circ h$. Obviously $\kappa(h) \in M'_{\xi}(X^{**})$. Thus the mapping

$$(20) \quad \kappa : M'_{\xi}(X) \ni h \rightarrow \kappa \circ h \in M'_{\xi}(X^{**})$$

is a linear isomorphism onto.

Finally, let us consider the mapping $J : M'_{\xi}(X) \rightarrow M_{\xi}(X^*)_{\tilde{n}}$ defined by

$$(21) \quad J = \bar{J} \circ \kappa.$$

Clearly J is a linear isomorphism onto.

Moreover

$$(22) \quad J(h) = F_h \quad \text{for } h \in M'_{\xi}(X),$$

where $F_h(g) = \int_{\Omega} \langle h(\omega), g(\omega) \rangle d\mu$ for $g \in M_{\xi}(X^*)$.

Indeed, for every $g \in M_\xi(X^*)$ we have

$$\begin{aligned} (J(h))(g) &= ((\bar{J} \circ \kappa)(h))(g) = (\bar{J}(\kappa(h)))(g) = (\bar{J}(\kappa \circ h))(g) = \bar{F}_{\kappa \circ h}(g) = \\ &= \int_{\Omega} \langle g(\omega), \kappa(h(\omega)) \rangle d\mu = \int_{\Omega} \langle h(\omega), g(\omega) \rangle d\mu = F_h(g) \end{aligned}$$

as we claimed.

Note, that for $f \in E(X)$ we have $\Pi(f) = J(f)$. ($E(X) \subset E''(X) \subset M'_\xi(X)$)

In fact, $\Pi(f) = (J_\xi^{-1} \circ \pi)(f) = J_\xi^{-1}(\pi(f)) = \pi_f \circ I_\xi$ for $f \in E(X)$, so for every $g \in M_\xi(X^*)$ we get

$$\begin{aligned} (\Pi(f))(g) &= (\pi_f \circ I_\xi)(g) = \pi_f(I_\xi(g)) = \pi_f(F_g) = F_g(f) \\ &= \int_{\omega} \langle f(\omega), g(\omega) \rangle d\mu = F_f(g) = (J(f))(g) \end{aligned}$$

as desired.

$$\begin{array}{ccc} E(X) & \xrightarrow{\pi} & (E(X)_\xi^*)_\sim \\ & \searrow \Pi & \downarrow J_\xi^{-1} \\ & & M_\xi(X^*)_\sim \end{array}$$

$J|_{E(X)}$

Thus $J|_{E(X)} = \Pi$. Hence

$$(23) \quad (J^{-1} \circ \Pi)(f) = f = id_{E(X)}(f) \quad \text{for } f \in E(X),$$

so $J^{-1} \circ \Pi : E(X) \rightarrow M'_\xi(X)$ is an inclusion map.

3. Main results. In this section, applying the identities established in Section 2, we examine the mutual relationships between the properties of the spaces $(E(X), \bar{\xi})$ and (E, ξ) .

THEOREM 3.1 *Let (E, ξ) be a locally convex-solid function space with a Lebesgue property and let X be a reflexive Banach space. Let M_ξ be an ideal of E' with $\text{supp } M_\xi = \Omega$ determined by ξ . Then the following equivalences hold:*

- (i) *The set $p(E)$ is order closed in $(E_\xi^*)_\sim$ if and only if the set $P(E)$ is order closed in M_ξ^\sim .*
- (ii) *The set $\pi(E(X))$ is order closed in $(E(X)_\xi^*)_\sim$ if and only if the set $\Pi(E(X))$ is order closed in $M_\xi(X^*)_\sim$.*

PROOF (i) (\implies) Choose a net (u_α) in E and ψ in M^\sim in such a way that $P(u_\alpha) \xrightarrow{(o)} \psi$ in M^\sim . Then $p(u_\alpha) = j_\xi(P(u_\alpha)) \xrightarrow{(o)} j_\xi(\psi)$ in $(E_\xi^*)^\sim$, because j_ξ is order continuous. Since $p(E)$ is order closed, $j_\xi(\psi) \in p(E)$. Thus there exists $u \in E$ such that $j_\xi(\psi) = p(u)$. Hence $\psi = j_\xi^{-1}(p(u)) = P(u) \in P(E)$, as desired.

(\impliedby) For the converse, choose a net (u_α) in E and $\Phi \in (E_\xi^*)^\sim$ with $p(u_\alpha) \xrightarrow{(o)} \Phi$ in $(E_\xi^*)^\sim$. Then $P(u_\alpha) = j_\xi^{-1}(p(u_\alpha)) \xrightarrow{(o)} j_\xi^{-1}(\Phi)$ in M^\sim , because j_ξ^{-1} is order continuous. Since $P(E)$ is order closed, we get $j_\xi^{-1}(\Phi) \in P(E)$. Hence $j_\xi^{-1}(\Phi) = P(u)$ for some $u \in E$, so $\Phi = j_\xi(P(u)) = p(u) \in p(E)$, as desired.

(ii) (\implies) Let us take a net (f_α) in $E(X)$ and $G \in M_\xi(X^*)^\sim$ such that $\Pi(f_\alpha) \xrightarrow{(o)} G$ in $M_\xi(X^*)^\sim$. Then $\pi(f_\alpha) = J_\xi(\Pi(f_\alpha)) \xrightarrow{(o)} J_\xi(G)$ in $(E(X)_\xi^*)^\sim$, because J_ξ is order continuous. By assumption $J_\xi(G) \in \pi(E(X))$, so $J_\xi(G) = \pi(f)$ for some $f \in E(X)$. Hence $G = J_\xi^{-1}(\pi(f)) = \Pi(f) \in \Pi(E(X))$, which proves our assertion.

(\impliedby) For the converse, choose a net (f_α) in $E(X)$ and $V \in (E(X)_\xi^*)^\sim$ in such a way that $\pi(f_\alpha) \xrightarrow{(o)} V$ in $(E(X)_\xi^*)^\sim$. Then $\Pi(f_\alpha) = J_\xi^{-1}(\pi(f_\alpha)) \xrightarrow{(o)} J_\xi^{-1}(V)$ in $M_\xi(X^*)^\sim$, because J_ξ^{-1} is order continuous. But $\Pi(E(X))$ is order closed in $M_\xi(X^*)^\sim$, so $J_\xi^{-1}(V) \in \Pi(E(X))$. Thus there exists f in $E(X)$ such that $J_\xi^{-1}(V) = \Pi(f)$, so $V = J_\xi(\Pi(f)) = \pi(f) \in \pi(E(X))$ and we are done. \blacksquare

Let M be an ideal of L^0 . Fix $\psi \in M_+^\sim$ and $x^* \in S_{X^*}$. Let us put $\overline{M} = \{v \otimes x^* : v \in M\}$. Then \overline{M} is a linear subspace of $M(X^*)$.

Let us consider the solid seminorm ϱ on $M(X^*)$ defined by

$$\varrho(g) = \psi(\tilde{g}) \quad \text{for } g \in M(X^*).$$

and the linear functional $G_\psi : \overline{M} \rightarrow \mathbb{R}$ defined by

$$G_\psi(v \otimes x^*) = \psi(v) \quad \text{for } v \in M.$$

Note, that for all $v \in M$ we have

$$|G_\psi(v \otimes x^*)| = |\psi(v)| \leq \psi(|v|) = \psi(\widetilde{|v| \otimes x^*}) = \varrho(v \otimes x^*).$$

By the Hahn-Banach extension theorem, there exists an extension of G_ψ to a linear functional on $M(X^*)$ (denoted by G_ψ again) and such that

$$|G_\psi(g)| \leq \varrho(g) \quad \text{for all } g \in M(X^*).$$

Fix $g_0 \in M(X^*)$. Then for each $g \in M(X^*)$ such that $\tilde{g} \leq \tilde{g}_0$ we have $|G_\psi(g)| \leq \varrho(g) \leq \varrho(g_0)$, because ϱ is solid, and so $|G_\psi|(g_0) = \sup\{|G_\psi(g)| : g \in M(X^*), \tilde{g} \leq \tilde{g}_0\} \leq \varrho(g_0)$. Thus we get

$$|G_\psi|(g) \leq \varrho(g) \quad \text{for each } g \in M(X^*).$$

One the other hand, for each $g \in M(X^*)$ we have

$$|G_\psi|(g) = |G_\psi|(\tilde{g} \otimes x^*) \geq |G_\psi(\tilde{g} \otimes x^*)| = |\psi(\tilde{g})| = \varrho(g).$$

It follows that

$$(*) \quad |G_\psi|(g) = \psi(\tilde{g}) \quad \text{for each } g \in M(X^*).$$

THEOREM 3.2 *Let (E, ξ) be a Hausdorff locally convex-solid function space with a Lebesgue property and let X be a reflexive Banach space. Let M_ξ be an ideal of E' with $\text{supp } M_\xi = \Omega$ determined by ξ .*

Then the subset $\Pi(E(X))$ of $M_\xi(X^)^\sim$ is order closed if and only if the subset $P(E)$ of M_ξ^\sim is order closed.*

PROOF (\implies) Assume that the subset $\Pi(E(X))$ of $M_\xi(X^*)^\sim$ is order closed. We will show that the subset $P(E)$ of M^\sim is order closed.

Choose a net (u_α) in E and $\psi \in M_\xi^\sim$ in such a way that $P(u_\alpha) \xrightarrow{(o)} \psi$ in M_ξ^\sim . Then $\psi \in (M_\xi)_n^\sim$, because $P(E) \subset (M_\xi)_n^\sim$ and $(M_\xi)_n^\sim$ is order closed in M_ξ^\sim . Hence there exists $w \in M_\xi'$ such that $\psi = j(w) = \psi_w$, i.e., $\psi(v) = \int_\Omega v(\omega)w(\omega) d\mu$ for $v \in M_\xi$.

Moreover, there exists a net (ψ_α) in $(M_\xi)_+^\sim$ such that $|P(u_\alpha) - \psi_w| \leq \psi_\alpha \downarrow_\alpha 0$ in M_ξ^\sim or equivalently $|P(u_\alpha) - \psi_w|(v) \leq \psi_\alpha(v) \downarrow 0$ for each $v \in M_\xi^+$. Note, that for each $v_0 \in M_\xi^+$ we have

$$\begin{aligned} |P(u_\alpha) - \psi_w|(v_0) &= \sup \{ |(P(u_\alpha) - \psi_w)(v)| : v \in M_\xi, |v| \leq v_0 \} \\ &= \sup \left\{ \left| \int_\Omega (u_\alpha(\omega) - w(\omega)) v(\omega) d\mu \right| : v \in M_\xi, |v| \leq v_0 \right\} \\ &= \sup \left\{ \left| \int_\Omega |u_\alpha(\omega) - w(\omega)| v(\omega) d\mu \right| : v \in M_\xi, |v| \leq v_0 \right\} \\ &= \int_\Omega |u_\alpha(\omega) - w(\omega)| v_0(\omega) d\mu. \end{aligned}$$

Fix $x \in S_X$.

Let us put $f_\alpha = u_\alpha \otimes x$ ($\in E(X)$), $h = w \otimes x$ ($\in M_\xi'(X)$) and $G_\alpha = G_{\psi_\alpha}$ ($\in M_\xi(X^*)^\sim$). Then $J(h) = F_h \in M_\xi(X^*)^\sim_n$.

Note, that for each $g_0 \in M_\xi(X^*)$ we have

$$\begin{aligned}
|\Pi(f_\alpha) - F_h|(g_0) &= \sup \{ |(\Pi(f_\alpha) - F_h)(g)| : g \in M_\xi(X^*), \tilde{g} \leq \tilde{g}_0 \} \\
&= \sup \left\{ \left| \int_\Omega \langle f_\alpha(\omega) - h(\omega), g(\omega) \rangle d\mu \right| : g \in M_\xi(X^*), \tilde{g} \leq \tilde{g}_0 \right\} \\
&= \sup \left\{ \left| \int_\Omega \langle g(\omega), \kappa(f_\alpha(\omega) - h(\omega)) \rangle d\mu \right| : g \in M_\xi(X^*), \tilde{g} \leq \tilde{g}_0 \right\} \\
&= \int_\Omega \|g_0(\omega)\|_{X^*} \|\kappa(f_\alpha(\omega) - h(\omega))\|_{X^{**}} d\mu \quad [15, \text{Theorem 2.4}] \\
&= \int_\Omega |u_\alpha(\omega) - w(\omega)| \tilde{g}_0(\omega) d\mu.
\end{aligned}$$

Hence for each $g \in M_\xi(X^*)$ ($\tilde{g} \in M_\xi^+$) we get:

$$\begin{aligned}
|\Pi(f_\alpha) - F_h|(g) &= \int_\Omega |u_\alpha(\omega) - w(\omega)| \tilde{g}(\omega) d\mu = |P(u_\alpha) - \psi_w|(\tilde{g}) \leq \psi_\alpha(\tilde{g}) \\
&\stackrel{(*)}{=} |G_{\psi_\alpha}|(g) = |G_\alpha|(g)
\end{aligned}$$

with $|G_\alpha|(g) = \psi_\alpha(\tilde{g}) \downarrow 0$, so $\Pi(f_\alpha) \xrightarrow{(o)} F_h$ in $M_\xi(X^*)^\sim$. Since $\Pi(E(X))$ is order closed, $F_h \in \Pi(E(X))$. Thus there exists $f \in E(X)$ such that $F_h = \Pi(f)$. It follows that $h = J^{-1}(F_h) = J^{-1}(\Pi(f)) \stackrel{(23)}{=} f \in E(X)$, so $|w| = \tilde{h} = \tilde{f} \in E$. Thus $w \in E$ and for each $v \in M_\xi$ we have

$$\psi(v) = \psi_w(v) = \int_\Omega v(\omega) w(\omega) d\mu = \varphi_v(w) = p_w(\varphi_v) = (p(w) \circ i_\xi)(v) = (P(w))(v),$$

which means that $\psi = P(w) \in P(E)$. Thus the subset $P(E)$ of M_ξ^\sim is order closed.

(\Leftarrow) Now, assume that the subset $P(E)$ of M_ξ^\sim is order closed. We will show that the subset $\Pi(E(X))$ of $M_\xi(X^*)^\sim$ is order closed.

Choose a net (f_α) in $E(X)$ and $G \in M_\xi(X^*)^\sim$ in such a way that $\Pi(f_\alpha) \xrightarrow{(o)} G$ in $M_\xi(X^*)^\sim$. Then $G \in M_\xi(X^*)^\sim_n$, because $\Pi(E(X)) \subset M_\xi(X^*)^\sim_n$ and by Proposition 1.1 $M_\xi(X^*)^\sim_n$ is order closed ideal of $M_\xi(X^*)^\sim$. Hence there exists $h \in M'_\xi(X)$ such that $G = J(h) = F_h$, i.e., $G(g) = \int_\Omega \langle h(\omega), g(\omega) \rangle d\mu$ for $g \in M_\xi(X^*)$.

Moreover, there exists a net (G_α) in $M_\xi(X^*)^\sim$ such that

$$|\Pi(f_\alpha) - F_h|(g) \leq |G_\alpha|(g) \downarrow_\alpha 0 \quad \text{for each } g \in M_\xi(X^*).$$

Note, that for each $g_0 \in M_\xi(X^*)$ we have

$$\begin{aligned}
|\Pi(f_\alpha) - F_h|(g_0) &= \sup \{ |(\Pi(f_\alpha) - F_h)(g)| : g \in M_\xi(X^*), \tilde{g} \leq \tilde{g}_0 \} \\
&= \sup \left\{ \left| \int_\Omega \langle f_\alpha(\omega) - h(\omega), g(\omega) \rangle d\mu \right| : g \in M_\xi(X^*), \tilde{g} \leq \tilde{g}_0 \right\} \\
&= \int_\Omega \widetilde{f_\alpha - h(\omega)} \tilde{g}_0(\omega) d\mu.
\end{aligned}$$

Then $(\tilde{f}_\alpha) \subset E$, $\tilde{h} \in M'_\xi$, so $j(\tilde{h}) = \psi_{\tilde{h}} \in (M_\xi)_n^\sim$. Further, let us put $\psi_\alpha = \psi_{G_\alpha}$ ($\in M_\xi^\sim$). Thus for each $v \in M_\xi^+$ and a fixed $x^* \in S_{X^*}$ we have

$$\begin{aligned}
|P(\tilde{f}_\alpha) - \psi_{\tilde{h}}|(v) &= \int_\Omega |\tilde{f}_\alpha(\omega) - \tilde{h}(\omega)| v(\omega) d\mu \\
&\leq \int_\Omega \widetilde{f_\alpha - h(\omega)} v(\omega) d\mu \\
&= |\Pi(f_\alpha) - F_h|(v \otimes x^*) \leq |G_\alpha|(v \otimes x^*) = \psi_\alpha(v)
\end{aligned}$$

with $\psi_\alpha(v) = |G_\alpha|(v \otimes x^*) \downarrow_\alpha 0$, so $P(f_\alpha) \xrightarrow{(o)} \psi_{\tilde{h}}$ in M_ξ^\sim . Since $P(E)$ is order closed, $\psi_{\tilde{h}} \in P(E)$, i.e., $\psi_{\tilde{h}} = P(u)$ for some $u \in E$. This means that $\tilde{h} = j^{-1}(\psi_{\tilde{h}}) = j^{-1}(P(u)) = u \in E$. It follows that $h \in E(X)$ and for each $g \in M_\xi(X^*)$ we have

$$G(g) = F_h(g) = \int_\Omega \langle h(\omega), g(\omega) \rangle d\mu = F_g(h) = \pi_h(F_g) = (\pi(h) \circ I_\xi)(g) = (\Pi(h))(g),$$

so $G = \Pi(h) \in \Pi(E(X))$. Thus the set $(\Pi(E(X)))$ of $M_\xi(X^*)^\sim$ is order closed, which completes the proof. \blacksquare

Combining Theorem 3.1 with Theorem 3.2, we obtain the following

COROLLARY 3.3 *Let (E, ξ) be a Hausdorff locally convex-solid function space with a Lebesgue property and let X be a reflexive Banach space. Then the set $\pi(E(X))$ is order closed in $(E(X)_{\xi}^*)^\sim$ if and only if the set $p(E)$ is order closed in $(E_\xi^*)^\sim$.*

Before the next theorem let us recall, that by $|\sigma|(E(X)_n^\sim, E(X))$ we mean the topology $\tau(\{\varrho_f : f \in E(X)\})$ generated by the family $\{\varrho_f : f \in E(X)\}$ of solid seminorms defined on $E(X)_n^\sim$ by $\varrho_f(F) = |F|(f)$ for $F \in E(X)_n^\sim$ (see [16, §3] for more details). For $f \in E(X)$ the linear functional $\pi_f : E(X)_n^\sim \rightarrow \mathbb{R}$ is defined by $\pi_f(F) = F(f)$ for $F \in E(X)_n^\sim$. It is easily seen that $|\pi_f|(F) = |F|(f)$ for $F \in E(X)_n^\sim$, $f \in E(X)$.

THEOREM 3.4 *Let E be an ideal of L^0 with $\text{supp } E = \text{supp } E' = \Omega$ and let X be a reflexive Banach space. Then $(E(X)_n^\sim, |\sigma|(E(X)_n^\sim, E(X)))^* = \{\pi_f : f \in E(X)\}$.*

PROOF In view of [16, Theorem 3.2] we know that $(E(X)_n^\sim, |\sigma|(E(X)_n^\sim, E(X)))^* = S(\{\pi_f : f \in E(X)\}) = \{V \in (E(X)_n^\sim)^\sim : |V| \leq |\pi_f| \text{ for some } f \in E(X)\} \subset (E(X)_n^\sim)^\sim$. Thus we have only to prove that for a functional $V \in (E(X)_n^\sim)^\sim$ such that $|V| \leq |\pi_f|$ for some $f \in E(X)$ there exists $h \in E(X)$ such that $V = \pi_h$.

By [15, Theorem 2.4 and Remark 2.1] we can define the natural mappings

$$K : E'(X^*) \ni g \mapsto F_g \in E(X)_n^\sim,$$

where $F_g(f) = \int_\Omega \langle f(\omega), g(\omega) \rangle d\mu$ for $f \in E(X)$ and

$$M : E''(X^{**}) \ni k \mapsto \bar{F}_k \in E'(X^*)^\sim_n,$$

where $\bar{F}_k(g) = \int_\Omega \langle g(\omega), k(\omega) \rangle d\mu$ for $g \in E'(X^*)$. It is easily seen that K, M are linear isomorphisms onto.

Proceeding as above (see definitions of $j_\xi, j'_\xi, J_\xi, J'_\xi$ in Section 2) let us consider the mappings

$$L : E'(X^*)^\sim_n \longrightarrow (E(X)_n^\sim)^\sim \quad \text{and} \quad L' : (E(X)_n^\sim)^\sim \longrightarrow E'(X^*)^\sim_n$$

defined by

$$L(\bar{F}) = \bar{F} \circ K^{-1} \quad \text{for } \bar{F} \in E'(X^*)^\sim_n \quad \text{and} \quad L'(V) = V \circ K \quad \text{for } V \in (E(X)_n^\sim)^\sim_n.$$

Clearly, L, L' are well defined, linear and $L' \circ L = id_{E'(X^*)^\sim_n}$, $L \circ L' = id_{(E(X)_n^\sim)^\sim_n}$.

Thus L, L' are bijections and $L' = L^{-1}$, so L and L^{-1} are linear isomorphism onto. Fix $V \in (E(X)_n^\sim)^\sim$ such that $|V| \leq |\pi_f|$ for some $f \in E(X)$. Obviously $V \in (E(X)_n^\sim)^\sim_n$. Since $(E(X)_n^\sim)^\sim_n = L(E'(X^*)^\sim_n)$, there exists a functional $\bar{F} \in E'(X^*)^\sim_n$ such that $V = \bar{F} \circ K^{-1}$. But $E'(X^*)^\sim_n = M(E''(X^{**}))$, so we can find $k \in E''(X^{**})$ such that $\bar{F} = M(k) = \bar{F}_k$. Moreover, for each $g_0 \in E'(X^*)$ we have $|\pi_f|(F_{g_0}) = |F_{g_0}|(f) = \int_\Omega \tilde{f}(\omega) \tilde{g}(\omega) d\mu$ and

$$\begin{aligned} |V|(F_{g_0}) &= \sup \{ |V(F_g)| : F_g \in E(X)_n^\sim, |F_g| \leq |F_{g_0}| \} \\ &= \sup \{ |(\bar{F}_k \circ K^{-1})(F_g)| : g \in E'(X^*), \tilde{g} \leq \tilde{g}_0 \} \\ &= \sup \{ |\bar{F}_k(g)| : g \in E'(X^*), \tilde{g} \leq \tilde{g}_0 \} \\ &= |\bar{F}_k|(g_0) = \int_\Omega \tilde{g}_0(\omega) \tilde{k}(\omega) d\mu. \end{aligned}$$

Thus we have

$$\int_\Omega \tilde{k}(\omega) \tilde{g}(\omega) d\mu = |V|(F_g) \leq |\pi_f|(F_g) = \int_\Omega \tilde{f}(\omega) \tilde{g}(\omega) d\mu \quad \text{for each } g \in E'(X^*).$$

Since $\text{supp } E' = \Omega$, we get $\tilde{k} \leq \tilde{f}$, and so $k \in E(X^{**})$. Let us put $h = \kappa^{-1} \circ k$. Thus $h \in E(X)$ and for each $g \in E'(X^*)$ we have

$$\begin{aligned} \pi_h(F_g) &= F_g(h) = \int_\Omega \langle h(\omega), g(\omega) \rangle d\mu = \int_\Omega \langle g(\omega), \kappa(h(\omega)) \rangle d\mu = \\ &= \int_\Omega \langle g(\omega), k(\omega) \rangle d\mu = \bar{F}_k(g) = (\bar{F}_k \circ K^{-1})(F_g) = V(F_g). \end{aligned}$$

This means that $V = \pi_h$, which completes the proof. \blacksquare

REMARK 3.5 In case E is perfect and X is reflexive, Theorem 3.4 was earlier proved by M. Nowak in [16]. Indeed, in that case the space $E(X)$ is perfect (see [16, Theorem 1.4]), so by [16, Theorem 3.3] we have $(E(X)_n^\sim, |\sigma|(E(X)_n^\sim, E(X)))^* = (E(X)_n^\sim)_n^\sim = \{\pi_f : f \in E(X)\}$.

Let M be an ideal of E' with $\text{supp } M = \Omega$ and let X be a reflexive Banach space. In the next Corollary, $|\sigma|(M(X^*)_n^\sim, M(X^*))$ is the topology $\tau(\{\varrho_g : g \in M(X^*)\})$ generated by the family $\{\varrho_g : g \in M(X^*)\}$ of solid seminorms defined on $M(X^*)_n^\sim$ by $\varrho_g(F) = |F|(g)$ for $F \in M(X^*)_n^\sim$ and for $g \in M(X^*)$ the linear functional $\varphi_g : M(X^*)_n^\sim \rightarrow \mathbb{R}$ is defined by $\varphi_g(F) = F(g)$ for $F \in M(X^*)_n^\sim$.

As a consequence of Theorem 3.4 we get:

COROLLARY 3.6 *Let M be an ideal of E' with $\text{supp } M = \Omega$ and let X be a reflexive Banach space. Then $(M(X^*)_n^\sim, |\sigma|(M(X^*)_n^\sim, M(X^*)))^* = \{\varphi_g : g \in M(X^*)\}$.*

4. Applications. We start by recalling the necessary and sufficient conditions for a Hausdorff locally convex-solid function space (E, ξ) to be an (order closed) ideal of its bidual space.

THEOREM 4.1 (see [1, Theorem 6.62]) *Let (E, ξ) be a Hausdorff locally convex-solid function space. Then the following statements are equivalent:*

- (i) ξ is a Lebesgue topology.
- (ii) Each order interval in E is $\sigma(E, E_\xi^*)$ -compact.
- (iii) E (embedded in a natural way) is an ideal of its bidual.

Recall that a locally solid topology ξ on a Riesz space E is called a *Levi topology* if every increasing ξ -bounded net of E^+ has a supremum in E (see [1, Definition 3.16]).

THEOREM 4.2 (see [1, Theorem 6.63]) *Let (E, ξ) be a Hausdorff locally convex-solid function space. Then the following statements are equivalent:*

- (i) ξ is a Lebesgue, Levi topology.
- (ii) $p(E) = (E_\xi^*)_n^\sim$.
- (iii) E (embedded in a natural way) is an order closed ideal of its bidual.
- (iv) E is $|\sigma|(E, E_\xi^*)$ -complete.

To give the norm-version of Theorem 4.2 recall, that a Banach lattice $(E, \|\cdot\|_E)$ is said to be a *KB-space* (*Kantorovich-Banach space*) whenever every increasing norm bounded sequence of E^+ is norm convergent. It is known that a Banach lattice $(E, \|\cdot\|_E)$ is a KB-space iff $\|\cdot\|_E$ has both the σ -Lebesgue property and the σ -Levi property. Moreover, in view of [13, Theorem 10.4.9], a Banach lattice $(E, \|\cdot\|_E)$ is a KB-space iff $\|\cdot\|_E$ has both the Lebesgue property and the Levi property, i.e. the norm topology on E is a Lebesgue and Levi topology.

It is also well known, that the bidual of a Banach space $(E, \|\cdot\|_E)$ is E^{**} , i.e. the second Banach dual of $(E, \|\cdot\|_E)$.

THEOREM 4.3 (see [2, Theorem 14.12], [1, Theorem 6.63]) *For a Banach lattice $(E, \|\cdot\|_E)$ the following statements are equivalent:*

- (i) $(E, \|\cdot\|_E)$ is a KB-space.
- (ii) $p(E) = (E^*)_{\sim}$.
- (iii) E (embedded in a natural way) is an order closed ideal of E^{**} .
- (iv) E is $|\sigma|(E, E^*)$ -complete.

Let (E, ξ) be a Hausdorff locally convex-solid function space. Then $E(X)_{\xi}^*$ is an ideal of $E(X)_{\sim}$ separating points of $E(X)$. By $\beta(E(X)_{\xi}^*, E(X))$ we will denote the strong topology on $E(X)_{\xi}^*$ with respect to the solid dual system $\langle E(X), E(X)_{\xi}^* \rangle$. It is known that $\beta(E(X)_{\xi}^*, E(X))$ is a Hausdorff locally convex-solid topology on $E(X)_{\xi}^*$ (see [16, §4] for more details). By [16, Theorem 2.1], the topological dual $(E(X)_{\xi}^*)_{\beta}^* = (E(X)_{\xi}^*, \beta(E(X)_{\xi}^*, E(X)))^*$ is an ideal of $(E(X)_{\xi}^*)_{\sim}$ separating points of $E(X)_{\xi}^*$. The space $(E(X)_{\xi}^*)_{\beta}^*$ will be called the bidual of $(E(X), \xi)$ (see [16, §4]).

In particular, if $(E, \|\cdot\|_E)$ is a Banach function space, then the strong topology $\beta(E(X)^*, E(X))$ on the Banach space $E(X)^*$ coincides with the topology of the norm $\|\cdot\|_{E(X)^*}$.

Theorem 4.1 can be extended to the vector-valued setting (see [19, Theorem 2.4]).

THEOREM 4.4 *Let (E, ξ) be a Hausdorff locally convex-solid function space and let X be a Banach space. Then the following statements are equivalent:*

- (i) ξ is a Lebesgue topology and X is reflexive.
- (ii) For each $u \in E^+$ the set $D_u := \{f \in E(X) : \tilde{f} \leq u\}$ is $\sigma(E(X), E(X)_{\xi}^*)$ -compact.
- (iii) $\pi(E(X))$ is an ideal of $(E(X)_{\xi}^*)_{\beta}^*$, i.e., $E(X)$ (embedded in a natural way) is an ideal of its bidual.

Now, applying the results of Section 3, we are able to prove the vector-valued version of Theorem 4.2.

THEOREM 4.5 *Let (E, ξ) be a Hausdorff locally convex-solid function space and let X be a Banach space. Then the following statements are equivalent:*

- (i) ξ is a Lebesgue, Levi topology and X is reflexive.
- (ii) $\pi(E(X)) = (E(X)_{\xi}^*)_{\sim}$.
- (iii) $\pi(E(X))$ is an order closed ideal of $(E(X)_{\xi}^*)_{\beta}^*$, i.e. $E(X)$ (embedded in a natural way) is an order closed ideal of its bidual.
- (iv) $E(X)$ is $|\sigma|(E(X), E(X)_{\xi}^*)$ -complete and X is reflexive.

PROOF (i) \implies (ii) Let M_{ξ} be an ideal of E' with $\text{supp } M_{\xi} = \Omega$ determined by ξ (see [19, Proposition 1.1]). By Theorem 4.2, we have $p(E) = (E_{\xi}^*)_{\sim}$. Hence $P(E) = (j_{\xi}^{-1} \circ p)(E) = j_{\xi}^{-1}((E_{\xi}^*)_{\sim}) = (M_{\xi})_{\sim} = j(M'_{\xi})$, so $(j^{-1} \circ P)(E) = M'_{\xi}$. Since $j^{-1} \circ P : E \rightarrow M'_{\xi}$ is an inclusion map, we have $M'_{\xi} = E$. We know that also $J^{-1} \circ \Pi : E(X) \rightarrow M'_{\xi}(X)$ is an inclusion map and $E(X) = M'_{\xi}(X)$, so we get $(J^{-1} \circ \Pi)(E(X)) = M'_{\xi}(X)$. It follows that $\Pi(E(X)) = J(M'_{\xi}(X)) = M_{\xi}(X^*)_{\sim}$. Thus $\pi(E(X)) = (J_{\xi} \circ \Pi)(E(X)) = J_{\xi}(M_{\xi}(X^*)_{\sim}) = (E(X)_{\xi}^*)_{\sim}$.

(ii) \implies (iii) It is obvious, because $\pi(E(X)) = (E(X)_{\xi}^*)_{\sim}$ is an order closed ideal of $(E(X)_{\xi}^*)_{\sim}$ and $\pi(E(X)) \subset (E(X)_{\xi}^*)_{\beta}^*$.

(iii) \implies (i) In view of Theorem 4.4, ξ is a Lebesgue topology and X is reflexive, because $\pi(E(X))$ is an ideal of $(E(X)_{\xi}^*)_{\beta}^*$. Since the set $\pi(E(X))$ is order closed in $(E(X)_{\xi}^*)_{\sim}$, then by Theorem 3.1, the set $\Pi(E(X))$ is order closed in $M_{\xi}(X^*)_{\sim}$. In view of Theorem 3.2, the set $P(E)$ is order closed in $(M_{\xi})_{\sim}$. Hence the set $p(E)$ is order closed in $(E_{\xi}^*)_{\sim}$. Consequently, the ideal $p(E)$ is order closed in $(E_{\xi}^*)_{\beta}^*$, so by Theorem 4.2 ξ is also a Levi topology.

(i) \implies (iv) Let M be an ideal of E' with $\text{supp } M = \Omega$ determined by ξ . By Corollary 3.5, we know that $(M(X^*)_{\sim}, |\sigma|(M(X^*)_{\sim}, (M(X^*))^*)^* = \{\varphi_g : g \in M(X^*)\}$. Note, that the topology $|\sigma|(M(X^*)_{\sim}, M(X^*))$ is the topology of uniform convergence on the family of all sets $D_v = \{g \in M(X^*) : \tilde{g} \leq v\}$, where $v \in M^+$. Moreover, all the sets D_v are $|\sigma|(M(X^*), M(X^*)_{\sim})$ -bounded, closed and convex, and both the topologies $\sigma(M(X^*), M(X^*)_{\sim})$ and $|\sigma|(M(X^*), M(X^*)_{\sim})$ are compatible (see [15, Theorem 4.4]), so they are also $\sigma(M(X^*), M(X^*)_{\sim})$ -bounded and closed. By the Grothendieck completeness theorem (see [20, Corollary 12.2.15]), in order to prove that $M(X^*)_{\sim}$ is $|\sigma|(M(X^*)_{\sim}, M(X^*))$ -complete, it is enough to show, that every $F \in M(X^*)^{\#}$ such that $F|_{D_v}$ is $\sigma(M(X^*), M(X^*)_{\sim})$ -continuous for every $v \in M^+$ is also $\sigma(M(X^*), M(X^*)_{\sim})$ -continuous, i.e. $F \in M(X^*)_{\sim}$.

Fix $F \in M(X^*)^{\#}$ such that $F|_{D_v}$ is $\sigma(M(X^*), M(X^*)_{\sim})$ -continuous for every $v \in M^+$. Choose a sequence (g_n) in $M(X^*)$ such that $\tilde{g}_n \xrightarrow{(o)} 0$ in M . Then $(g_n) \subset D_{v_0}$ for some $v_0 \in M^+$. Since $|\sigma|(M(X^*), M(X^*)_{\sim})$ is a Lebesgue topology (see [15, Theorem 4.2], [18, Theorem 2.4]), we get $g_n \rightarrow 0$ in $|\sigma|(M(X^*), M(X^*)_{\sim})$, so $g_n \rightarrow 0$ in $\sigma(M(X^*), M(X^*)_{\sim})$. Thus $F|_{D_{v_0}}(g_n) \rightarrow 0$ i.e., $F(g_n) \rightarrow 0$. It means that $F \in M(X^*)_{\sim} = M(X^*)_{\sim}$ (see [15, Theorem 2.3]). Consequently the space $(M(X^*)_{\sim}, |\sigma|(M(X^*)_{\sim}, M(X^*)))$ is complete. Therefore the space

$((E(X)_\xi^*)_\sim, |\sigma|((E(X)_\xi^*)_\sim, E(X)_\xi^*))$ is complete. By (ii) we have $(E(X)_\xi^*)_\sim = \pi(E(X))$, so also the space $(\pi(E(X)), |\sigma|(\pi(E(X)), E(X)_\xi^*))$ is complete. But this space is linearly homeomorphic to the space $(E(X), |\sigma|(E(X), E(X)_\xi^*))$, so $E(X)$ is $|\sigma|(E(X), E(X)_\xi^*)$ -complete.

(iv) \implies (i) In view of Theorem 4.2, it is enough to prove that the topology $|\sigma|(E, E_\xi^*)$ is complete.

First, let (u_α) be a Cauchy net in E^+ for $|\sigma|(E, E_\xi^*)$. Fix $x \in S_X$. Then $(u_\alpha \otimes x)$ is a Cauchy net for $|\sigma|(E(X), E(X)_\xi^*)$. Therefore there exists $f \in E(X)$ such that $u_\alpha \otimes x \longrightarrow f$ in $|\sigma|(E(X), E(X)_\xi^*)$, or equivalently, $|F_g|(u_\alpha \otimes x - f) \longrightarrow 0$ for each $g \in M(X^*)$. Hence $|F_g|(u_\alpha \otimes x - f \otimes x) \longrightarrow 0$ for each $g \in M(X^*)$, because $|u_\alpha - \tilde{f}| = |u_\alpha \otimes x - \tilde{f}| \leq u_\alpha \otimes x - f$. Thus $\varphi_v(|u_\alpha - \tilde{f}|) \longrightarrow 0$ for each $v \in M^+$, or equivalently, $|u_\alpha - \tilde{f}| \longrightarrow 0$ in $|\sigma|(E, E_\xi^*)$, i.e., $u_\alpha \longrightarrow \tilde{f}$ in $|\sigma|(E, E_\xi^*)$.

Now, let (u_α) be a Cauchy net in E for $|\sigma|(E, E_\xi^*)$. Since $|\sigma|(E, E_\xi^*)$ is locally solid, the nets $(|u_\alpha|)$, (u_α^+) , (u_α^-) of positive elements of E are also Cauchy nets for $|\sigma|(E, E_\xi^*)$ ($||u_{\alpha_1}| - |u_{\alpha_2}|| \leq |u_{\alpha_1} - u_{\alpha_2}|$ by the triangle inequality and $|u_{\alpha_1}^+ - u_{\alpha_2}^+| \leq |u_{\alpha_1} - u_{\alpha_2}|$, $|u_{\alpha_1}^- - u_{\alpha_2}^-| \leq |u_{\alpha_1} - u_{\alpha_2}|$ by the Birkhoff's inequality). Thus there exist $u^+, u^- \in E^+$ such that $u_\alpha^+ \longrightarrow u^+$, $u_\alpha^- \longrightarrow u^-$ in $|\sigma|(E, E_\xi^*)$. Let us put $u := u^+ - u^-$. Clearly $u \in E$ and $u_\alpha = u_\alpha^+ - u_\alpha^- \longrightarrow u^+ - u^- = u$ in $|\sigma|(E, E_\xi^*)$. It means that the topology $|\sigma|(E, E_\xi^*)$ is complete, as desired. \blacksquare

Combining Theorem 4.3, Theorem 4.2 and Theorem 4.5, we obtain the following vector-valued version of Theorem 4.3.

COROLLARY 4.6 *Assume that $(E, \|\cdot\|_E)$ is a Banach function space and $(X, \|\cdot\|_X)$ is a Banach space. Then the following statements are equivalent:*

- (i) E is a KB-space and X is reflexive.
- (ii) $\pi(E(X)) = (E(X)_\xi^*)_\sim$.
- (iii) $E(X)$ (embedded in a natural way) is an order closed ideal of $E(X)^{**}$.
- (iv) $E(X)$ is $|\sigma|(E(X), E(X)_\xi^*)$ -complete and X is reflexive.

As an application of Theorem 4.5, we obtain a characterization of perfectness of the space $E(X)$.

Let us recall that an ideal E of L^0 is said to be perfect whenever a natural embedding

$$p : E \ni u \longmapsto p_u \in (E_n^\sim)_n, \quad p_u(\varphi) = \varphi(u) \quad \text{for } \varphi \in E_n^\sim$$

is onto, i.e. $p(E) = (E_n^\sim)_n$ (see [2, p. 58–59]).

It is well known that E is perfect if and only if $E'' = E$.

Now we recall the notion of perfectness for vector-valued function spaces $E(X)$ (see [16, Definition 1.4]). For $f \in E(X)$ let us put $\pi_f(F) = F(f)$ for $F \in E(X)_n^\sim$.

Then $\pi_f \in (E(X)_n^\sim)_n^\sim$ (see [16, p. 220]). The space $E(X)$ is said to be *perfect*, whenever the natural embedding

$$\pi : E(X) \ni f \longrightarrow \pi_f \in (E(X)_n^\sim)_n^\sim$$

is onto, i.e. $\pi(E(X)) = (E(X)_n^\sim)_n^\sim$.

THEOREM 4.7 *Assume that $\text{supp } E' = \Omega$. Then the following statements are equivalent:*

- (i) E is perfect and X is reflexive.
- (ii) $E(X)$ is perfect.

PROOF (i) \implies (ii) It follows from [16, Theorem 1.4].

(ii) \implies (i) Let us consider the topology $\xi = |\sigma|(E, E_n^\sim)$. Clearly ξ is a Hausdorff locally convex-solid topology on E . Thus the associated topology $\bar{\xi}$ on $E(X)$ is the absolute weak topology $|\sigma|(E(X), E(X)_n^\sim)$, i.e., $\bar{\xi} = |\sigma|(E(X), E(X)_n^\sim)$, so $E(X)_{\bar{\xi}}^* = E(X)_n^\sim$ (see [15, Theorem 4.4]). Since $E(X)$ is perfect we get $\pi(E(X)) = (E(X)_n^\sim)_n^\sim = (E(X)_{\bar{\xi}}^*)_n^\sim$. By Theorem 4.5, ξ is a Lebesgue and Levi topology on E and X is reflexive. But $E_\xi^* = E_n^\sim$, so in view of Theorem 4.2, we get $p(E) = (E_\xi^*)_n^\sim = (E_n^\sim)_n^\sim$. This means that E is perfect, which completes the proof. ■

REMARK. In case when X^* has the Radon-Nikodym property the implication (ii) \implies (i) of Theorem 4.7, can be found in [16, Theorem 1.4].

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(Received: 23.04.2007)
