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Korovkin theorem in modular spaces

Abstract. In this paper we obtain an extension of the classical Korovkin theorem in abstract modular spaces. Applications to some discrete and integral operators are discussed.

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1. Introduction. The Korovkin classical Theorem ([17], see also [10]) states the uniform convergence in $C([a,b])$, the space of the continuous real functions defined in $[a,b]$, of a sequence of positive linear operators by stating the convergence only on three test functions $\{1,x,x^2\}$. This celebrated result allows us to check the convergence with a minimum of computations. The work of Korovkin was inspired by the Bernstein proof of the Weierstrass theorem ([9]). Here the author established the uniform convergence of the Bernstein polynomials of the function f by stating it only on the functions $\{1,x,x^2\}$. There is also a trigonometric version of the Korovkin theorem using the test functions $\{1,\cos x,\sin x\}$ (see [18], [11]).

Later on several extensions of the Korovkin theorem were obtained in various settings. We quote here the books [12], [20], [13] and [1] and the extensive survey [15] which contains a wide list of references . Other interesting generalizations were obtained in [16], [26].

Recently, versions of the Korovkin theorem were obtained in different functional spaces, namely L^p-spaces or abstract Lebesgue spaces (see e.g. [14], [25]). Note that for L^p-spaces in general is not possible to get the convergence in L^p for all L^p-functions, but it is necessary to consider suitable subspaces, depending on the form of the positive linear operators involved.

Here we extend the Korovkin theorem in the abstract setting of the modular spaces. These spaces were extensively studied by J. Musielak (see [23], see also [22]) and later on in [19]. An abstract approximation theory in modular spaces was developed in [7] for the general case of nonlinear integral operators. This class of spaces is very wide and contains as particular case the L^p-spaces, the Orlicz spaces, the Musielak-Orlicz spaces and others. For a general sequence of positive linear operators (T_n), defined on suitable subspaces of a modular space, we determine a class of functions with the property that if $(T_n e_i)$ converges to e_i, where $e_i(t) = t^i$, $i = 0, 1, 2$, with respect to the Luxemburg norm in the modular space, then it converges with respect to the modular topology, on every function of the class. Key tools for this result are a density property of the space of the continuous functions in the modular space (see [21]) and a kind of "approximate" modular continuity assumption on the sequence $(T_n)_{n\in\mathbb{N}}$ over the class. In particular we obtain, as a special case, a version of the Korovkin theorem in L^p spaces and in Orlicz or Musielak-Orlicz spaces.

In Section 3 we give a general Korovkin theorem in abstract modular spaces while Section 4 is devoted to the study of this theorem, in Orlicz spaces, in the special case of particular discrete operators, including the classical Bernstein type polynomials. In Section 5, we apply our general theory to Kantorovich type operators in Orlicz spaces. Finally in Section 6 we give applications to a special sequence of convolution type operators. It is important to remark that in the last two instances the subspace of convergence is the whole Orlicz space.

Here, we will obtain a one-dimensional extension of Korovkin's Theorem in modular spaces and we will work with a compact interval $[a, b] \subset \mathbb{R}$ as a base space.

2. Notations and definitions. Let $I = [a, b]$ be a bounded interval of the real line $\mathbb{R}$, provided with the Lebesgue measure. We will denote by $X(I)$ the space of all real-valued measurable functions $f : I \to \mathbb{R}$ provided with equality a.e., by $C(I)$ the space of all continuous functions and by $C^\infty(I)$ the space of all infinitely differentiable functions.

A functional $\varrho : X(I) \to \overline{\mathbb{R}_0^+}$ is said to be a modular on $X(I)$ if

i) $\varrho(f) = 0 \Leftrightarrow f = 0$, a.e. in I,

ii) $\varrho(-f) = \varrho(f)$, for every $f \in X(I)$,

iii) $\varrho(\alpha f + \beta g) \leq \varrho(f) + \varrho(g)$, for every $f, g \in X(I), \alpha, \beta \geq 0, \alpha + \beta = 1$.

We will say that a modular ϱ is N-quasi convex if there is constant $N \geq 1$ such that

$$\varrho(\alpha f + \beta g) \leq N\alpha\varrho(Nf) + N\beta\varrho(Ng),$$

for every $f, g \in X(I), \alpha, \beta \geq 0, \alpha + \beta = 1$. If $N = 1$ we will say that ϱ is convex.

By means of the functional ϱ, we introduce the vector subspace of $X(I)$, denoted by $L^\varrho(I)$, defined by

$$L^\varrho(I) = \{f \in X(I) : \lim_{\lambda\to 0^+} \varrho(\lambda f) = 0\}.$$

The subspace $L^\varrho(I)$ is called the modular space generated by ϱ. The subspace of $L^\varrho(I)$ defined by

$$E^\varrho(I) = \{f \in L^\varrho(I) : \varrho(\lambda f) < +\infty \text{ for all } \lambda > 0\}$$

is called the space of the finite elements of $L^\varrho(I)$, see [23].

The following assumptions on modulars will be used

a) ϱ is monotone, i.e. for $f, g \in X(I)$ and $|f| \leq |g|$ then $\varrho(f) \leq \varrho(g)$.

b) ϱ is finite, i.e. $\chi_I \in L^\varrho(I)$, where χ_I denotes the characteristic function of the set I.

c) ϱ is absolutely finite, i.e. ϱ is finite and for every $\varepsilon > 0$, $\lambda > 0$ there is $\delta > 0$ such that $\varrho(\lambda\chi_B) < \varepsilon$ for any measurable subset $B \subset I$ with $|B| < \delta$.

d) ϱ is strongly finite, i.e. $\chi_I \in E^\varrho(I)$.

e) ϱ is absolutely continuous, i.e. there exists $\alpha > 0$ such that for every $f \in X(I)$, with $\varrho(f) < +\infty$, the following condition is satisfied: for every $\varepsilon > 0$ there is $\delta > 0$ such that $\varrho(\alpha f \chi_B) < \varepsilon$, for every measurable subset $B \subset I$ with $|B| < \delta$.

For the above notions see [23], [24] and [7].

Note that, since $|I| < +\infty$, if ϱ is strongly finite and absolutely continuous then it is also absolutely finite (see [4]).

Classical examples of modular spaces are given by the Orlicz spaces, generated by a $\varphi-$function φ or more generally, by any Musielak-Orlicz space generated by a φ-function φ depending on a parameter, satisfying some growth conditions with respect to the parameter, (see [23], [19], [7]). The modular functional generating the above spaces satisfy all the previous assumptions.

We say that a sequence of functions $(f_n)_{n\in\mathbb{N}} \subset L^\varrho(I)$ is modularly convergent to a function $f \in L^\varrho(I)$, if there exists $\lambda > 0$ such that

$$\lim_{n\to+\infty} \varrho[\lambda(f_n - f)] = 0.$$

This notion extends the norm-convergence in L^p-spaces. Moreover it is weaker than the F-norm-convergence induced by the Luxemburg F-norm generated by ϱ and defined by

$$\|f\|_\rho \equiv \inf\{u > 0 : \varrho(f/u) \leq u\}.$$

We recall that a sequence of functions $(f_n)_{n\in\mathbb{N}}$ is F-norm-convergent (or strongly convergent) to f iff

$$\lim_{n\to+\infty} \varrho[\lambda(f_n - f)] = 0$$

for every $\lambda > 0$. The two notions of convergence are equivalent if and only if the modular satisfies a Δ_2-condition, i.e. there exists a constant $M > 0$ such that $\varrho[2f] \leq M\varrho[f]$, for every $f \in X(I)$, (see [23]). For example, this happens for every L^p-spaces and Orlicz spaces generated by φ-functions with the Δ_2-regularity condition (see [23], [7]).

The modular convergence induces a topology on $L^\varrho(I)$, called modular topology. Given a subset $\mathcal{A} \subset L^\varrho(I)$, we will denote by $\overline{\mathcal{A}}$ the closure of $\mathcal{A}$ with respect to the modular topology. Then $f \in \overline{\mathcal{A}}$ if there is a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ such that f_n is modularly convergent to f.

Let us remark that $C(I) \subset L^\varrho(I)$ whenever ϱ is monotone and finite. Indeed, for $\lambda > 0$ we have

$$\varrho(\lambda f) \leq \varrho(\lambda \|f\|_\infty \chi_I),$$

and so, since $\chi_I \in L^\varrho(I)$, we have

$$\lim_{\lambda\to 0^+} \varrho(\lambda f) = 0,$$

that is $f \in L^\varrho(I)$.

Analogously, if ϱ is monotone and strongly finite, then $C(I) \subset E^\varrho(I)$.

We have the following (see [21] and [7]).

Proposition 2.1 *Let ϱ be a monotone, absolutely finite and absolutely continuous modular on $X(I)$. Then $\overline{C^\infty(I)} = L^\varrho(I)$.*

3. Modular Korovkin theorem. In this section we need the following notion of convexity for the modular ϱ. We will say that ϱ is N-quasi-semiconvex if there exists a constant $N \geq 1$ such that

$$\varrho[af] \leq Na\varrho[Nf],$$

for every nonnegative function $f \in X(I)$ and $0 < a \leq 1$. It is easy to see that when ϱ is N-quasi-semiconvex then we have the following characterization of the modular space $L^\varrho(I)$:

$$L^\varrho(I) = \{f \in X(I) : \varrho[\lambda f] < +\infty \text{ for some } \lambda > 0\},$$

see for example [23] and [7]. Clearly every N-quasi-semiconvex modular is N-quasi convex.

Let us consider a family $\boldsymbol{T} = (T_n)_{n\in\mathbb{N}}$ of positive linear operators $T_n : D \to X(I)$, where $D \subset L^\varrho(I)$ contains $L^\infty(I)$. We will assume that the family $(T_n)_{n\in\mathbb{N}}$ satisfies the following property $(*)$: there exists a subset $X_{\boldsymbol{T}} \subset D$ containing $C^\infty(I)$ such that for every function $f \in X_{\boldsymbol{T}}$ we have

$$\limsup_{n\to+\infty} \varrho[\lambda(T_n f)] \leq P\varrho[\lambda f]$$

for every $\lambda > 0$ and an absolute constant P.

We have the following lemma

Lemma 3.1 *If ϱ is N-quasi-semiconvex, then $T_n f \in L^\varrho(I)$ for sufficiently large n whenever $f \in X_{\boldsymbol{T}}$.*

Proof Let $f \in X_{\boldsymbol{T}}$ be fixed and let $\lambda > 0$ such that $\varrho[\lambda f] < +\infty$. There exists $\overline{n}$ depending on λ such that for every $n \geq \overline{n}$

$$\varrho[\lambda T_n f] \leq P\varrho[\lambda f] + 1 < +\infty.$$

This means that $T_n f \in L^{\varrho}(I)$. ∎

In the following for every $s, t \in I$ we will put $g(s,t) = g_s(t) = (s-t)^2$. Moreover we will denote by $e_i(t)$ the functions t^i, for $i = 0,1,2$ and for $t \in I$. The main theorem of this section is as follows

Theorem 3.2 *Let ϱ be a monotone, strongly finite, absolutely continuous and N-quasi-semiconvex modular on $X(I)$. Let $\boldsymbol{T} = (T_n)_{n\in\mathbb{N}}$ be a sequence of positive linear operators satisfying the above assumption. Then if*

$$\lim_{n\to+\infty} T_n e_i = e_i \quad i = 0,1,2$$

strongly in $L^{\varrho}(I)$ then

$$\lim_{n\to+\infty} T_n f = f$$

modularly in $L^{\varrho}(I)$ for each $f \in L^{\varrho}(I)$ such that $f - C^{\infty}(I) \subset X_{\boldsymbol{T}}$.

Proof At first, we consider $f \in C(I)$. Then there exists a constant $M > 0$ such that $|f(t)| \leq M$ for every $t \in I$. Given $\varepsilon > 0$, we can choose $\delta > 0$ such that $|s-t| < \delta$ implies $|f(s) - f(t)| < \varepsilon$ where $s, t \in I$. It follows easily that for all $s, t \in I$

$$|f(s) - f(t)| < \varepsilon + \frac{2M}{\delta^2}(s-t)^2$$

or

$$-\varepsilon - \frac{2M}{\delta^2} g_s(t) \leq f(s) - f(t) \leq \varepsilon + \frac{2M}{\delta^2} g_s(t).$$

Since T_n is a positive linear operator we have

$$-\varepsilon(T_n e_0)(s) - \frac{2M}{\delta^2}(T_n g_s)(s) \leq f(s)(T_n e_0)(s) - (T_n f)(s) \leq \varepsilon(T_n e_0)(s) + \frac{2M}{\delta^2}(T_n g_s)(s).$$

The above inequalities are equivalent to

$$|(T_n f)(s) - f(s)(T_n e_0)(s)| \leq \varepsilon(T_n e_0)(s) + \frac{2M}{\delta^2}[e_2(s)(T_n e_0)(s) - 2e_1(s)(T_n e_1)(s) + (T_n e_2)(s)].$$

Thus

$$\begin{aligned} & |(T_n f)(s) - f(s)| \leq |(T_n f)(s) - f(s)(T_n e_0)(s)| + |f(s)(T_n e_0)(s) - f(s)| \\ \leq \; & \varepsilon(T_n e_0)(s) + \frac{2M}{\delta^2}[e_2(s)(T_n e_0)(s) - 2e_1(s)(T_n e_1)(s) + (T_n e_2)(s)] \\ + \; & M|(T_n e_0)(s) - e_0(s)|. \end{aligned}$$

Applying the modular we have

$$\begin{aligned} &\varrho[T_n f - f] \leq \varrho[3\varepsilon T_n e_0] + \varrho[3M|(T_n e_0)(\cdot) - e_0(\cdot)|] \\ + \quad &\varrho[\frac{6M}{\delta^2}(e_2(\cdot)(T_n e_0)(\cdot) - 2e_1(\cdot)(T_n e_1)(\cdot) + (T_n e_2)(\cdot))] := J_1 + J_2 + J_3. \end{aligned}$$

Now we evaluate J_3. We can write

$$\begin{aligned} &e_2(s)(T_n e_0)(s) - 2e_1(s)(T_n e_1)(s) + (T_n e_2)(s) \\ = \quad &e_2(s)((T_n e_0)(s) - e_0(s)) + ((T_n e_2)(s) - e_2(s)) - 2e_1(s)((T_n e_1)(s) - e_1(s)), \end{aligned}$$

thus

$$\begin{aligned} &|e_2(s)(T_n e_0)(s) - 2e_1(s)(T_n e_1)(s) + (T_n e_2)(s)| \\ \leq \quad &\|e_2\|_\infty |(T_n e_0)(s) - e_0(s)| + |(T_n e_2)(s) - e_2(s)| + 2\|e_1\|_\infty |(T_n e_1)(s) - e_1(s)|. \end{aligned}$$

Hence, denoting by $K = \max\{\|e_2\|_\infty, 2\|e_1\|_\infty\}$

$$J_3 \leq \varrho[\frac{18MK}{\delta^2}|T_n e_0 - e_0|] + \varrho[\frac{18MK}{\delta^2}|T_n e_1 - e_1|] + \varrho[\frac{18M}{\delta^2}|T_n e_2 - e_2|].$$

By assumptions, we have

$$\lim_{n\to+\infty} \varrho[\lambda(T_n e_i - e_i)] = 0, \quad i = 0, 1, 2$$

for every $\lambda > 0$ and so for every constant $\mu > 0$

$$\limsup_{n\to+\infty} \varrho[\mu(T_n f - f)] \leq P\varrho[3\mu\varepsilon e_0].$$

Now, since ϱ is N-quasi-semiconvex and strongly finite, we have, assuming $\varepsilon < 1$,

$$\varrho[3\mu\varepsilon e_0] \leq N\varepsilon\varrho[3N\mu e_0].$$

So we deduce easily the strong convergence for the continuous function f.

As a final step, let $f \in L^\varrho(I)$ be a function such that $f - C^\infty(I) \subset X_{\boldsymbol{T}}$. By Proposition 2.1, there is a $\lambda > 0$ and a sequence $(f_k)_{k\in\mathbb{N}} \subset C^\infty(I)$ such that $\varrho[3\lambda f] < +\infty$ and

$$\lim_{n\to+\infty} \varrho[3\lambda(f_k - f)] = 0.$$

Let $\varepsilon > 0$ be fixed and let $\overline{k}$ be such that for every $k \geq \overline{k}$, $\varrho[3\lambda(f_k - f)] < \varepsilon$. Fix now $\overline{k}$, then we have

$$\varrho[\lambda(T_n f - f)] \leq \varrho[3\lambda T_n(f - f_{\overline{k}})] + \varrho[3\lambda(T_n f_{\overline{k}} - f_{\overline{k}})] + \varrho[3\lambda(f_{\overline{k}} - f)].$$

Passing to limsup, taking into account the first part of the proof and the assumption on the family (T_n), we obtain

$$\limsup_{n\to+\infty} \varrho[\lambda(T_n f - f)] \leq \varepsilon(P + 1)$$

and the assertion follows from the arbitrariness of $\varepsilon > 0$. ∎

If the modular ϱ satisfies the Δ_2 condition then the space $C^\infty(I)$ is strongly dense in $L^\varrho(I)$ and so we obtain $T_n f - f$ tends to zero strongly for every f such that $f - C^\infty(I) \subset X_{\boldsymbol{T}}$. Thus Theorem 3.2 becomes

THEOREM 3.3 *Under the assumptions of Theorem 3.2, if moreover ϱ satisfies a Δ_2-condition, then the following assertions are equivalent*

i) $T_n e_i \to e_i, \quad i = 0, 1, 2,$ *strongly in* $L^\varrho(I)$

ii) $T_n f \to f$ *strongly in* $L^\varrho(I)$, *for every function* f *such that* $f - C^\infty(I) \subset X_{\boldsymbol{T}}$.

In particular this holds for L^p-spaces and Orlicz spaces generated by φ-functions with the Δ_2-regularity condition (see [23], [7]).

4. Application to discrete operators.

4.1. General theory.

Let $(r(n))_{n\in\mathbb{N}}$ be an increasing sequence of natural numbers. For every fixed $n \in \mathbb{N}$, by $\Gamma_n = (\nu_{n,k})_{k=0,1,\ldots,r(n)} \subset I$ we denote a finite sequence of points such that

$$0 < \lambda_{n,k} := \nu_{n,k+1} - \nu_{n,k} \leq b_n, \quad k = 0, 1, \ldots, r(n) - 1,$$

where b_n are positive real numbers with $\lim_{n\to+\infty} b_n = 0$.

Let us consider a sequence $S = (S_n)_{n\in\mathbb{N}}$ of positive operators of the form

$$(S_n f)(s) = \sum_{k=0}^{r(n)} K_n(s, \nu_{n,k}) f(\nu_{n,k}), \quad n \in \mathbb{N}, \ s \in I \tag{1}$$

where $(K_n)_{n\in\mathbb{N}}$ $K_n : I \times \Gamma_n \to \mathbb{R}$ is a sequence of nonnegative functions such that

$$\sum_{k=0}^{r(n)} K_n(s, \nu_{n,k}) = 1 \ \text{ for every } n \in \mathbb{N}, \ s \in I.$$

Note that the domain of the operator (1) contains the space $X(I)$, due to the nature of the operator. Here $X(I)$ is the space of all real valued measurable functions which are everywhere defined in I (i.e. we distinguish two equivalent but different functions).

For $j \in \mathbb{N}$ we put

$$m_j(K_n, s) := \sum_{k=0}^{r(n)} K_n(s, \nu_{n,k})(\nu_{n,k} - s)^j.$$

As in the previous section we denote by $e_i(t)$ the functions t^i, for $i = 0, 1, 2$ and for $t \in I$. According to the above assumptions we have immediately $S_n e_0 = e_0 = 1$, for every $n \in \mathbb{N}$. We have the following

Proposition 4.1 *Let ϱ be a monotone modular on $X(I)$. Then a necessary and sufficient condition that*

$$\lim_{n\to+\infty} m_j(K_n,\cdot)=0,\ \ j=1,2 \tag{2}$$

strongly in $L^\varrho(I)$ is that

$$\lim_{n\to+\infty} S_n e_j = e_j,\ \ j=1,2$$

strongly in $L^\varrho(I)$.

Proof We can assume $\lambda=1$. First we prove the necessary condition. It is obvious that $(S_ne_1)(s)-e_1(s)=m_1(K_n,s)$. Moreover

$$(S_ne_2)(s)-e_2(s)=m_2(K_n,s)+2e_1(s)m_1(K_n,s).$$

So passing to the modular we have

$$\varrho[(S_ne_1)-e_1]=\varrho[m_1(K_n,\cdot)]$$
$$\varrho[(S_ne_2)-e_2]\le\varrho[2m_2(K_n,\cdot)]+\varrho[4\|e_1\|_\infty m_1(K_n,\cdot)]$$

and so the assertion follows. For the sufficient condition, note that

$$m_2(K_n,s)=(S_ne_2)(s)-e_2(s)-2e_1(s)((S_ne_1)(s)-e_1(s))$$

and so applying the modular we obtain the assertion. ■

We have the following corollary

Corollary 4.2 *Let ϱ be a monotone, strongly finite, absolutely continuous and N-quasi-semiconvex modular on $X(I)$. Assume that the family $(S_n)_{n\in\mathbb{N}}$ satisfies property $(*)$ and (2) holds. Then*

$$\lim_{n\to+\infty} S_n f = f$$

modularly in $L^\varrho(I)$ for each $f\in L^\varrho(I)$ such that $f-C^\infty(I)\subset X_S$, where X_S is the corresponding class given in property $(*)$.

In the next section we will describe the class X_S in some particular cases.

4.2. A description of the class X_S in Orlicz spaces. Let Φ be the class of all functions $\varphi:\mathbb{R}_0^+\to\mathbb{R}_0^+$ such that

i) φ is a convex function

ii) $\varphi(0)=0$, $\varphi(u)>0$ for $u>0$ and $\lim_{u\to+\infty}\varphi(u)=+\infty$.

For $\varphi \in \Phi$, we define the functional

$$\varrho^{\varphi}[f] = \int_I \varphi(|f(s)|)ds$$

for every $f \in X(I)$.
As it is well known, ϱ^{φ} is a convex modular on $X(I)$ and the subspace

$$L^{\varphi}(I) = \{f \in X(I) : \varrho^{\varphi}[\lambda f] < +\infty \text{ for some } \lambda > 0\}$$

is the Orlicz space generated by φ, (see [23]). The subspace of $L^{\varphi}(I)$, defined by

$$E^{\varphi}(I) = \{f \in X(I) : \varrho^{\varphi}[\lambda f] < +\infty \text{ for every } \lambda > 0\},$$

is called the space of finite elements of $L^{\varphi}(I)$. For example every bounded function belongs to $E^{\varphi}(I)$. Note that this modular satisfies all the assumptions listed in Section 2.

Let us consider the operator

$$(S_n f)(s) = \sum_{k=0}^{r(n)} K_n(s, \nu_{n,k}) f(\nu_{n,k}),$$

introduced in the previous section. Let us assume that

$$\int_I K_n(s, \nu_{n,k})ds \leq \xi_n$$

where ξ_n is a bounded sequence of positive numbers independent of k. For every $n \in I\!N$, we define

$$\varrho_n^{\varphi}[f] = \sum_{k=0}^{r(n)} \varphi(|f(\nu_{n,k})|), \;\; f \in X(I).$$

Now, let us denote by $\mathcal{F}_{\varphi}$ the class of all functions in $L^{\varphi}(I)$ such that

$$\limsup_{n\to+\infty} \xi_n \varrho_n^{\varphi}[\lambda f] \leq P \varrho^{\varphi}[\lambda f],$$

for every $\lambda > 0$ and an absolute constant $P > 0$ independent of f and λ. We have the following

Proposition 4.3 $\mathcal{F}_{\varphi} \subset X_{\mathcal{S}}$.

Proof Let $\lambda > 0$ be fixed. Using the Jensen inequality and the assumptions on the kernel $(K_n)_{n \in I\!N}$, we get

$$\varrho^{\varphi}[\lambda S_n f] \leq \int_I \sum_{k=0}^{r(n)} \varphi(\lambda |f(\nu_{n,k})|) K_n(s, \nu_{n,k}) ds \leq \xi_n \sum_{k=0}^{r(n)} \varphi(\lambda |f(\nu_{n,k})|) = \xi_n \varrho_n^{\varphi}[\lambda f],$$

and so, passing to the limsup, we obtain immediately

$$\limsup_{n\to+\infty} \varrho^{\varphi}[\lambda S_n f] \leq P \varrho^{\varphi}[\lambda f],$$

i.e. the assertion. ∎

Example 4.4 For example let us consider $I = [0,1]$ and the operator

$$(S_n f)(s) = \sum_{k=0}^{n} K_n(s, \nu_{n,k}) f(\nu_{n,k}),$$

where for every $n \in I\!N$, $(\nu_{n,k})_{k=0,\dots,n}$ is a finite partition of $[0,1]$ and $0 < a_n \leq \nu_{n,k+1} - \nu_{n,k} \leq b_n$, for every $k = 0, \dots, n-1$. Here the sequences a_n, b_n are taken such that $\lim_{n\to+\infty} \frac{\xi_n}{a_n} = \ell < +\infty$ and $\lim_{n\to+\infty} b_n = 0$. In this case we see that the class $\mathcal{F}_\varphi$ contains the space of all the Riemann integrable functions in $[0,1]$. Indeed we have, with $\nu_{n,n+1} := b_n + 1$,

$$\limsup_{n\to+\infty} \xi_n \varrho_n^\varphi[\lambda f] = \limsup_{n\to+\infty} \xi_n \sum_{k=0}^{n} \varphi(\lambda|f(\nu_{n,k})|) \leq \lim_{n\to+\infty} \frac{\xi_n}{a_n} \sum_{k=0}^{n} \varphi(\lambda|f(\nu_{n,k})|)(\nu_{n,k+1} - \nu_{n,k}).$$

But the last sum is a Riemann sum of $\varphi \circ \lambda f$ which is obviously Riemann integrable if f is so. Then in this case we have that $f \in \mathcal{F}_\varphi$ with $P = \ell$.

5. Application to Kantorovich-type operators. Using the notations of the previous section, we consider here a finite sequence $\Gamma_n = (\nu_{n,k})_{k=0,1,\dots,r(n),r(n)+1} \subset I$, satisfying the following assumption:

$$0 < a_n \leq \lambda_{n,k} := \nu_{n,k+1} - \nu_{n,k} \leq b_n, \quad k = 0, 1, \dots r(n),$$

where a_n, b_n are positive real numbers and $\lim_{n\to+\infty} b_n = 0$.

Let us consider a sequence $U = (U_n)_{n\in I\!N}$ of positive operators of the form

$$(U_n f)(s) = \sum_{k=0}^{r(n)} K_n(s, \nu_{n,k}) \frac{1}{\lambda_{n,k}} \int_{\nu_{n,k}}^{\nu_{n,k+1}} f(t)dt, \quad n \in I\!N, \; s \in I \tag{3}$$

where $(K_n)_{n\in I\!N}$, $K_n : I \times \Gamma_n \to I\!R$, is a sequence of nonnegative functions such that

$$\sum_{k=0}^{r(n)} K_n(s, \nu_{n,k}) = 1 \text{ for every } n \in I\!N, \; s \in I.$$

In this case the domain of the operator U_n contains the space of all locally integrable functions in I. In particular it easy to see that it contains $L^\infty(I)$.

We assume that

$$\lim_{n\to+\infty} m_j(K_n, \cdot) = 0, \; j = 1, 2 \tag{4}$$

strongly in $L^\varphi(I)$ and

$$\int_I K_n(s, \nu_{n,k}) ds \leq \xi_n$$

where ξ_n is a bounded sequence of positive numbers independent of k.

We begin with the following lemma

Lemma 5.1 *Let $\varphi \in \Phi$. Let us assume that $\xi_n/a_n \leq M$, for every $n \in \mathbb{N}$ and an absolute constant $M > 0$. Then for every $f \in L^\varphi(I)$*

$$\varrho^\varphi[U_n f] \leq M \varrho^\varphi[f].$$

Proof Applying twice the Jensen inequality we have

$$\varrho^\varphi[U_n f] \leq \sum_{k=0}^{r(n)} \frac{1}{\lambda_{n,k}} \left(\int_{\nu_{n,k}}^{\nu_{n,k+1}} \varphi(|f(t)|)dt \right) \int_I K_n(s, \nu_{n,k})ds$$

$$\leq \sum_{k=0}^{r(n)} \frac{\xi_n}{a_n} \int_{\nu_{n,k}}^{\nu_{n,k+1}} \varphi(|f(t)|)dt \leq M \varrho^\varphi[f].$$

In particular $U_n f \in L^\varphi(I)$ whenever $f \in L^\varphi(I)$. Moreover in this instance the property $(*)$ is satisfied with $X_{\boldsymbol{U}} = L^\varphi(I)$. ■

Hence under the above assumptions on the kernel $(K_n)_{n\in\mathbb{N}}$ we have the following corollary

Corollary 5.2 *If (4) is satisfied, for every $f \in L^\varphi(I)$ there holds*

$$\lim_{n\to+\infty} U_n f = f$$

modularly in $L^\varphi(I)$.

Proof Firstly note that by assumptions, $U_n e_0 = e_0$. Moreover, taking e_1 we have

$$(U_n e_1)(s) = \frac{1}{2} \sum_{k=0}^{r(n)} K_n(s, \nu_{n,k}) \frac{1}{\lambda_{n,k}} (\nu_{n,k+1}^2 - \nu_{n,k}^2)$$

$$= \frac{1}{2} \sum_{k=0}^{r(n)} K_n(s, \nu_{n,k})(\nu_{n,k+1} + \nu_{n,k}) = \frac{1}{2} \sum_{k=0}^{r(n)} K_n(s, \nu_{n,k})\nu_{n,k+1} + \frac{1}{2}(S_n e_1)(s)$$

$$= J_1 + J_2.$$

Now, by assumptions on the moments of the kernel $(K_n)_{n\in\mathbb{N}}$, using Proposition 4.1, we get $J_2 \to \frac{1}{2}e_1$ strongly. Next we can write

$$J_1 = \frac{1}{2}(S_n e_1)(s) + \frac{1}{2} \sum_{k=0}^{r(n)} K_n(s, \nu_{n,k})(\nu_{n,k+1} - \nu_{n,k}).$$

The last term in the right hand side of the previuous relation is estimated by

$$\frac{1}{2} b_n \sum_{k=0}^{r(n)} K_n(s, \nu_{n,k}) = \frac{1}{2} b_n$$

which tends to zero. This easily implies that $J_1 \to \frac{1}{2}e_1$ stronlgy and so finally $U_n e_1 \to e_1$ strongly. Let us take now the function e_2. We have

$$(U_n e_2)(s) = \sum_{k=0}^{r(n)} K_n(s,\nu_{n,k}) \frac{1}{\lambda_{n,k}} \int_{\nu_{n,k}}^{\nu_{n,k+1}} t^2 dt$$

$$= \frac{1}{3}\sum_{k=0}^{r(n)} K_n(s,\nu_{n,k})(\nu_{n,k+1}^2 + \nu_{n,k}^2 + \nu_{n,k+1}\nu_{n,k}) = J_1 + J_2 + J_3.$$

According to the assumptions on the kernel $(K_n)_{n\in I\!N}$ we have $J_2 \to \frac{1}{3}e_2$. Next

$$J_1 = \frac{1}{3}\sum_{k=0}^{r(n)} K_n(s,\nu_{n,k})\nu_{n,k+1}^2$$

$$= \frac{1}{3}\sum_{k=0}^{r(n)} K_n(s,\nu_{n,k})\nu_{n,k}^2 + \frac{1}{3}\sum_{k=0}^{r(n)} K_n(s,\nu_{n,k})(\nu_{n,k+1}^2 - \nu_{n,k}^2)$$

$$= \frac{1}{3}(S_n e_2)(s) + \frac{1}{3}\sum_{k=0}^{r(n)} K_n(s,\nu_{n,k})(\nu_{n,k+1} - \nu_{n,k})(\nu_{n,k+1} + \nu_{n,k}).$$

The last term in the previuous relation is estimated by

$$\frac{2b_n \max\{|a|,|b|\}}{3}\sum_{k=0}^{r(n)} K_n(s,\nu_{n,k}) = \frac{2b_n \max\{|a|,|b|\}}{3}$$

which tends to zero. So we get $J_1 \to \frac{1}{3}e_2$ strongly. Finally

$$J_3 = \frac{1}{3}(S_n e_2)(s) + \frac{1}{3}\sum_{k=0}^{r(n)} K_n(s,\nu_{n,k})\nu_{n,k}(\nu_{n,k+1} - \nu_{n,k})$$

and as before the last term tends to zero. So $J_3 \to \frac{1}{3}e_2$ strongly and hence $U_n e_2 \to e_2$ strongly. The assertion follows from Theorem 3.2. ∎

A classical particular case is given by the Bernstein-Kantorovich operator (see [3]), defined by:

$$(U_n f)(s) = \sum_{k=0}^{n} \binom{n}{k} s^k (1-s)^{n-k}(n+1)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt, \quad s \in I = [0,1].$$

Here we put $r(n) = n$, $\nu_{n,k} = \frac{k}{n+1}$, and

$$K_n(s,\nu_{n,k}) = \binom{n}{k} s^k (1-s)^{n-k}(n+1),$$

for every $s \in I = [0,1]$ and $k = 0,1,\ldots n$. It is well-known that in this instance, $(U_n e_0)(s) = e_0(s)$ for every $s \in I$ and

$$m_1(K_n,s) = 0, \quad m_2(K_n,s) = \frac{s(1-s)}{n}$$

(see [20], [13], [3]) and so we can apply our previous theory to this operator.

6. Application to moment-type operators. Let us consider $I = [0, 1]$ and $(K_n)_{n\in \mathbb{N}}$ be a sequence of kernel functions $K_n : I \to \mathbb{R}_0^+$ with $K_n(t)t^{-1} \in L^1(I)$ and the following properties

$$\int_0^1 K_n(t)dt = 1 \text{ and } \int_0^1 K_n(t)t^{-1}dt \le W$$

for every $n \in \mathbb{N}$ and where W is an absolute constant.

Let $\varphi \in \Phi$ be fixed and let $L^\varphi(I)$ be the corresponding Orlicz space (see the previous section). For any function $f \in L^\varphi(I)$ we define the positive linear operator

$$(T_n f)(s) = \int_0^1 K_n(t)f(ts)dt, \;\; s \in I.$$

A typical example of such operators is generated by the "moment" kernel defined by $K_n(t) = (n+1)t^n, \;\; t \in I$. The properties of this operator were studied extensively by several authors (see e.g. [2], [8]). A first result on these operators is given by the following proposition (see also [8]).

PROPOSITION 6.1 *$T_n f \in L^\varphi(I)$ whenever $f \in L^\varphi(I)$ and*

$$\varrho^\varphi[T_n f] \le W \varrho^\varphi[f].$$

PROOF Using the Jensen inequality and the Fubini-Tonelli theorem, we have

$$\varrho^\varphi[T_n f] \le \int_0^1 K_n(t) \left[\int_0^1 \varphi(|f(ts)|)ds\right] dt$$

$$\le \int_0^1 K_n(t)t^{-1} \varrho^\varphi[f]dt \le W \varrho^\varphi[f]. \qquad \blacksquare$$

We define the integral moments $m_1(K_n, s)$ and $m_2(K_n, s)$ on putting

$$m_1(K_n, s) = s\int_0^1 K_n(t)(t-1)dt$$

and

$$m_2(K_n, s) = s^2\int_0^1 K_n(t)(t-1)^2dt.$$

As in the discrete case we have

PROPOSITION 6.2 *A necessary and sufficient condition that*

$$\lim_{n\to+\infty} m_j(K_n, \cdot) = 0, \;\; j = 1, 2$$

strongly in $L^\varphi([0,1])$ is that

$$\lim_{n\to+\infty} T_n e_j = e_j, \;\; j = 1, 2$$

strongly in $L^\varphi([0,1])$.

PROOF The proof follows from the identities

$$m_1(K_n, s) = (T_n e_1 - e_1)(s)$$

and

$$m_2(K_n, s) = (T_n e_2 - e_2)(s) - 2s\ m_1(K_n, s).$$

As a consequence we get the following corollary

COROLLARY 6.3 *If the moments* $m_1(K_n, \cdot)$ *and* $m_2(K_n, \cdot)$ *are strongly convergent to zero then*

$$\lim_{n\to+\infty} T_n f = f$$

modularly in $L^\varphi(I)$ *for each* $f \in L^\varphi(I)$.

PROOF We only remark that in this instance by Proposition 6.1 $X_{\mathbf{T}} = L^\varphi(I)$. ■

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