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Existance of solutions of nonlinear integral equations and Henstock–Kurzweil integrals

Abstract. We prove an existence theorems for the nonlinear integral equation

$$x(t) = f(t) + \int_0^a k_1(t,s)x(s)ds + \int_0^a k_2(t,s)g(s,x(s))ds, \quad t \in I_a = [0,a], \quad a \in R_+,$$

where f, g, x are functions with values in Banach spaces. Our fundamental tools are: measures of noncompactness and properties of the Henstock-Kurzweil integral.

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1. Introduction. Let $(E, \|\cdot\|)$ be a Banach space, $I_a = [0,a]$, $a > 0$ and let $g : I_a \times E \to E$ be a function such that $g(t,x(t)) = t^2 x(t) + h(t)$, where $|t| < 1$, $\|x\| < 1$, $h(t) = \begin{cases} \frac{d}{dt}(t^2 \sin t^{-2}), & t \neq 0 \\ 0, & t = 0 \end{cases}$.

Consider a problem

$$x(t) = f(t) + \int_0^a k_1(t,s)x(s)ds + \int_0^a k_2(t,s)g(s,x(s))ds, \quad t \in I_a = [0,a], \quad a \in R_+,$$

where $x, f : I_a \to E$, $k_1, k_2 : I_a \times I_a \to R_+$ be measurable functions such that $k_1(t,\cdot)$, $k_2(t,\cdot)$ are continuous.

Because functions $k_1(t,s)x(s)$ and $k_2(t,s)g(s,x(s))$ are not Bochner integrable , so it is necessary to introduce more general integral.

The Henstock-Kurzweil integral encompasses the Newton, Riemann and Lebesgue integrals [10,12,18]. A particular feature of this integral is that integrals of highly oscillating functions such as $F'(t)$, where $F(t) = t^2 \sin t^{-2}$ on (0,1] and $F(0) = 0$, can be defined.

This integral was introduced by Henstock and Kurzweil independently in 1957 - 58 and has since proved useful in the study of ordinary differential equations [8, 9, 15, 17, 25, 26].

It is well known that Henstock's Lemma plays an important role in the theory of the Henstock-Kurzweil integral in the real - valued case. On the other hand, in connection with the Henstock-Kurzweil integral for Banach space valued functions, S. S. Cao pointed out in [7] that Henstock's Lemma holds for the case of finite dimensions, but it does not always hold in infinite dimensions. So in this paper we will use the HL integral which satisfies Henstock's Lemma and which is more general than the Bochner integral.

Let $(E, \|\cdot\|)$ be a Banach space, $(E_1, \|\cdot\|)$ a separable Banach space and let $I_a = [0, a]$, $a \in R_+$. We study an existence of a solution of the following functional integral equation:

(1)

$$x(t) = f(t) + \int_0^a k_1(t,s)x(s)ds + \int_0^a k_2(t,s)g(s,x(s))ds, \quad t \in I_a = [0,a], \quad a \in R_+,$$

where f, g, x are functions with values in a Banach space E (or in a separable Banach space E_1) and integrals are taken in the sense of HL.

Note that the integral equation (1) can be considered as a nonlinear Fredholm equation expressed as a perturbed linear equation.

We should mention that an extensive work has been done in the study of the solutions of particular cases of (1) (see, e.g., [1, 2, 3, 13, 14, 20, 21, 23, 24]).

A Mönch fixed point theorem [22] and the techniques of the theory of the measure of noncompactness are used to prove the existence of solution of problem (1). By using some conditions expressed in terms of the measure of noncompactness which the function g satisfies, we define a completely continuous operator F over the Banach space $C([0, a])$, whose fixed points are solutions of (1). The fixed point theorem of Mönch is used to prove the existence of a fixed point of the operator F.

Our fundamental tools are the Kuratowski and Hausdorff measures of noncompactess [6, 16].

For any bounded subset A of E we denote by $\alpha(A)$ *the Kuratowski measure of noncompactness* of A, i.e. the infimum of all $\varepsilon > 0$ such that there exists a finite covering of A by sets of diameter smaller than ε.

For any bounded subset A of E we denote by $\alpha_1(A)$ *the Hausdorff measure of noncompactness* of A, i.e. the infimum of all $\varepsilon > 0$ such that A can be covered by a finite number of balls of radius smaller than ε .

The properties of the measure of noncompactness $\gamma = \alpha$, α_1 are:

(i) if $A \subset B$ then $\gamma(A) \le \gamma(B)$;

(ii) $\gamma(A) = \gamma(\bar{A})$, where $\bar{A}$ denotes the closure of A;

(iii) $\gamma(A) = 0$ if and only if A is relatively compact;
(iv) $\gamma(A \cup B) = \max\{\gamma(A), \gamma(B)\}$;
(v) $\gamma(\lambda A) = |\lambda|\gamma(A)$ $(\lambda \in R)$;
(vi) $\gamma(A + B) \leq \gamma(A) + \gamma(B)$;
(vii) $\gamma(convA) = \gamma(A)$, where $conv(A)$ denotes the convex extension of A,
(viii) $\alpha_1(A) \leq \alpha(A) \leq 2\alpha_1(A)$.
We will need the following lemma:

Lemma 1.1 (5) . *Let $H \subset C(I_a, E)$ be a family of strongly equicontinuous functions. Let, for $t \in I_a, H(t) = \{h(t) \in E, h \in H\}$. Then $\alpha_C(H) = \sup_{t \in I_a} \alpha(H(t)) = \alpha(H(I_a))$, where $\alpha_C(H)$ denotes the measure of noncompactness in $C(I_a, E)$ and the function $t \mapsto \alpha(H(t))$ is continuous.*

We will apply the following fixed point theorem.

Theorem 1.2 (22) . *Let D be a closed convex subset of E , and let F be a continuous map from D into itself. If for some $x \in D$ the implication*
$\bar{V} = \overline{conv}(\{x\} \cup F(V)) \quad \Rightarrow \quad V$ is relatively compact, holds for every countable subset V of D , then F has a fixed point.

Now we present the definition of ACG_* function.

Definition 1.3 (10) . A family F of functions F is said to be *uniformly absolutely continuous in the restricted sense on* A, for short, *uniformly* $AC_*(A)$, if for every $\varepsilon > 0$ there exists $\eta > 0$ such that for every F in F and for every finite or infinite sequence of nonoverlapping intervals $\{[a_i, b_i]\}$ with $a_i, b_i \in A$ and satisfying $\sum_i |b_i - a_i| < \eta$, we have $\sum_i \omega(F, [a_i, b_i]) < \varepsilon$, where $\omega(F, [a_i, b_i])$ denotes the oscillation of F over $[a_i, b_i]$.

A family F of functions F is said to be *uniformly generalized absolutely continuous in the restricted sense on* $[a, b]$, or *uniformly* ACG_*, if $[a, b]$ is the union of a sequence of closed sets A_i such that on each A_i the function F is uniformly $AC_*(A_i)$.

2. Henstock–Kurzweil integral.

In this part we present the Henstock-Kurzweil integral in a Banach space and we give some properties of this integral.

Definition 2.1 (10) . Let δ be a positive function defined on the interval $[a, b]$. *A tagged interval* $(x, [c, d])$ consists of an interval $[c, d] \subset [a, b]$ and a point $x \in [c, d]$. The tagged interval $(x, [c, d])$ is *subordinate* to δ if $[c, d] \subset [x - \delta(x), x + \delta(x)]$.

The letter P will be used to denote finite collections of nonoverlapping tagged intervals. Let $P = \{(s_i, [c_i, d_i]) : 1 \leq i \leq n,\ n \in N\}$be such a collection in $[a, b]$. Then

(i) The points $\{s_i : 1 \leq i \leq n\}$ are called the *tags* of P.

(ii) The intervals $\{[c_i, d_i] : 1 \leq i \leq n\}$ are called the intervals of P.

(iii) If $\{(s_i, [c_i, d_i]) : 1 \leq i \leq n\}$ is subordinate to δ for each i, then we say that P is sub δ.

(iv) If $[a, b] = \bigcup\limits_{i=1}^{n} [c_i, d_i]$ then P is called a tagged partition of $[a, b]$.

(v) If P is a tagged partition of $[a, b]$ and if P is sub δ, then P is sub δ on $[a, b]$.

(vi) If $f : [a,\ b] \quad \rightarrow \quad E$ then $f(P) := \sum\limits_{i=1}^{n} f(s_i)(d_i - c_i)$.

If F is a function defined on the subintervals of $[a, b]$, then
$F(P) = \sum\limits_{i=1}^{n} F([c_i, d_i]) = \sum\limits_{i=1}^{n} (F(d_i) - F(c_i))$.
If $F : [a, b] \rightarrow E$, then F can be treated as a function of intervals by defining $F([c, d]) = F(d) - F(c)$. For such a function, $F(P) = F(b) - F(a)$ if P is a tagged partition of $[a, b]$.

Definition 2.2 (7) . The function $f : [a, b] \quad \rightarrow \quad E$ is *Henstock-Kurzweil integrable on* $[a, b]$ $(f \in HK([a, b], E))$ if there exists a vector z in E with the following property: for each $\varepsilon > 0$ there exists a positive function δ on $[a, b]$ such that $\|f(P) - z\| < \varepsilon$ whenever P is sub δ on $[a, b]$. The function f is *Henstock-Kurzweil integrable on a measurable set* $A \subset [a, b]$ if the function $f\chi_A$ is Henstock-Kurzweil integrable on $[a, b]$. The vector z is the *Henstock-Kurzweil integral of f.*

We remark that this definition includes the generalized Riemann integral defined by Gordon in [11]. In a special case, when δ is a constant function, we get the Riemann integral.

Definition 2.3 (7) . A function $f : [a, b] \rightarrow E$ is *HL integrable* on $[a, b](f \in HL([a, b], E))$ if there exists a function $F : [a, b] \rightarrow E$ with the following property: given $\varepsilon > 0$ there exists a positive function δ on $[a, b]$ such that if $P = \{(s_i, [c_i, d_i] : 1 \leq i \leq n\}$ is a tagged partition of $[a, b]$, sub δ, then

$$\sum_{i=1}^{n} \|f(s_i)(d_i - c_i) - F([c_i, d_i])\| < \varepsilon.$$

We note that by the triangle inequality $f \in HL([a, b], E)$ implies $f \in HK([a, b], E)$. In general, the converse is not true. For real - valued functions, the two integrals are equivalent.

Definition 2.4 (7) . Let $f : I_a \rightarrow E$ be Henstock-Kurzweil integrable on $[a, b]$. Then a function $F(t) = \int\limits_{a}^{t} f(s)ds$, which is defined on subintervals of $[a, b]$ and the integral is in the sense of Henstock – Kurzweil, is called the *primitive of f.*

We now mention Henstock's Lemma for real - valued Henstock-Kurzweil integrable functions. For the proof, see [17, Theorem 3.7].

THEOREM 2.5 (HENSTOCK'S LEMMA) . *If f is Henstock-Kurzweil integrable on $[a,b]$ with primitive F, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any tagged partition $P = \{(s_i, [c_i, d_i]), \quad 1 \le i \le n\}$ of $[a,b]$ sub δ, we have:*

$$\sum_{i=1}^{n} |f(s_i)(d_i - c_i) - F([c_i, d_i])| < \varepsilon.$$

Theorem 2.5 says that in the definition of the Henstock – Kurzweil integral for real – valued functions [10], we may put the absolute value sign $|\cdot|$ inside the summation sign $\sum$. We know [7] that this is no longer true if we replace $|\cdot|$ with $\|\cdot\|$, i.e. Henstock's Lemma is not satisfied by Henstock – Kurzweil integrable Banach – valued functions. By the definition of HL integral, an HL integrable function with primitive F satisfies Henstock's Lemma with $|\cdot|$ replaced $\|\cdot\|$.

For the Henstock - Kurzweil integral, in particular for the HL integral, we have the following theorems.

THEOREM 2.6 (7) . *Let $f : [a,b] \to E$. If $f = 0$ almost everywhere on $[a,b]$, then f is HL integrable on $[a,b]$ and $\int_a^b f(t)dt = 0$.*

THEOREM 2.7 (7) . *Let $f : [a,b] \to E$ be HL integrable on $[a,b]$ and let $F(x) = \int_a^x f(t)dt$ for each $x \in [a,b]$. Then*

(i) F is continuous on $[a,b]$,

(ii) $F$is differentiable almost everywhere on $[a,b]$and $F' = f$,

(iii) f is measurable.

THEOREM 2.8 (27,THEOREM 5) .*Suppose that $f_n : [a,b] \to E, n = 1,2,...$ is a sequence of HL integrable functions satisfying the following conditions:*

(i) $f_n(x) \to f(x)$almost everywhere in $[a,b]$, as $n \to \infty$;

(ii) the set of primitives of $f_n, \{F_n(x)\}$, where $F_n(x) = \int_a^x f_n(s)ds$, is uniformly ACG_*in n;

(iii) the primitives F_n are equicontinuous on $[a,b]$.

Then f is HL integrable on $[a,b]$ and $\int_a^x f_n \to \int_a^x f$ uniformly on $[a,b]$, as $n \to \infty$.

We remark that this theorem for Denjoy-Bochner integrals is mentioned in [27] without proof. It is also true for HL integrals. The proof is similar to that of Theorem 7.6 in [18] (see also [26, Theorem 1.8]).

LEMMA 2.9 (25) . *Let E_1 be a separable Banach space. Suppose that V is a countable set of HL integrable functions. Let $F = \left\{ \int_0^t x(s)ds : x \in V,\ t \in I_a \right\}$ be an equicontinuous, equibounded and uniformly ACG_* on I_a. Then $\alpha_1 \left(\int_0^t V(s)ds \right) \leq \int_0^t \alpha_1(V(s))ds,\ t \in I_a$, whenever $\alpha_1(V(s)) \leq \varphi(s)$, for a.e. $s \in I_a$, φ is a Lebesgue integrable function and α_1 denotes the Hausdorff measure of noncompactness.*

THEOREM 2.10 *If the function $f : I_a \to E$ is HL integrable, then*

$$\int_I f(t)dt \in |I|\overline{conv} f(I),$$

where I is an arbitrary subinterval of I_a and $|I|$is the lenght of I.

The proof is similar to that of Lemma 2.1.3 in [19]. See also [4, Theorem 10.4, p. 268].

3. Main result. We recall that a function $f : I_a \times E \quad \to \quad E$ is a *Carathéodory function* if for each $x \in E$, $f(t, x)$ is measurable in $t \in I_a$ and for almost all $t \in I_a$, $f(t, x)$ is continuous with respect to x.

I. In this section we prove an existence theorem for functions with values in a separable Banach space E_1.

For $x \in C(I_a, E)$, we define the norm of x by: $\|x\|_C = \sup\{\|x(t)\|,\ t \in I_a\}$.

Let $B(p) = \{x \in C(I_a, E) :\ \|x\|_C \leq \|f(\cdot)\|_C + p\},\ p > 0$. Note that this set is closed and convex.

We define the operator $F : C(I_a, E) \to C(I_a, E)$ by

$$F(x)(t) = f(t) + \int_0^a k_1(t,s)x(s)ds + \int_0^a k_2(t,s)g(s,x(s))ds, \quad t \in I_a, \quad x \in B(p),$$

where integrals are taken in the sense of HL.

Moreover, let $\Gamma(p) = \{F(x) \in E : x \in B(p)\}$, $p > 0$.

A continuous function $x : I_a \to E$ is said to be *a solution of the problem* (1) if it satisfies:

$x(t) = f(t) + \int_0^a k_1(t,s)x(s)ds + \int_0^a k_2(t,s)g(s,x(s))ds$, for every $t \in I_a = [0, a]$.

THEOREM 3.1 *Assume that for each continuous function $x : I_a \to E_1$, $g(\cdot, x(\cdot))$ is HL integrable and g is a Carathéodory function. Let $k_1, k_2 : I_a \times I_a \to R_+$ be measurable functions such that $k_1(t, \cdot)$, $k_2(t, \cdot)$ are continuous. Moreover, let there exists $p_0 > 0$ and a Carathéodory function $\omega : I_a \times R_+ \to R_+$ with*

(2) $\qquad \alpha(g(s, X)) \leq \omega(s, \alpha(X))$ *for a.e.* $s \in I_a$ *and* $X \subset B(p_0)$,

such that the zero function is the unique continuous solution of the inequality

$$q(t) \le 2[\int_0^c k_1(t,s)q(s)ds + \int_0^c k_2(t,s)\omega(s,q(s))ds].$$

Suppose that $\Gamma(p_0)$ *is equicontinuous, equibounded and uniformly* ACG_* *on* I_a*. Then there exists at least one solution of the problem (1) on* I_c *for some* $0 < c \le a$ *with continuous initial function* f*.*

PROOF By equicontinuity and equiboundedness of $\Gamma(p_0)$ there exists some number c $(0 < c \le a)$, such that $\left\| \int_0^c [k_1(t,s)x(s) + k_2(t,s)g(s,x(s)]ds \right\| \le p_0$, for $t \in I_c$ and $x \in B(p_0)$.

By our assumptions, the operator F is well defined and maps $B(p_0)$ into $B(p_0)$.

We now show that the operator F is continuous. Let $x_n \to x$ in $C(I_c, E)$. Because the function g is a Carathéodory function (continuous with respect to the second variable), so from equality

$$\|F(x_n) - F(x)\| = \left\| \int_0^c (k_1(t,s)(x_n(s) - x(s)) + k_2(t,s)(g(s,x_n(s)) - g(s,x(s))))ds \right\|$$

we have $F(x_n) \to F(x)$ in $C(I_c, E)$.

Observe that a fixed point of F is the solution of the problem (1). Now we prove that F has a fixed point using Theorem 1.2.

Suppose that $V \subset B(p_0)$ satisfies the condition $\bar{V} = \overline{conv}(\{x\} \cup F(V))$, for some $x \in B(p_0)$. Let, for $t \in I_c$, $V(t) = \{v(t) \in E : v \in \mathrm{V}\}$. Since $V \subset B(p_0)$, $F(V) \subset \Gamma(p_0)$. Then $V \subset \overline{V} = \overline{conv}(\{x\} \cup F(V))$ is equicontinuous. By Lemma 1.1, $t \mapsto v(t) = \alpha(V(t))$ is continuous on I_c.

Let $\int_0^c Z(s)ds = \left\{ \int_0^c x(s)ds : x \in Z \right\}$ for any $Z \subset C(I_c, E)$ and let $\tilde{g}$ denote the mapping defined by $\tilde{g}(x(s)) = g(s, x(s))$, for each $x \in B(p_0)$ and $s \in I_c$. Obviously, $\tilde{g}(V(s)) = g(s, V(s))$, $F(V(t)) = f(t) + \int_0^c [k_1(t,s)V(s) + k_2(t,s)\tilde{g}(V(s))]\, ds$.

Using (2), Lemma 2.9 and the properties of the measure of noncompactness α we have:

$$\alpha(F(V)(t)) = \alpha\left(f(t) + \int_0^c [k_1(t,s)V(s) + k_2(t,s)\tilde{g}(V(s))]ds \right) \le$$

$$2\alpha_1\left(\int_0^c [k_1(t,s)V(s) + k_2(t,s)\tilde{g}(V(s))]ds \right) \le$$

$$2\int_0^c [k_1(t,s)\alpha_1(V(s)) + k_2(t,s)\alpha_1(g(s,V(s))]ds \le$$

$$2\int_0^c [k_1(t,s)\alpha(V(s)) + k_2(t,s)\alpha(g(s,V(s)))]ds \leq$$

$$2\int_0^c [k_1(t,s)\alpha(V(s)) + k_2(t,s)\omega(s,\alpha(V(s)))]ds.$$

Because $V = \overline{conv}(\{x\} \cup F(V))$, so

$$v(t) \leq 2[\int_0^c k_1(t,s)v(s)ds + \int_0^c k_2(t,s)\omega(s,v(s))ds].$$

By our assumptions, because the zero function is the unique continuous solution of the last inequality, so we get $v(t) = \alpha(V(t)) = 0$. By Arzela-Ascoli‘s theorem, V is relatively compact. So, by Theorem 1.2, F has a fixed point which is a solution of the problem (1). ∎

II. Now we present an existence theorem for the problem (1) for a real Banach space E.

Let $r(K)$ be the spectral radius of the integral operator K defined by:

$$K(u)(t) = \int_0^a (k_1(t,s) + k_2(t,s))u(s)ds, \quad u \in B(p_0), \quad t \in I_a.$$

THEOREM 3.2 *Assume that for each continuous function* $x: \; I_a \to E, \; g(\cdot, x(\cdot))$ *is HL integrable, gis a Carathéodory function and* $k_1, k_2 : \; I_a \times I_a \to R_+$ *are measurable functions such that* $k_1(t,\cdot)$, $k_2(t,\cdot)$ *are continuous. Moreover, let there exists* $p_0 > 0$ *and* $L > 0$ *such that*

$$(3) \qquad \alpha(g(I,X)) \leq L\alpha(X) \text{for each } X \subset B(p_0), I \subset I_a \quad .$$

Suppose that $\Gamma(p_0)$ *is equicontinuous, equibounded and uniformly* ACG_* *on* I_a *and* $(1+L)r(K) < 1$.

Then there exists at least one solution of the problem (1)on I_c *for some* $0 < c \leq a$ *with continuous initial function* f.

PROOF By equicontinuity and equiboundedness of $\Gamma(p_0)$ there exists some number c $(0 < c \leq a)$, such that $\left\| \int_0^c [k_1(t,s)x(s) + k_2(t,s)g(s,x(s)]ds \right\| \leq p_0$ for $t \in I_c$ and $x \in B(p_0)$.

By our assumptions, the operator F is well defined and maps $B(p_0)$ into $B(p_0)$.

We now show that the operator F is continuous. Let $x_n \to x$ in $C(I_c, E)$. Because the function g is a Carathéodory function (continuous with respect to the second variable), so from equality

$$\|F(x_n) - F(x)\| = \left\| \int_0^c (k_1(t,s)(x_n(s) - x(s)) + k_2(t,s)(g(s,x_n(s)) - g(s,x(s))))ds \right\|$$

we have $F(x_n) \to F(x)$ in $C(I_c, E)$.

Observe that a fixed point of F is the solution of the problem (1). Now we prove that F has a fixed point using Theorem 1.2.

Suppose that $V \subset B(p_0)$ satisfies the condition $\bar{V} = \overline{conv}(\{x\} \cup F(V))$ for some $x \in B(p_0)$.

Let, for $t \in I_c$, $V(t) = \{v(t) \in E,\ v \in V\}$.

Since $V \subset B(p_0)$, $F(V) \subset \Gamma(p_0)$. Then $V \subset \overline{V} = \overline{conv}(\{x\} \cup F(V))$ is equicontinuous. By Lemma 1.1, $t \mapsto v(t) = \alpha(V(t))$ is continuous on I_c.

We divide the interval I_c: $0 = t_0 < t_1 < ... < t_m = c$, where $t_i = \frac{ic}{m}$, $i = 0, 1, ..., m$. Let $V([t_i, t_{i+1}]) = \{u(s) \in E : u \in V,\ t_i \le s \le t_{i+1}\},\ i = 0, 1, ..., m-1$. By Lemma 1.1 and the continuity of v there exists $s_i \in T_i = [t_i, t_{i+1}]$ such that

$$\alpha(V([t_i, t_{i+1}])) = \sup\{\alpha(V(s)) : t_i \le s \le t_{i+1}\} =: v(s_i). \tag{4}$$

On the other hand, by the definition of the operator F and Theorem 2.10 we have

$$F(u)(t) = f(t) + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} [k_1(t,s)u(s) + k_2(t,s)g(s,u(s))]ds$$

$$\in f(t) + \sum_{i=0}^{m-1} (t_{i+1} - t_i)\overline{conv}[k_1(t,T_i)V(T_i) + k_2(t,T_i)g(T_i, V(T_i))],$$

for each $u \in V$, where $k_m(t,T_i) = \{k_m(t,s) : t, s \in T_i\}$, $m = 1, 2$ and $g(T_i, V(T_i)) = \{g(t,x(t)) : t \in T_i, x \in V\}$.

Therefore

$$F(V)(t) \subset f(t) + \sum_{i=0}^{m-1} (t_{i+1} - t_i)\, \overline{conv}[k_1(t,T_i)V(T_i) + k_2(t,T_i)g(T_i, V(T_i))].$$

Using (3), (4) and the properties of the measure of noncompactness α we obtain

$$\alpha(F(V)(t)) \le \sum_{i=0}^{m-1} (t_{i+1} - t_i)[k_1(t,T_i)v(s_i) + k_2(t,T_i)Lv(s_i)]$$

$$= \sum_{i=0}^{m-1} (t_{i+1} - t_i)k_1(t,T_i)v(s_i) + L\sum_{i=0}^{m-1} (t_{i+1} - t_i)k_2(t,T_i)v(s_i)$$

$$\le \sum_{i=0}^{m-1} (t_{i+1} - t_i) \sup_{s_i \in T_i} k_1(t,s_i)v(s_i) + L\sum_{i=0}^{m-1} (t_{i+1} - t_i) \sup_{s_i \in T_i} k_2(t,s_i)v(s_i)$$

$$= \sum_{i=0}^{m-1} (t_{i+1} - t_i) k_1(t, p_i) v(s_i) + L \sum_{i=0}^{m-1} (t_{i+1} - t_i) k_2(t, q_i) v(s_i),$$

where $s_i, p_i, q_i \in T_i$, hence

$$\alpha(F(V)(t)) \leq \sum_{i=0}^{m-1} (t_{i+1} - t_i) k_1(t, p_i) v(p_i) + \sum_{i=0}^{m-1} (t_{i+1} - t_i)[k_1(t, p_i)(v(s_i) - v(p_i))]$$

$$+L \sum_{i=0}^{m-1} (t_{i+1} - t_i) k_2(t, q_i) v(q_i) + L \sum_{i=0}^{m-1} (t_{i+1} - t_i)[k_2(t, q_i)(v(s_i) - v(q_i))]$$

$$= \sum_{i=0}^{m-1} (t_{i+1} - t_i) k_1(t, p_i) v(p_i) + \tfrac{c}{m} \sum_{i=0}^{m-1} [k_1(t, p_i)(v(s_i) - v(p_i))]$$

$$+L \sum_{i=0}^{m-1} (t_{i+1} - t_i) k_2(t, q_i) v(q_i) + \frac{Lc}{m} \sum_{i=0}^{m-1} [k_2(t, q_i)(v(s_i) - v(q_i))].$$

By continuity of v we have $v(s_i) - v(p_i) < \varepsilon_1$ and $\varepsilon_1 \to 0$ as $m \to \infty$ and $v(s_i) - v(q_i) < \varepsilon_2$ and $\varepsilon_2 \to 0$ as $m \to \infty$.

So $\alpha(F(V)(t)) < \int_0^c k_1(t, s) v(s) ds + c \sup_{p \in I_c} k_1(t, p) \varepsilon_1 + L \int_0^c k_2(t, s) v(s) ds + Lc \sup_{q \in I_c} k_2(t, q) \varepsilon_2$. Therefore

$$\alpha(F(V(t))) \leq (1 + L) \int_0^c [k_1(t, s) + k_2(t, s)] v(s) ds, \text{ for } t \in I_c. \tag{5}$$

Since $V = \overline{conv}(\{u\} \cup F(V))$, by the property of the measure of the noncompactness α we have $\alpha(V(t)) \leq \alpha(F(V)(t))$ and so in view of (5) it follows that $v(t) \leq (1+L) \int_0^c [k_1(t, s) + k_2(t, s)] v(s) ds, \quad for\ t \in I_c$. Because this inequality holds for every t$\in I_c$ and $(1+L) r(K) < 1$, by applying Gronwall's inequality, we conclude that $\alpha(V(t)) = 0$ for t$\in I_c$. Hence Arzela-Ascoli's theorem proves that the set V is relatively compact. Consequently, by Theorem 1.2, F has a fixed point which is a solution of the problem (1). ■

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