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Impulsive functional-differential equations of first order

Abstract. In this paper we present some existence results for impulsive functional-differential equations of first order.

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1. Introduction. In recent years, problems with impulses have been studied by a number of authors, for example [1-5]. This work was motivated by [1],[3], where impulsive functional differential equations of first order and second order were considered. In this paper we discuss the integro-differential equations of first order with impulses at fixed moments. By the method of upper and lower solutions we obtain existence result in a sector, then using the monotone iterative scheme we prove the existence of extremal solutions. Finally, we investigate the existence and the uniqueness of the global solutions for impulsive integro-differential equations of first order. Our consideration is based on fixed point theorem.

2. Preliminaries. In this section we introduce notations and definitions which are used throughout this paper.

Let $J = [0,T]$, $0 = t_0 < t_1 < ... < t_p < t_{p+1} = T$, $J' = J \setminus \{t_1, ...t_p\}$, $J_k = [t_{k-1}, t_k], k = 1, ..., p+1$.

We define the following class of functions:

$$PC(J,\mathbb{R}) = \{u : J \to \mathbb{R} : u \in C(J',\mathbb{R}), \text{ there exist } u(t_k^+) \text{ and } u(t_k^-) = u(t_k), \\ k = 1, 2, ..., p\}.$$

$$PC^1(J,\mathbb{R}) = \{u \in PC(J,\mathbb{R}) : \ u_{|(t_{k-1},t_k)} \in C^1\left((t_{k-1},t_k),\mathbb{R}\right), k = 1,...,p+1,$$
$$\text{there exist } u'(0^+), u'(T^-), u'(t_k^+)\ , k = 1,2,...,p\}.$$

Let us consider the functional differential problem of the form

$$(1) \qquad u'(t) = f(t,[\mathcal{T}u](t),u(t)), \quad t \in int\ J_k,\ k = 1,...,p+1,$$
$$(2) \qquad \Delta u(t_k) = I_k(u(t_k)),\ k = 1,...,p,$$
$$(3) \qquad u(0) = u_0,$$

where $\mathcal{T} : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ is a Voltera operator, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\Delta u(t_k) = u(t_k^+) - u(t_k), I_k : \mathbb{R} \to \mathbb{R}$ for each $k = 1,...,p$. We assume that $\mathcal{T}$ is continuous, monotone nondecreasing and for any bounded set $E \subset PC(J,\mathbb{R}), \mathcal{T}E$ is bounded.

By a solution of (1)-(3) we mean a function $u \in PC^1(J,\mathbb{R})$ satisfying (1)-(3).

Definition 2.1 A function $\alpha \in PC^1(J,\mathbb{R})$ is said to be a lower solution of problem (1)-(3) if

$$\alpha'(t) \le f(t,[\mathcal{T}\alpha](t),\alpha(t)), \quad t \in int\ J_k,\ k = 1,...,p+1,$$
$$\Delta\alpha(t_k) \le I_k(\alpha(t_k)),\ k = 1,...,p,$$
$$\alpha(0) \le u_0,$$

and an upper solution of (1)-(3) if the above inequalities are reversed.

If $\alpha, \beta \in PC(J,\mathbb{R})$ satisfy $\alpha(t) \le \beta(t), t \in J$, then we write $\alpha \le \beta$ and define the sector

$$[\alpha,\beta] = \{v \in PC(J,\mathbb{R}) : \alpha \le v \le \beta\}.$$

We introduce the following assumptions:

(H1) $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous at each point $(t,x,y) \in J' \times \mathbb{R} \times \mathbb{R}$. We assume that for all $x, y \in \mathbb{R}$ there exist the limits

$$\lim_{t\to t_k^-} f(t,x,y) = f(t_k,x,y)$$

for $k = 1,...,p+1$; and

$$\lim_{t\to t_k^+} f(t,x,y)$$

for $k = 0,...,p$.

(H2) $I_k : \mathbb{R} \to \mathbb{R}$ are continuous and nondecreasing for each $k = 1,...,p$.

(H3) There exist $\alpha, \beta \in PC^1(J,\mathbb{R})$ such that $\alpha(t) \le \beta(t), t \in J$, where α, β are lower and upper solution of (1)-(3) respectively.

(H4) $f(t,u,v)$ is nondecreasing in u for each (t,v) such that $t \in J, \alpha(t) \le v \le \beta(t)$.

Note that problems (1)-(3) are equivalent to the integral equations.

LEMMA 2.2 *If $f \in C(J' \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, then $u \in PC(J, \mathbb{R})$ is a solution of (1)-(3) if and only if $u \in PC^1(J, \mathbb{R})$ is a solution of following equation*

$$u(t) = u_0 + \int_0^t f(s, [\mathcal{T}u](s), u(s))ds + \sum_{0<t_k<t} I_k(u(t_k)),\ t \in J. \tag{4}$$

THEOREM 2.3 *Let the assumptions (H1)-(H4) hold. Then the problem (1)-(3) has a solution $u \in [\alpha, \beta]$.*

PROOF Let $v \in \mathbb{R}, t \in J$. Denote

$$p(v) = \max[\alpha(t), \min(v, \beta(t))],$$

$$\gamma(v) = \begin{cases} \dfrac{-v + \beta(t)}{1 + v^2}, & v \geq \beta(t) \\ 0, & \alpha(t) \leq v \leq \beta(t) \\ \dfrac{\alpha(t) - v}{1 + v^2}, & v \leq \alpha(t). \end{cases}$$

If $u : J \to \mathbb{R}$, then by $p(u)$ we denote the function $p(u) : J \ni t \to p(u(t)) \in \mathbb{R}$, Consider the initial problem

$$u'(t) = F(t, [\mathcal{T}u](t), u(t)),\ t \in [0, t_1], \tag{5}$$

$$u(0) = u_0, \tag{6}$$

where

$$F(t, [\mathcal{T}u](t), u(t)) = f(t, [\mathcal{T}p(u)](t), p(u(t))) + \gamma(u(t)).$$

From the definition of $p(u)$ and the monotone character of $\mathcal{T}$ we have

$$\alpha(t) \leq p(u)(t) \leq \beta(t),\ t \in [0, t_1],$$

$$[\mathcal{T}\alpha](t) \leq [\mathcal{T}p(u)](t) \leq [\mathcal{T}\beta](t), t \in [0, t_1]. \tag{7}$$

F is bounded and continuous on $[0, t_1] \times \Sigma \times \Omega = \big\{(t, v, u) \in \mathbb{R}^3 : t \in [0, t_1],$ $[\mathcal{T}\alpha](t) \leq v \leq [\mathcal{T}\beta](t), \alpha(t) \leq u \leq \beta(t)\big\}$, i.,e. there exists $M > 0$ such that for any $(t, v, u) \in [0, t_1] \times \Sigma \times \Omega$ we have $|F(t, v, u)| < M$. If we replace in (4) $f(s, [\mathcal{T}u](s), u(s))$ by $F(s, [\mathcal{T}u](s), u(s))$ and $t \in [0, t_1]$ then (4) is equivalent to (5)-(6). Next we define an operator $S : C([0, t_1], \mathbb{R}) \to C([0, t_1], \mathbb{R})$. For any $u \in C([0, t_1], \mathbb{R})$, let

$$(Su)(t) = u_0 + \int_0^t F(s, [\mathcal{T}u](s), u(s))ds,\ t \in [0, t_1].$$

The operator S is continuous. Let

$$B = \left\{ u \in C([0, t_1], \mathbb{R}) : \sup_{t \in [0, t_1]} |u(t)| \leq |u_0| + Mt_1 \right\}.$$

Then B is convex, closed and bounded. For $u \in B, t \in [0, t_1]$, we get

$$|(Su)(t)| = \left| u_0 + \int_0^t F(s, [\mathcal{T}u](s), u(s))ds \right| \leq |u_0| + Mt.$$

Then

$$\sup_{t \in [0,t_1]} |(Su)(t)| \leq |u_0| + Mt_1.$$

So $SB \subseteq B$. By an application of Schauder's fixed point theorem, the initial problem (5)-(6) has a solution u_1 on $[0, t_1]$.

We shall now prove that $\alpha(t) \leq u_1(t) \leq \beta(t), t \in [0, t_1]$. Put $m(t) = \alpha(t) - u_1(t), t \in [0, t_1]$. If the inequality $\alpha(t) \leq u_1(t), t \in [0, t_1]$ is not true, then there exists a $\tilde{t} \in (0, t_1]$ such that

$$m(\tilde{t}) = \max_{t \in [0,t_1]} m(t) = \varepsilon > 0.$$

We consider two following cases:

Case I. Suppose $\tilde{t} \in (0, t_1)$. In consequence

$$m'(\tilde{t}) = 0.$$

Since $\alpha(\tilde{t}) > u_1(\tilde{t})$, then $p(u_1(\tilde{t})) = \alpha(\tilde{t})$ and

$$(8) \qquad \begin{aligned} 0 = m'(\tilde{t}) = \alpha'(\tilde{t}) - u_1'(\tilde{t}) \leq f(\tilde{t}, [\mathcal{T}\alpha](\tilde{t}), \alpha(\tilde{t})) \\ -f(\tilde{t}, [\mathcal{T}p(u_1)](\tilde{t}), p(u_1(\tilde{t}))) - \frac{\varepsilon}{1 + u_1^2(\tilde{t})}. \end{aligned}$$

Moreover, in view of (7) and (H4),

$$f(\tilde{t}, [\mathcal{T}\alpha](\tilde{t}), \alpha(\tilde{t})) - f(\tilde{t}, [\mathcal{T}p(u_1)](\tilde{t}), \alpha(\tilde{t})) \leq 0.$$

From this and (8) we obtain the contradiction

$$0 = m'(\tilde{t}) \leq -\frac{\varepsilon}{1 + u_1^2(\tilde{t})} < 0.$$

Case II. Suppose that $\tilde{t} = t_1$ i.e., $m(t_1) = \alpha(t_1) - u_1(t_1) = \varepsilon$. Since $m(t) < \varepsilon$ for $t \in [0, t_1)$, then there exists a sequence $\tau_\nu \in [0, t_1), \tau_\nu < \tau_{\nu+1}$ such that $\lim_{\nu \to \infty} \tau_\nu = t_1$ and $D_- m(\tau_\nu) = \liminf_{h \to 0^-} \dfrac{m(\tau_\nu + h) - m(\tau_\nu)}{h} \geq 0$.

Let N_0 be such positive integer that for $\nu \geq N_0$ we have $\alpha(\tau_\nu) - u_1(\tau_\nu) > 0$. Then it follows that

$$(9) \qquad \begin{aligned} 0 \leq D_- m(\tau_\nu) \leq \alpha'(\tau_\nu) - u_1'(\tau_\nu) \leq f(\tau_\nu, [\mathcal{T}\alpha](\tau_\nu), \alpha(\tau_\nu)) \\ -f(\tau_\nu, [\mathcal{T}p(u_1)](\tau_\nu), p(u_1(\tau_\nu))) - \frac{\varepsilon}{1 + u_1^2(\tau_\nu)}. \end{aligned}$$

Since $\alpha(\tau_\nu) > u_1(\tau_\nu)$, then $p(u_1(\tau_\nu)) = \alpha(\tau_\nu)$. Moreover, in view of (7) and (H4),

$$f(\tau_\nu, [\mathcal{T}\alpha](\tau_\nu), \alpha(\tau_\nu)) - f(\tau_\nu, [\mathcal{T}p(u_1)](\tau_\nu), \alpha(\tau_\nu)) \leq 0.$$

From this and (9) we obtain the contradiction

$$0 \leq m'(\tau_\nu) \leq -\frac{\varepsilon}{1+u_1^2(\tau_\nu)} < 0.$$

Similarly, we can prove $u_1 t) \leq \beta(t),\ t \in [0, t_1]$.
Since $\alpha(t_1) \leq u_1(t_1) \leq \beta(t_1)$ and I_1 is nondecreasing, we get

$$I_1(\alpha(t_1)) \leq I_1(u_1(t_1)) \leq I_1(\beta(t_1)).$$

From this, by the definition of lower and upper solution, we have

$$\alpha(t_1^+) \leq \alpha(t_1) + I_1(\alpha(t_1) \leq u_1(t_1) + I_1(u_1(t_1)) \leq \beta(t_1) + I_1(\beta(t_1)) \leq \beta(t_1^+).$$

Then

$$\alpha(t_1^+) \leq u_1(t_1) + I_1(u_1(t_1)) \leq \beta(t_1^+).$$

Repeating the same arguments, we can show that the problem

$$\begin{aligned} &u'(t) = F(t, [\mathcal{T}u](t), u(t)),\ t \in (t_1, t_2], \\ &u(t_1^+) = u_1(t_1) + I_1(u_1(t_1)), \end{aligned}$$

where $u(s) = u_1(s), s \in [0, t_1]$, has a solution u_2 such that $\alpha(t) \leq u_2(t) \leq \beta(t)$, $t \in (t_1, t_2]$.

So forth and so on, for $t \in (t_p, t_{p+1}]$, we consider the initial problem

$$\begin{aligned} &u'(t) = F(t, [\mathcal{T}u](t), u(t)),\ t \in (t_p, t_{p+1}], \\ &u(t_p^+) = u_p(t_p) + I_p(u_p(t_p)). \end{aligned}$$

Similarly, we can prove that this problem has a solution u_{p+1} such that $\alpha(t) \leq u_{p+1}(t) \leq \beta(t),\ t \in (t_p, t_{p+1}]$. Continuing the proof, let

$$u(t) = \begin{cases} u_1(t), & t \in [0, t_1]; \\ u_2(t), & t \in (t_1, t_2]; \\ \dots, & \dots \\ u_{p+1}(t), & t \in (t_p, t_{p+1}]. \end{cases}$$

Then u is a solution of problem (1)-(3) and $\alpha(t) \leq u(t) \leq \beta(t),\ t \in J$.

Next we consider the impulsive equation

$$u'(t) = f(t, [\mathcal{T}u](t), u(t)), \quad t \in int\ J_k,\ k = 1, ..., p+1, \tag{10}$$

$$\Delta u(t_k) = I_k(u(t_k)),\ k = 1, ..., p, \tag{11}$$

subject to the periodic boundary condition

$$u(0) = u(T). \tag{12}$$

DEFINITION 2.4 A function $\alpha \in PC^1(J, \mathbb{R})$ is said to be a lower solution of problem (10)-(12) if

$$\alpha'(t) \leq f(t, [\mathcal{T}\alpha](t), \alpha(t)), \quad t \in int\ J_k,\ k = 1, ..., p+1,$$
$$\Delta\alpha(t_k) \leq I_k(\alpha(t_k)),\ k = 1, ..., p,$$
$$\alpha(0) \leq \alpha(T)$$

and an upper solution of (10)-(12) if the above inequalities are reversed.

By applying Definition 2.4 and Theorem 2.3 we have the following result:

THEOREM 2.5 *Let the assumptions (H1),(H2) and (H4) hold. Assume there exist* $\alpha, \beta \in PC^1(J, \mathbb{R})$ *such that* α, β *are lower and upper solutions of (10)-(12) respectively,* $\alpha(t) \leq \beta(t), t \in J$ *and* $\alpha(0) = \beta(0)$. *Then problem (10)-(12) has a solution* $u \in [\alpha, \beta]$.

PROOF Let $u(\cdot; \alpha(0)) \in [\alpha, \beta]$ denotes the solution of the following problem

$$u'(t) = f(t, [\mathcal{T}u](t), u(t)), \quad t \in int\ J_k,\ k = 1, ..., p+1,$$
$$\Delta u(t_k) = I_k(u(t_k)),\ k = 1, ..., p,$$
$$u(0) = \alpha(0).$$

The existence of a solution to the above problem follows from Theorem 2.3.
Hence, by the definition of the lower and upper solution we have

$$\alpha(0) \leq \alpha(T) \leq u(T; \alpha(0)),$$
$$\beta(0) \geq \beta(T) \geq u(T; \alpha(0)).$$

Since $\alpha(0) = \beta(0)$, then

$$\alpha(0) \leq u(T; \alpha(0)) \leq \beta(T) \leq \beta(0) = \alpha(0).$$

Thus

$$u\,(T; \alpha(0)) = \alpha(0).$$

We prove that $u = u\,(\cdot; \alpha(0))$ is a solution of (10)-(12). ■

3. Monotone iterative method. In this section we establish existence criteria for extremal solutions of the problem (1)-(3) by the method of lower and upper solutions and the monotone method. Now we introduce simple result, which plays an important role in monotone iterative technique.

LEMMA 3.1 *Let* $u \in PC^1(J, \mathbb{R}), m \in PC(J, R)$ *such that*

$$u'(t) \leq m(t)u(t), \quad t \in J',$$
$$\triangle u(t_k) \leq 0, \quad k = 1, ..., p,$$
$$u(0) \leq 0.$$

Then $u(t) \leq 0, t \in J$.

Proof We first note that if $u \in PC^1(J,\mathbb{R}), m, \sigma \in PC(J,\mathbb{R}), d_k \in \mathbb{R}$, $k = 1, ..., p$ verify

$$(13)\qquad \begin{aligned} &u'(t) = m(t)u(t) + \sigma(t), t \in J', \\ &\triangle u(t_k) = d_k, \quad k = 1, ..., p \\ &u(0) = u_0, \end{aligned}$$

then u can be expressed as

$$(14)\qquad u(t) = u_0 e^{M(t)} + \int_0^t e^{M(t)-M(s)}\sigma(s)ds + \sum_{\{k:0<t_k<t\}} \left(e^{M(t)-M(t_k)}\right) d_k,$$

where $M(t) = \int_0^t m(r)dr$. Apply (13) and (14) with $\sigma(t) \le 0, t \in J, u_0 \le 0$ and $d_k \le 0,\ k = 1, ..., p$ we obtain our result. ∎

Theorem 3.2 *Let the assumptions (H1)-(H4) hold and there exists $M \ge 0$ such that the function f satisfy the following condition*

$$(15)\qquad f(t,u,v_1) - f(t,u,v_2) \ge -M(v_1 - v_2),$$

for $t \in J, [\mathcal{T}\alpha](t) \le u \le [\mathcal{T}\beta](t), \beta(t) \ge v_1 \ge v_2 \ge \alpha(t)$.
Then there exist monotone sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$, such that $\lim_{n\to\infty} \alpha_n(t) = \rho(t)$, $\lim_{n\to\infty} \beta_n(t) = r(t)$ monotonically and piecewise uniformly on J, where ρ and r are the minimal and maximal solutions of (1)-(3), respectively.

Proof Let us consider the following problem

$$(16)\qquad \begin{aligned} &u'(t) = F(t,u(t)), t \in J', \\ &\Delta u(t_k) = I_k(\xi(t_k)),\ k = 1, ..., p, \\ &u(0) = u_0, \end{aligned}$$

where

$$F(t,u(t)) = f(t, [\mathcal{T}\xi](t), \xi(t)) - M(u(t) - \xi(t))$$

for any $t \in J, \xi \in [\alpha,\beta]$.
This problem has for every $\xi \in [\alpha,\beta]$ a unique solution $u \in PC^1(J,\mathbb{R})$. Then we can define the operator $B : [\alpha,\beta] \to PC^1(J,\mathbb{R})$ by

$$(17)\qquad [B\xi](t) = u(t),\ t \in J,$$

where u is the unique solution of (16).
The mapping B defined by (17) satisfies

- **(i)** $\alpha \le B\alpha$ and $\beta \ge B\beta$ on J.
- **(ii)** For $\xi_1, \xi_2 \in [\alpha,\beta], \xi_1 \le \xi_2$ on J implies $B\xi_1 \le B\xi_2$ on J (i.e., B is a monotone operator on the segment $[\alpha,\beta]$.)

To prove (i) let us set $B\alpha = \alpha_1$ and $B\beta = \beta_1$, where α_1, β_1 are the unique solutions of the problem (16) corresponding to $\xi = \alpha$ and $\xi = \beta$, respectively. We first prove that $\alpha(t) \le \alpha_1(t), t \in J$. Indeed, if we consider $v(t) = \alpha(t) - \alpha_1(t), t \in J$, then

$$\begin{aligned} v'(t) &= \alpha'(t) - \alpha_1'(t) \\ &\le f(t, [\mathcal{T}\alpha](t), \alpha(t)) - f(t, [\mathcal{T}\alpha](t), \alpha(t)) + M(\alpha_1(t) - \alpha(t)) \\ &= -Mv(t), \quad t \in J', \\ \triangle v(t_k) &= \triangle \alpha(t_k) - \triangle \alpha_1(t_k) \le I_k(\alpha(t_k)) - I_k(\alpha(t_k)) = 0, \ k = 1, ..., p, \\ v(0) &= \alpha(0) - \alpha_1(0) \le u_0 - u_0 = 0. \end{aligned}$$

Thus, Lemma 3.1 implies that $v = \alpha - \alpha_1 \le 0$ on J. Analogously one can show that $\beta \ge \beta_1$ on J.

Now, to prove (ii), let us set $B\xi_1 = y_1$ and $B\xi_2 = y_2$, where $\xi_1, \xi_2 \in [\alpha, \beta], \xi_1 \le \xi_2$ on J and y_1, y_2 are the unique solutions of the problem (16) with $\xi = \xi_1$ and $\xi = \xi_2$, respectively. Let $v(t) = y_1(t) - y_2(t), t \in J$. Using (15),(H4) and (H2) we have

$$\begin{aligned} v'(t) &= y_1'(t) - y_2'(t) \\ &= f(t, [\mathcal{T}\xi_1](t), \xi_1(t)) - M(y_1(t) - \xi_1(t)) \\ &\quad - f(t, [\mathcal{T}\xi_2](t), \xi_2(t)) + M(y_2(t) - \xi_2(t)) \\ &= f(t, [\mathcal{T}\xi_1](t), \xi_1(t)) - f(t, [\mathcal{T}\xi_2](t), \xi_1(t)) \\ &\quad + f(t, [\mathcal{T}\xi_2](t), \xi_1(t)) - f(t, [\mathcal{T}\xi_2](t), \xi_2(t)) \\ &\quad - M(y_1(t) - \xi_1(t)) + M(y_2(t) - \xi_2(t)) \\ &\le M(\xi_2(t) - \xi_1(t)) - M(y_1(t) - \xi_1(t)) + M(y_2(t) - \xi_2(t)) \\ &= -M(y_1(t) - y_2(t)) \\ &= -Mv(t), \quad t \in J', \\ \Delta v(t_k) &= \Delta y_1(t_k) - \Delta y_2(t_k) = I_k(\xi_1(t_k)) - I_k(\xi_2(t_k)) \le 0, k = 1, ..., p, \\ v(0) &= y_1(0) - y_2(0) = 0. \end{aligned}$$

Lemma 3.1 implies that $v = y_1 - y_2 \le 0$ on J.

The mapping B defined by (17) generates monotone sequences $\{\alpha_n\}, \{\beta_n\}$, where

$$\begin{aligned} &\alpha_1 = \alpha, \\ &\alpha_{n+1} = B\alpha_n, \quad n > 1 \end{aligned}$$

and

$$\begin{aligned} &\beta_1 = \beta, \\ &\beta_{n+1} = B\beta_n, \quad n > 1. \end{aligned}$$

In fact, we have

$$\alpha(t) \le \alpha_1(t) \le ... \le \alpha_n(t) \le \beta_n(t) \le ... \le \beta_1(t) \le \beta(t), \quad t \in J.$$

Note that the problem (16) is equivalent to the integral equation (4) if we replace in (4), $f(s, [\mathcal{T}u](s), u(s))$ by $F(s, u(s))$. The sequences $\{\alpha_n\}, \{\beta_n\}$, are uniformly bounded and completely continuous, therefore the sequences are uniformly convergent in $(t_k, t_{k+1}], k = 0, ..., p$. There exist $\rho, r \in PC^1(J, \mathbb{R})$ such that $\{\alpha_n\} \nearrow \rho$ and $\{\beta_n\} \searrow r$ and $\rho, r \in [\alpha, \beta]$. Employing the integral representation of the solution of (16) we conclude that ρ and r are solutions of the problem (1)-(3). It is easy to prove by an induction argument following an argument similar to that employed to prove (ii) that

$$\alpha_n(t) \leq y(t) \leq \beta_n(t), t \in J, n \in N,$$

where $y \in [\alpha, \beta]$ is any solution of (1)-(3). Taking the limit as $n \to \infty$ we get

$$\rho(t) \leq y(t) \leq r(t), t \in J,$$

what means that ρ, r are respectively minimal and maximal solutions of the (1)-(3). ∎

4. Global solutions. Let $J = [0, \infty)$, $0 < t_1 < t_2 < ... < t_n < ...$, $\lim_{n\to\infty} t_n = +\infty$. Consider the functional-differential problem

$$x'(t) = f(t, [\mathcal{T}x](t), x(t)), \quad t \geq 0, t \neq t_1, t_2, ..., t_k, ..., \tag{18}$$

$$\Delta x(t_k) = I_k(x(t_k)), \; k = 1, 2, ... \tag{19}$$

$$x(0) = x_0, \tag{20}$$

where $f, \mathcal{T}, I_k (k = 1, 2, ...)$ are the same as in (1)-(3).

We require the following assumptions on function f and operator $\mathcal{T}$.

(A1) Function $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and there exist $a, b \in C(J, \mathbb{R}^+)$ such that

$$|f(t, x, w) - f(t, y, z)| \leq a(t)|x - y| + b(t)|w - z|$$

for any $x, y, w, z \in \mathbb{R}, t \in J$.

(A2) There exists $c \in C(J, \mathbb{R}^+)$ such that

$$|[\mathcal{T}x](t) - [\mathcal{T}y](t)| \leq c(t) \max_{s\in[t_{k-1},t]} |x(s) - y(s)|, \; t \in [t_{k-1}, t_k]$$

for any $x, y \in PC(J, \mathbb{R})$ such that $x = y$ in $[0, t_{k-1}], k = 1, 2, ...$

Now we enunciate result parallel to Theorem 2.2 in [3]. There are some differences but the idea and technique of proof are the same.

THEOREM 4.1 *Suppose that (A1),(A2) hold. Then equation (18)-(20) has a unique solution* $x \in PC(J, \mathbb{R})$.

PROOF Let $x \in C([0, t_1], \mathbb{R})$. Consider the following norm in $C([0, t_1], \mathbb{R})$:

$$\| x \|_* = \max_{t\in[0,t_1]} \left\{ e^{-M_1 t} \max_{s\in[0,t]} |x(s)| \right\},$$

where $M_1 = N_1 + K,\ K > 0,\ N_1 = \max_{t\in[0,t_1]} \{a(t)c(t) + b(t)\}$.
Consider the operator $A_1 : C([0, t_1], \mathbb{R}) \to C([0, t_1], \mathbb{R})$ defined by

$$(A_1x)(t) = x_0 + \int_0^t f(s, [\mathcal{T}x](s), x(s))\, ds.$$

Then for $x, y \in C([0, t_1], \mathbb{R})$ and $t \in [0, t_1]$, we have that

$$\begin{aligned}
|(A_1x)(t) &- (A_1y)(t)| \le \int_0^t |f(s, [\mathcal{T}x](s), x(s)) - f(s, [\mathcal{T}y](s), y(s))|\, ds \\
&\le \int_0^t [a(s)|[\mathcal{T}x](s) - [\mathcal{T}y](s)| + b(s)|x(s) - y(s)|]\, ds \\
&\le \int_0^t \left[a(s)c(s) \max_{\tau\in[0,s]} |x(\tau) - y(\tau)| + b(s) \max_{\tau\in[0,s]} |x(\tau) - y(\tau)|\right] ds \\
&= \int_0^t (a(s)c(s) + b(s))\, e^{M_1s} e^{-M_1s} \max_{\tau\in[0,s]} |x(\tau) - y(\tau)| ds \\
&\le \int_0^t (a(s)c(s) + b(s))\, e^{M_1s} \max_{s\in[0,t]} \left\{ e^{-M_1s} \max_{\tau\in[0,s]} |x(\tau) - y(\tau)| \right\} ds \\
&\le \| x - y \|_* \int_0^t (a(s)c(s) + b(s))\, e^{M_1s} ds \\
&\le N_1 \| x - y \|_* \int_0^t e^{M_1s} ds \\
&= \frac{N_1}{M_1} \left(e^{M_1t} - 1\right) \| x - y \|_* \\
&\le \frac{N_1}{M_1} e^{M_1t} \| x - y \|_*
\end{aligned}$$

Thus

$$\max_{s\in[0,t]} |(A_1x)(s) - (A_1y)(s)| \le \frac{N_1}{M_1} \| x - y \|_* \max_{s\in[0,t]} e^{M_1s} \le \frac{N_1}{M_1} e^{M_1t} \| x - y \|_*,$$

$$e^{-M_1t} \max_{s\in[0,t]} |(A_1x)(s) - (A_1y)(s)| \le \frac{N_1}{M_1} \| x - y \|_*,$$

$$\max_{t\in[0,t_1]} \left\{ e^{-M_1t} \max_{s\in[0,t]} |(A_1x)(s) - (A_1y)(s)| \right\} \le \frac{N_1}{M_1} \| x - y \|_*,$$

i.e.,

$$\| A_1x - A_1y \|_* \le \frac{N_1}{M_1} \| x - y \|_* .$$

Thus A_1 is a contractive operator and by Banach fixed point theorem, A_1 has a unique fixed point $x_1^* \in C([0, t_1], \mathbb{R})$.

For $x \in C([t_1, t_2], \mathbb{R})$ let

$$\| x \|_* = \max_{t\in[t_1,t_2]} \left\{ e^{-M_2(t-t_1)} \max_{s\in[t_1,t]} |x(s)| \right\},$$

where $M_2 = N_2 + K,\ K > 0,\ N_2 = \max\limits_{t\in[t_1,t_2]} \{a(t)c(t) + b(t)\}$.
Consider the operator $A_2 : C([t_1, t_2], \mathbb{R}) \to C([t_1, t_2], \mathbb{R})$ defined by

$$(A_2 x)(t) = x_1^*(t_1) + I_1(x_1^*(t_1)) + \int_{t_1}^{t} f(s, [\mathcal{T}x](s), x(s))\, ds,$$

where $x(\tau) = x_1^*(\tau), \tau \in [0, t_1]$.
Then for $x, y \in C([t_1, t_2], \mathbb{R})$ and $t \in [t_1, t_2]$, we have

$$\begin{aligned}
|(A_2x)(t) &- (A_2y)(t)| \le \int_{t_1}^{t} |f(s, [\mathcal{T}x](s), x(s)) - f(s, [\mathcal{T}y](s), y(s))|\, ds \\
&\le \int_{t_1}^{t} [a(s)|[\mathcal{T}x](s) - [\mathcal{T}y](s)| + b(s)|x(s) - y(s)|]\, ds \\
&\le \int_{t_1}^{t} \left[a(s)c(s) \max_{\tau\in[t_1,s]} |x(\tau) - y(\tau)| + b(s) \max_{\tau\in[t_1,s]} |x(\tau) - y(\tau)|\right] ds \\
&= \int_{t_1}^{t} (a(s)c(s) + b(s))\, e^{M_2(s-t_1)} e^{-M_2(s-t_1)} \max_{\tau\in[t_1,s]} |x(\tau) - y(\tau)| ds \\
&\le \int_{t_1}^{t} (a(s)c(s) + b(s))\, e^{M_2(s-t_1)} \max_{s\in[t_1,t]} \left\{ e^{-M_2(s-t_1)} \max_{\tau\in[t_1,s]} |x(\tau) - y(\tau)| \right\} ds \\
&\le \| x - y \|_* \int_{t_1}^{t} (a(s)c(s) + b(s))\, e^{M_2(s-t_1)} ds \\
&\le N_2 \| x - y \|_* \int_{t_1}^{t} e^{M_2(s-t_1)} ds \\
&= \frac{N_2}{M_2} \left(e^{M_2(t-t_1)} - 1\right) \| x - y \|_* \\
&\le \frac{N_2}{M_2} e^{M_2(t-t_1)} \| x - y \|_*
\end{aligned}$$

Thus

$$\max_{s\in[t_1,t]} |(A_2x)(s) - (A_2y)(s)| \le \frac{N_2}{M_2} \| x - y \|_* \max_{s\in[t_1,t]} e^{M_2(s-t_1)} \le \frac{N_2}{M_2} e^{M_2(t-t_1)} \| x - y \|_*,$$

$$e^{-M_2(t-t_1)} \max_{s\in[t_1,t]} |(A_2x)(s) - (A_2y)(s)| \le \frac{N_2}{M_2} \| x - y \|_*,$$

$$\max_{t\in[t_1,t_2]} \left\{ e^{-M_2(t-t_1)} \max_{s\in[t_1,t]} |(A_2x)(s) - (A_2y)(s)| \right\} \le \frac{N_2}{M_2} \| x - y \|_*,$$

i.e.,

$$\| A_2x - A_2y \|_* \le \frac{N_2}{M_2} \| x - y \|_* .$$

Thus A_2 is a contractive operator and by Banach fixed point theorem, A_2 has a unique fixed point $x_2^* \in C([t_1, t_2], \mathbb{R})$.

So forth and so on, for $x \in C\left([t_n, t_{n+1}], \mathbb{R}\right)$, let

$$\| x \|_{*}= \max_{t\in[t_n,t_{n+1}]} \left\{ e^{-M_{n+1}(t-t_n)} \max_{s\in[t_n,t]} |x(s)| \right\},$$

where $M_{n+1} = N_{n+1} + K,\ K > 0,\ N_{n+1} = \max\limits_{t\in[t_n,t_{n+1}]} \{a(t)c(t) + b(t)\}$
and

$$\left(A_{n+1}x\right)(t) = x_n^*(t_n) + I_n(x_n^*(t_n)) + \int_{t_n}^{t} f\left(s, [\mathcal{T}x](s), x(s)\right) ds,$$

where $x(\tau) = x_1^*(\tau), \tau \in (0, t_1], ..., x(\tau) = x_n^*(\tau), \tau \in (t_{n-1}, t_n]$.
Similarly, we can prove that A_{n+1} has a unique fixed point $x_{n+1}^* \in C\left([t_n, t_{n+1}], \mathbb{R}\right)$.
The function

$$x^*(t) = \begin{cases} x_1^*(t), & t \in [0, t_1] \\ x_2^*(t), & t \in (t_1, t_2] \\ & . \\ & . \\ & . \\ x_n^*(t), & t \in (t_n, t_{n+1}] \\ & . \\ & . \\ & . \end{cases}$$

is the unique solution of the problem (18)-(20). ∎

Example.

Consider the equation

$$\text{(21)} \qquad \begin{cases} x'(t) = t - x(t)\cos t + t^2 \int_0^t e^{\tau - t} x(\tau) d\tau, & t \geq 0,\ t \neq 1, 2, ..., k, ...; \\ \Delta x(k) = \dfrac{1}{2} x(k), \quad k = 1, 2, ...; \\ x(0) = 0. \end{cases}$$

It is easy to verify that the function $f(t, x, w) = t - x\cos t + t^2 w$, where $t \geq 0, x, w \in \mathbb{R}$ satisfies Assumption (A1) and the Voltera operator $[\mathcal{T}x](t) = \int_0^t e^{\tau - t} x(\tau) d\tau$ satisfies assumption (A2). Hence, by Theorem 4.1, (21) has a unique global solutions.

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