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Numerical approximations of parabolic functional differential equations on unbounded domains

Abstract. The paper is concerned with initial problems for nonlinear parabolic functional differential equations. A general class of difference methods is constructed. A theorem on the error estimate of approximate solutions for difference functional equations of the Volterra type with an unknown function of several variables is presented. The convergence of explicit difference schemes is proved by means of consistency and stability arguments. It is assumed that given function satisfy nonlinear estimates of the Perron type with respect to a functional variable. Results obtained in the paper can be applied to differential integral problems and equations with retarded variables. Numerical examples are presented.

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1. Introduction. In recent years, a number of papers concerning numerical methods for partial functional differential equations have been published.

Difference methods for nonlinear parabolic functional differential equations were considered in [6], [10] - [12] [17], [18]. Numerical approximations of classical solutions to first order partial functional differential equations were investigated in [1], [7], [8], [13], [14]. The monograph [9] contains an exposition of recent developments of difference methods for hyperbolic functional differential problems.

It is easy to construct an explicit or implicit Euler's type difference method for a nonlinear problem which satisfies the consistency condition on all sufficiently regular solutions of a functional differential equation. The main task in these considerations is to find a finite difference approximation of a original problem which is stable. The method of difference inequalities or simple theorems on recurrent inequalities are used in the investigation of the stability of nonlinear difference functional problems. The proofs of the convergence of difference methods were also based on a general theorem on error estimates of approximate solutions to functional difference equations of the Volterra type with initial boundary conditions and with unknown functions of several variables.

These considerations as a rule involved a lot of calculations to reach the convergence result so the main property of the corresponding operators was not easy to be seen. The aim of the present paper is to show that the results mentioned above as well as many others theorems are consequences of a very simple result concerned with an abstract nonlinear difference functional equation with unknown function of several variables.

Now we formulate differential functional problems. For any metric spaces X and Y we denote by $C(X,Y)$ the class of all continuous functions from X into Y. We use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. Let $M_{n\times n}$ denote the set of $n\times n$ matrices with real element. Write

$$E=[0,a]\times\mathbb{R}^n,\quad E_0=[-d_0,0]\times\mathbb{R}^n,\quad D=[-d_0,0]\times[-d,d]$$

where $a>0$, $d_0\in\mathbb{R}_+$, $\mathbb{R}_+=[0,+\infty)$ and $d=(d_1,\ldots,d_n)\in\mathbb{R}^n_+$. For a function $z:E_0\cup E\to\mathbb{R}$ and for a point $(t,x)\in E$ we define a function $z_{(t,x)}:D\to\mathbb{R}$ as follows: $z_{(t,x)}(\tau,y)=z(t+\tau,x+y)$, $(\tau,y)\in D$. The maximum norm in the space $C(D,\mathbb{R})$ is denoted by $\|\cdot\|_D$. Write

$$\Sigma=E\times C(D,\mathbb{R})\times\mathbb{R}^n\times M_{n\times n}$$

and suppose that $F:\Sigma\to\mathbb{R}$ and $\varphi:E_0\to\mathbb{R}$ are given functions. We consider the problem consisting of the functional differential equation

$$\partial_t z(t,x)=F(\,t,x,z_{(t,x)},\partial_x z(t,x),\partial_{xx}z(t,x)\,) \tag{1}$$

and the initial condition

$$z(t,x)=\varphi(t,x)\ \text{ on }\ E_0 \tag{2}$$

where $x=(x_1,\ldots,x_n)$ and

$$\partial_x z=(\partial_{x_1}z,\ldots,\partial_{x_n}z),\quad \partial_{xx}z=\big[\,\partial_{x_ix_j}z\,\big]_{i,j=1,\ldots,n}.$$

Differential equations with deviated variables and differential integral equations can be obtained as particular cases of (1) by suitable definitions of the operator F. Existence and uniqueness results for classical or generalized solutions for (1), (2) are given in [2], [3], [4], [5].

We are interested in establishing a method of approximation of classical solutions to problem (1), (2) by means of solutions of associated difference functional problems and in estimating of the difference between the exact and approximate solutions. The results presented in [6], [10] - [12] [17], [18] are not applicable to (1), (2).

The paper is organized as follows. In Section 2 we propose a general method for the investigation of the stability of difference schemes generated by initial problems for nonlinear functional differential equations. We prove a theorem on error estimates of approximate solutions to functional difference equations of the Volterra type with unknown function of several variables. The error of an approximate solution is estimated by a solution of an initial problem for a nonlinear differential equation. In Section 3 we apply the above general idea to the investigation of the convergence of difference schemes for (1), (2). Numerical examples are given in the last part of the paper.

We use in the paper general ideas for finite difference equations which were introduced in [9], [15], [16].

2. Stability of difference functional problems. For any two sets U and W we denote by $\mathbb{F}(U,W)$ the class of all functions defined on U and taking values in W. If $f : U \to W$ and $V \subset U$ then $f\,|_V$ is the restriction of f to the set V. Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of natural numbers and integers respectively. We define a mesh on $E_0 \cup E$ in the following way. Suppose that $(h_0, h') = h$, $h' = (h_1, \ldots, h_n)$, stand for steps of the mesh. For $(r, m) \in \mathbb{Z}^{1+n}$ where $m = (m_1, \ldots, m_n)$, we define nodal points as follows:

$$t^{(r)} = rh_0, \;\; x^{(m)} = (x_1^{(m_1)} \ldots, x_n^{(m_n)}) = (m_1h_1, \ldots, m_nh_n).$$

Let us denote by H the set of all h such that there are $K_0 \in \mathbb{Z}$ and $K = (K_1, \ldots, K_n) \in \mathbb{Z}^n$ satisfying the conditions: $K_0h_0 = d_0$ and $(K_1h_1, \ldots, K_nh_n) = d$. Write

$$R_h^{1+n} = \{\, (t^{(r)}, x^{(m)}) : \; (r,m) \in \mathbb{Z}^{1+n} \,\}$$

and

$$E_h = E \cap \mathbb{R}_h^{1+n}, \;\; E_{0.h} = E_0 \cap \mathbb{R}_h^{1+n}, \;\; D_h = D \cap R_h^{1+n}.$$

Let $N_0 \in \mathbb{N}$ be defined by the relations: $N_0h_0 \le a < (N_0+1)h_0$ and

$$E'_h = \{\, (t^{(r)}, x^{(m)}) \in E_h : \; 0 \le r \le N_0 - 1 \,\}.$$

Set $L = (L_1, \ldots, L_n)$ where $L_i = \max\{\, 1, K_i \,\}$ for $i = 1, \ldots, n$ and

$$\Omega_h = \{\, (t^{(r)}, x^{(m)}) : \;\; -K_0 \le r \le 0, \; -L \le m \le L \,\}.$$

For functions $w : \Omega_h \to \mathbb{R}$ and $z : E_{0.h} \cup E_h \to \mathbb{R}$ we write $w^{(r,m)} = w(t^{(r)}, x^{(m)})$ on Ω_h and $z^{(r,m)} = z(t^{(r)}, x^{(m)})$ on $E_{0.h} \cup E_h$. If $z : E_{0.h} \cup E_h \to \mathbb{R}$ and $(t^{(r)}, x^{(m)}) \in E_h$ then the function $z_{\langle r,m \rangle} : \Omega_h \to \mathbb{R}$ is given by

$$z_{\langle r,m \rangle}(\tau, y) = z(t^{(r)} + \tau, x^{(m)} + y), \;\; (\tau, y) \in \Omega_h.$$

Then $z_{\langle r,m \rangle}$ is the restriction of z to the set

$$\big(\, [t^{(r)} - d_0, t^{(r)}] \times [x^{(m)} - b, x^{(m)} + b] \,\big) \cap \mathbb{R}_h^{1+n}$$

where $b = (L_1h_1, \ldots, L_nh_n)$ and this restriction is shifted to the set Ω_h.

Suppose that the operator $F_h : E'_h \times \mathbb{F}(\Omega_h, \mathbb{R}) \to \mathbb{R}$ is given. For $(t^{(r)}, x^{(m)}, w) \in E'_h \times \mathbb{F}(\Omega_h, \mathbb{R})$ we write

$$F_h[\, w \,]^{(r,m)} = F_h(t^{(r)}, x^{(m)}, w).$$

Given $\varphi_h \in \mathbb{F}(E_{0.h}, \mathbb{R})$, we consider the functional difference equation

$$z^{(r+1,m)} = F_h[\, z_{\langle r,m\rangle}\,]^{(r,m)} \tag{3}$$

with the initial boundary condition

$$z^{(r,m)} = \varphi_h^{(r,m)} \ \text{ on } \ E_{0.h}. \tag{4}$$

It is clear that there exists exactly one solution $z_h : E_{0.h} \cup E_h \to \mathbb{R}$ of (3), (4).

Suppose that the functions $v_h : E_{0.h} \cup E_h \to \mathbb{R}$ and $\tilde{\alpha}$, $\tilde{\gamma} : H \to \mathbb{R}_+$ satisfy the conditions

$$\big| v_h^{(r+1,m)} - F_h[\, (v_h)_{\langle r,m\rangle}\,]^{(r,m)} \big| \le \tilde{\gamma}(h) \ \text{ on } \ E'_h$$

$$\big| \varphi_h^{(r,m)} - v_h^{(r,m)} \big| \le \tilde{\alpha}(h) \ \text{ on } \ E_{0.h} \ \text{ and } \ \lim_{h\to 0} \tilde{\alpha}(h) = 0, \ \ \lim_{h\to 0} \tilde{\gamma}(h) = 0.$$

The function v_h satisfying the above relations is considered as an approximate solution of (3), (4). We give a theorem on the estimate of the difference between the exact and approximate solutions of (3), (4).

Write

$$Y_h = \{ (t^{(r)}, x^{(m)}) \in \Omega_h : \ \ r = 0, \ \ -1 \le m_i \le 1 \ \text{ for } \ i = 1, \ldots, n \}.$$

For a function $w : \Omega_h \to \mathbb{R}$ we put

$$\| w\, |_{D_h} \|_h = \max \{ \, |w^{(r,m)}| : \ (t^{(r)}, x^{(m)}) \in D_h \, \},$$

$$\| w\, |_{Y_h} \|_h = \max \{ \, \|w^{(r,m)}\| : \ (t^{(r)}, x^{(m)}) \in Y_h \, \}.$$

For $z : E_{0.h} \cup E_h \to \mathbb{R}$ we define the seminorms

$$\|z\|_{h.r} = \sup \big\{ \, |z^{(i,m)}| : \ \ -K_0 \le i \le r, \ \ m \in \mathbb{Z}^n \, \big\}, \ \ 0 \le r \le N_0.$$

Set

$$A_h = \big\{ \, t^{(0)}, t^{(1)}, \ldots t^{(N_0)} \, \big\}.$$

For $\beta : A_h \to \mathbb{R}$ we write $\beta^{(i)} = \beta(t^{(i)})$ on A_h.

The following assumption will be needed throughout the paper.

Assumption $H[\sigma]$**.** The function $\sigma : [0,a] \times \mathbb{R}_+ \to \mathbb{R}_+$ of the variables (t,p) satisfies the conditions:

1) σ is continuous and it is nondecreasing with respect to the both variables,

2) $\sigma(t,0) = 0$ for $t \in [0,a]$ and the function $\tilde{\omega}(t) = 0$ for $t \in [0,a]$ is the maximal solution of the Cauchy problem

$$\omega'(t) = \sigma(\, t, \omega(t)), \quad \omega(0) = 0.$$

We give a theorem on the estimate of the difference between the exact and approximate solutions to problem (3), (4).

Theorem 2.1 *Suppose that* $F_h : E'_h \times \mathbb{F}(\Omega_h, \mathbb{R}) \to \mathbb{R}$, $\varphi_h : E_{0.h} \to \mathbb{R}$ *and*

1) $u_h : E_{0.h} \cup E_h \to \mathbb{R}$ *is the solution of (3), (4),*

2) $v_h : E_{0.h} \cup E_h \to \mathbb{R}$ *and there are* $\alpha_0, \gamma : H \to \mathbb{R}_+$ *such that*

$$(5) \qquad \left| v_h^{(r+1,m)} - F_h[\,(v_h)_{\langle r,m\rangle}\,]^{(r,m)} \right| \le h_0\gamma(h) \text{ on } E'_h \text{ and } \lim_{h\to 0}\gamma(h) = 0,$$

$$(6) \qquad \left| \varphi_h^{(r,m)} - v_h^{(r,m)} \right| \le \alpha_0(h) \text{ on } E_{0.h} \text{ and } \lim_{h\to 0}\alpha_0(h) = 0.$$

3) *there exists* $\sigma : [0,a] \times \mathbb{R}_+ \to \mathbb{R}_+$ *such that Assumption* $H[\sigma]$ *is satisfied and*

$$(7) \qquad \left| F_h[\,w\,]^{(r,m)} - F_h[\,\bar{w}\,]^{(r,m)} \right| \le \|(w-\bar{w})\,|_{Y_h}\,\|_h + h_0\sigma(\,t^{(r)}, \|(w-\bar{w})\,|_{D_h}\,\|_h\,).$$

on $E'_h \times \mathbb{F}(\Omega_h, \mathbb{R})$.

Then

$$(8) \qquad \|u_h - v_h\|_{h.r} \le \omega(t^{(r)}, h) \text{ for } 0 \le r \le N_0,$$

where $\omega(\,\cdot\,, h) : [0,a] \to \mathbb{R}_+$ *is the maximal solution of the Cauchy problem*

$$\omega'(t) = \sigma(t, \omega(t)) + \gamma(h), \;\; \omega(0) = \alpha_0(h).$$

Proof Let us denote by $\beta_h : A_h \to \mathbb{R}_+$ the solution of the difference problem

$$\beta^{(r+1)} = \beta^{(r)} + h_0\sigma(\,t^{(r)}, \beta^{(r)}\,) + h_0\gamma(h),\; 0 \le r \le N_0 - 1,$$
$$\beta^{(0)} = \alpha_0(h).$$

It follows easily that

$$(9) \qquad \|u_h - v_h\|_{h.r} \le \beta_h^{(r)} \text{ for } 0 \le r \le N_0.$$

The function $\omega(\,\cdot\,, h)$ is convex on $[0,a]$. Then we have

$$\omega(t^{(r+1)}, h) \ge \omega(t^{(r)}, h) + h_0\sigma(\,t^{(r)}, \omega(t^{(r)}, h)\,) + h_0\gamma(h), \;\; 0 \le r \le N_0 - 1.$$

This gives

$$\beta_h^{(r)} \le \omega(t^{(r)}, h) \text{ for } 0 \le r \le N_0.$$

The above inequality and (9) imply (8). This proves the theorem. ∎

Remark 2.2 If all the assumption of Theorem 2.1 are satisfied then there is $\alpha : H \to \mathbb{R}_+$ such that

$$(10) \qquad \|u_h - v_h\|_{h.r} \le \alpha(h) \text{ for } 0 \le r \le N_0 \text{ and } \lim_{h\to 0}\alpha(h) = 0.$$

It follows from (8) that the above condition is satisfied with $\alpha(h) = \omega(a,h)$.

REMARK 2.3 Suppose that $\sigma(t,p) = L_0 p$ on $[0,a] \times \mathbb{R}_+$ where $L_0 \in \mathbb{R}_+$. Then assumption (7) has the form

$$\left| F_h[\,w\,]^{(r,m)} - F_h[\,\bar{w}\,]^{(r,m)} \right| \le \|(w-\bar{w})\,|_{Y_h}\,\|_h + h_0 L_0 \|(w-\bar{w})\,|_{D_h}\,\|_h$$

on $E'_h \times \mathbb{F}(\Omega_h,\mathbb{R})$. Then assertion (10) has the form

$$\|u_h - v_h\|_{h.r} \le \tilde{\alpha}(h) \text{ for } 0 \le r \le N_0$$

where

$$\tilde{\alpha}(h) = \alpha_0(h)\exp\big[L_0 a\big] + \gamma(h)\frac{\exp\big[L_0 a\big]-1}{L_0} \;\; if \;\; L_0 > 0, \tag{11}$$

and

$$\tilde{\alpha}(h) = \alpha_0(h) + a\gamma(h) \;\; if \;\; L_0 = 0. \tag{12}$$

The above estimates are important in applications.

3. Difference schemes for parabolic functional differential equations. We formulate a difference method for initial value problem (1), (2). Write

$$\Gamma = \{\,(i,j) \in \mathbb{N}^2 : \;\; 1 \le i,j \le n, \;\; i \ne j\,\}$$

and suppose that we have defined the sets $\Gamma_+, \Gamma_- \subset \Gamma$ such that $\Gamma_+ \cup \Gamma_- = \Gamma$, $\Gamma_+ \cap \Gamma_- = \emptyset$. In particular, it may happens that $\Gamma_+ = \emptyset$ or $\Gamma_- = \emptyset$. Moreover we assume that $(i,j) \in \Gamma_+$ when $(j,i) \in \Gamma_+$.

For $1 \le i \le n$ we define $e_i = (0,\ldots,0,1,0,\ldots,0) \in \mathbb{R}^n$ with 1 standing on the $i-$th place. We consider the difference operators δ_i^+ and δ_i^-, $1 \le i \le n$, defined by

$$\delta_i^+ z^{(r,m)} = \frac{1}{h_i}\big[\, z^{(r,m+e_i)} - z^{(r,m)}\,\big],$$

$$\delta_i^- z^{(r,m)} = \frac{1}{h_i}\big[\, z^{(r,m)} - z^{(r,m-e_i)}\,\big].$$

Let δ_0 and $(\delta_1,\ldots,\delta_n) = \delta$ be difference operators given by

$$\delta_0 z^{(r,m)} = \frac{1}{h_0}\big[z^{(r+1,m)} - z^{(r,m)}\big],$$

$$\delta_i z^{(r,m)} = \frac{1}{2h_i}\big[z^{(r,m+e_i)} - z^{(r,m-e_i)}\big], \;\; 1 \le i \le n.$$

We apply the difference operators

$$\delta^{(2)} = \big[\,\delta_{i\,j}\,\big]_{i,j=1,\ldots,n}$$

defined in the following way

$$\delta_{ii} z^{(r,m)} = \delta_i^+\,\delta_i^-\,z^{(r,m)}, \;\; i = 1,\ldots,n,$$

and

$$\delta_{ij} z^{(r,m)} = \frac{1}{2}\big[\, \delta_i^+ \, \delta_j^- \, z^{(r,m)} + \delta_i^- \, \delta_j^+ \, z^{(r,m)} \,\big] \ \text{ for } \ (i,j) \in \Gamma_-,$$

$$\delta_{ij} z^{(r,m)} = \frac{1}{2}\big[\, \delta_i^+ \, \delta_j^+ \, z^{(r,m)} + \delta_i^- \, \delta_j^- \, z^{(r,m)} \,\big] \ \text{ for } \ (i,j) \in \Gamma_+.$$

In the same way we define the difference expressions

$$\delta w^{(0,\theta)} = (\delta_1 w^{(0,\theta)}, \ldots, \delta_n w^{(0,\theta)}), \ \ \delta^{(2)} w^{(0,\theta)} = \big[\, \delta_{ij} w^{(0,\theta)} \,\big]_{i,j=1,\ldots,n},$$

where $w : \Omega_h \to \mathbb{R}$ and $\theta = (0, \ldots, 0) \in \mathbb{R}^n$.

We need the following discrete version of the operator $(t,x) \to z_{(t,x)}$. Suppose that $z : E_{0.h} \cup E_h \to \mathbb{R}$ and $(t^{(r)}, x^{(m)}) \in E_h$. Then $z_{[r,m]} : D_h \to \mathbb{R}$ is a function defined by

$$z_{[r,m]}(\tau, y) = z(t^{(r)} + \tau, x^{(m)} + y), \ \ (\tau, y) \in D_h.$$

The maximum norm in the space $\mathbb{F}(D_h, \mathbb{R})$ is denoted by $\| \cdot \|_{D_h}$.

Equation (3) contains the functional variable $z_{(t,x)}$ which is an element of the space $C(D, \mathbb{R})$. Since solutions of difference equations are functions defined on the mesh then we need an interpolating operator $T_h : \mathbb{F}(D_h, \mathbb{R}) \to C(D, \mathbb{R})$.

Given $\varphi : E_{0.h} \to \mathbb{R}$ and the operator $T_h : \mathbb{F}(D_h, \mathbb{R}) \to C(D, \mathbb{R})$. We approximate classical solutions of (1), (2) with solutions of the difference functional equation

$$\delta_0 z^{(r,m)} = F\big(\, t^{(r)}, x^{(m)}, T_h z_{[r,m]}, \delta z^{(r,m)}, \delta^{(2)} z^{(r,m)} \,\big) \tag{13}$$

with the initial condition

$$z^{(r,m)} = \varphi_h^{(r,m)} \ \text{ on } \ E_{0.h}. \tag{14}$$

It is clear that there exists exactly one solution $u_h : E_{0.h} \cup E_h \to \mathbb{R}$ of (13), (14).

We claim that the difference method (13), (14) is a particular case of (3), (4). Let $F_h : E'_h \times \mathbb{F}(\Omega_h, \mathbb{R}) \to \mathbb{R}$ be defined by

$$F_h[\, w \,]^{(r,m)} = w^{(0,\theta)} + h_0 F\big(\, t^{(r)}, x^{(m)}, T_h w, \delta w^{(0,\theta)}, \delta^{(2)} w^{(0,\theta)} \,\big) \tag{15}$$

with the above defined $\delta w^{(0,\theta)}$ and $\delta^{(2)} w^{(0,\theta)}$. It is clear that (13), (14) is equivalent to (3), (4) with F_h defined by (15).

4. Convergence of difference methods. Our basic assumptions on F and T_h are the following.

Assumption $H[F]$. The function $F : \Sigma \to \mathbb{R}$ of the variables (t, x, w, q, s) where

$$q = (q_1, \ldots, q_n), \ \ s = [s_{ij}]_{i,j=1,\ldots,n}$$

is continuous and

1) the partial derivatives

$$\partial_q F = \big(\partial_{q_1} F, \ldots, \partial q_n F\big), \ \ \partial_s F = \big[\, \partial_{s_{ij}} F \,\big]_{i,j=1,\ldots,n}$$

exist on Σ and $\partial_q f \in C(\Sigma, \mathbb{R}^n)$, $\partial_s F \in C(\Sigma, M_{n \times n})$,

2) for each $P = (t, x, w, q, s) \in \Sigma$ the matrix $\partial_s F(P)$ is symmetric and

$$\partial_{s_{ij}} F(P) \geq 0 \text{ for } (i,j) \in \Gamma_+, \tag{16}$$

$$\partial_{s_{ij}} F(P) \leq 0 \text{ for } (i,j) \in \Gamma_-, \tag{17}$$

$$1 - 2h_0 \sum_{i=1}^{n} \frac{1}{h_i^2} \partial_{s_{ii}} F(P) + h_0 \sum_{(i,j)\in\Gamma} \frac{1}{h_i h_j} \big|\partial_{s_{ij}} F(P)\big| \geq 0, \tag{18}$$

$$\frac{1}{h_i} \partial_{s_{ii}} F(P) - \sum_{j=1 j\neq i}^{n} \frac{1}{h_j} \big|\partial_{s_{ij}} F(P)\big| - \frac{1}{2} \big|\partial_{q_i} F(P)\big| \geq 0,\ 1 \leq i \leq n, \tag{19}$$

3) there is $\sigma : [0, a] \times \mathbb{R}_+ \to \mathbb{R}_+$ such that Assumption $H[\sigma]$ is satisfied and

$$\big|\, F(t, x, w, q, s) - F(t, x, \bar{w}, q, s) \,\big| \leq \sigma(\, t, \|w - \bar{w}\|_D \,) \text{ on } \Sigma. \tag{20}$$

Assumption $H[T_h]$**.** The interpolating operator $T_h : \mathbb{F}(D_h, \mathbb{R}) \to C(D, \mathbb{R})$ satisfies the conditions:

1) for $w, \bar{w} \in \mathbb{F}(D_h, \mathbb{R})$ we have

$$\|T_h w - T_h \bar{w}\|_D \leq \|w - \bar{w}\|_{D_h},$$

2) if $\tilde{w}$ is of class C^2 then there is $\beta : h \to \mathbb{R}_+$ such that

$$\|T_h \tilde{w}_h - \tilde{w}\|_D \leq \beta(h) \text{ and } \lim_{h\to 0} \beta(h) = 0,$$

where $\tilde{w}_h$ is the restriction of $\tilde{w}$ to the set D_h.

Remark 4.1 An example of T_h satisfying Assumption $H[T_h]$ can be found in [9], Chapter V.

Now we prove a theorem on the convergence of the method (13), (14).

Theorem 4.2 *Suppose that Assumption $H[F]$ is satisfied and*

1) there is $\tilde{c} > 0$ such that $h_i \leq \tilde{c} h_j$ for $i, j = 1, \ldots, n$,

2) $u_h : E_{0.h} \cup E_h \to \mathbb{R}$ is a solution of (13), (14) and there is $\alpha_0 : H \to \mathbb{R}_+$ such that

$$\big|\varphi^{(r,m)} - \varphi_h^{(r,m)}\big| \leq \alpha_0(h) \text{ on } E_{0.h} \text{ and } \lim_{h\to 0} \alpha_0(h) = 0, \tag{21}$$

3) $v : E_0 \cup E \to \mathbb{R}$ is a classical solution of (1), (2) and v is of class C^2 on $E_0 \cup E$ and the function $\partial_{xx} v$ is bounded on $E_0 \cup E$.

Then there is $\alpha : H \to \mathbb{R}$ such that

$$\left|(v_h - u_h)^{(r,m)}\right| \le \alpha(h) \text{ on } E_h \text{ and } \lim_{h\to 0} \alpha(h) = 0, \tag{22}$$

where v_h is the restriction of v to the set $E_{0.h} \cup E_h$.

PROOF We apply Theorem 2.1 to prove (22). Suppose that F_h is given by (15). It follows that $u_h : E_{0.h} \cup E_h \to \mathbb{R}$ satisfies (3), (4) and there is $\gamma : H \to \mathbb{R}_+$ such that condition (5) is satisfied. From (21) we conclude that condition (6) holds.

Now we consider the difference $F_h[w] - F_h[\bar{w}]$ where $w, \bar{w} \in \mathbb{F}(\Omega_h, \mathbb{R})$. Write

$$\Lambda_h^{(r,m)} = h_0\Big[F\big(t^{(r)}, x^{(m)}, T_h[w\,|_{D_h}], \delta w^{(0,\theta)}, \delta^{(2)} w^{(0,\theta)}\big) \\ -F\big(t^{(r)}, x^{(m)}, T_h[\bar{w}\,|_{D_h}], \delta w^{(0,\theta)}, \delta^{(2)} w^{(0,\theta)}\big)\Big].$$

It follows from Assumption $H[F]$ that there are

$$\tilde{S}_h : E'_h \to \mathbb{R}^n_+, \quad \tilde{S}_h = (\tilde{S}_{h.1}, \ldots, \tilde{S}_{h.n}),$$
$$\bar{S}_h : E'_h \to \mathbb{R}^n_+, \quad \bar{S}_h = (\bar{S}_{h.1}, \ldots, \bar{S}_{h.n}),$$

and

$$Q_h : E'_h \to M_{n\times n}, \quad Q_h = \big[\, Q_{h.ij} \,\big]_{i,j=1,\ldots,n}$$

such that

$$Q_{h.ij}^{(r,m)} \ge 0 \text{ for } (i,j) \in \Gamma,$$
$$\sum_{i=1}^{n} Q_{h.ii}^{(r,m)} \le 1 \text{ on } E'_h$$

and

$$F_h[w]^{(r,m)} - F_h[\bar{w}]^{(r,m)} = \Lambda_h^{(r,m)} + \big[1 - \sum_{i=1}^{n} Q_{h.ii}^{(r,m)}\big](w-\bar{w})^{(0,\theta)} \\ + \sum_{i=1}^{n} \tilde{S}_{h.i}^{(r,m)} (w-\bar{w})^{(0,e_i)} + \sum_{i=1}^{n} \bar{S}_{h.i}^{(r,m)} (w-\bar{w})^{(0,-e_i)} \\ + \sum_{(i,j)\in\Gamma_-} Q_{h.ij}^{(r,m)} \big[(w-\bar{w})^{(0,e_i-e_j)} + (w-\bar{w})^{(0,-e_i+e_j)}\big] \\ + \sum_{(i,j)\in\Gamma_+} Q_{h.ij}^{(r,m)} \big[(w-\bar{w})^{(0,e_i+e_j)} + (w-\bar{w})^{(0,-e_i-e_j)}\big].$$

Moreover we have

$$\sum_{i=1}^{n} Q_{h.ii}^{(r,m)} + \sum_{i=1}^{n} \tilde{S}_{h.i}^{(r,m)} + \sum_{i=1}^{n} \bar{S}_{h.i}^{(r,m)} + 2 \sum_{(i,j)\in\Gamma} Q_{h.ij}^{(r,m)} = 0$$

where $(t^{(r)}, x^{(m)}) \in E'_h$.

The above relations and (20) imply (7). Thus we see that all the assumptions of Theorem 2.1 are satisfied and the assertion (22) follows. ■

REMARK 4.3 Suppose that all the assumption of Theorem 4.2 are satisfied and

$$\sigma(t,p) = L_0 p \ \ on \ \ [0,a]\times\mathbb{R}_+$$

where $L_0 \in \mathbb{R}_+$. Then we have

$$\left|(u_h - v_h)^{(r,m)}\right| \le \tilde{\alpha}(h) \ \ on \ \ E_h$$

where $\tilde{\alpha} : H \to \mathbb{R}_+$ is given by (11), (12).

REMARK 4.4 Results presented in the paper can be extended on weakly coupled differential functional systems.

5. Numerical examples. Put $h = (h_0, h_1, h_2)$ and $t^{(r)} = rh_0$, $(x^{(m_1)}, y^{(m_2)}) = (m_1h_1, m_2h_2)$ where $r \in \mathbb{N}$, $(N_1, N_2) \in \mathbb{Z}^2$. Write

$$\Omega_h = \{(t^{(r)}, x^{(m_1)}, y^{(m_2)}) : \ 0 \le r \le N_0,$$
$$(-N_1 + r, -N_2 + r) \le (m_1, m_2) \le (N_1 - r, N_2 - r)\}$$

where $N_0h_0 = a$, $(N_1h_1, N_2h_2) = (b_1, b_2)$ and $N_1 > N_0$, $N_2 > N_0$.

EXAMPLE 5.1 Put $n = 2$. Consider the differential equation with deviated variables

$$\begin{aligned}\partial_t z(t,x,y) &= \partial_{xx}z(t,x,y) + \partial_{yy}z(t,x,y) + \partial_{xy}z(t,x,y)\\ &+\frac{1}{2}\sin\left[\partial_{xx}z(t,x,y) + \partial_{yy}z(t,x,y) + 2\partial_{xy}z(t,x,y)\right]\\ &+z(t,0.5x,0.5y) - z(0.5t,x,y) + (x - y - t^2)z(t,x,y)\end{aligned} \tag{23}$$

and the initial condition

$$z(0,x,y) = 1 \ \ (x,y) \in (-b_1,b_1)\times(-b_2,b_2), \tag{24}$$

where $b_1 > 0$, $b_2 > 0$. The function $v(t,x,y) = \exp[t(x-y)]$ is the solution of (23), (24). Let $z_h : \Omega_h \to \mathbb{R}$ denote the function which is obtained by using the difference scheme for (23), (24). Write

$$M(r) = [2(N_1 - r) + 1][2(N_2 - r) + 1]$$

and

$$\varepsilon_h^{(r)} = \frac{1}{M(r)} \sum_{\nu=-N_1+r}^{N_1-r} \sum_{\mu=-N_2+r}^{N_2-r} \left|(z_h - v_h)^{(r,\nu,\mu)}\right|, \ \ 0 \le r \le N_0, \tag{25}$$

where v_h is the restriction of v to the set Ω_h. The numbers $\varepsilon_h^{(r)}$ are the arithmetical means of the errors with fixed $t^{(r)}$.

In the table we give experimental values of the function ε_h for the following parameters: $a = 0.3$, $b_1 = b_2 = 15$, $h_0 = 10^{-3}$, $h_1 = h_2 = 5\cdot 10^{-2}$.

Table I

$t^{(r)}$	$\varepsilon_h^{(r)}$
0.12	0.007703351393
0.14	0.007604331355
0.16	0.006870163962
0.18	0.005711592060
0.20	0.004392285379
0.22	0.003172335124
0.24	0.002226460011
0.26	0.001608708800
0.28	0.001262877956
0.30	0.001262935748

The results shown in the table are consistent with our mathematical analysis.

Example 5.2 Consider the differential integral equation

$$\partial_t z(t,x,y) = \arctan\Big[\partial_{xx} z(t,x,y) + \partial_{yy} z(t,x,y) - \partial_{xy} z(t,x,y) + 2z(t,x,y) + \frac{1}{4}\int_{-x}^{x}\int_{-y}^{y} z(t,\xi,\eta)\,d\eta d\xi\Big] + z(t,x,y) + \cos x\,\cos y. \tag{26}$$

and the initial condition

$$z(0,x,y) = 0,\quad (x,y) \in [-b_1,b_1]\times[-b_2,b_2]. \tag{27}$$

The solution of the above problem is known, it is

$$v(t,x,y) = \big(e^t - 1\big)\cos x\,\cos y.$$

Let $z_h : \Omega_h \to \mathbb{R}$ denote the function which is obtained by using the difference scheme for (26), (27). We consider the errors ε_h defined by (25) for the above problem.

In the Table II we give experimental values of the function ε_h for the following parameters: $a = 0.3$, $b_1 = b_2 = 15$, $h_0 = 10^{-3}$, $h_1 = h_2 = 5 \cdot 10^{-2}$.

Table II

$t^{(r)}$	$\varepsilon_h^{(r)}$
0.12	0.009864483038
0.14	0.013089607365
0.16	0.015369589723
0.18	0.022974892126
0.20	0.028555330021
0.22	0.028097648984
0.24	0.041702538920
0.26	0.056714130083
0.28	0.031326660304
0.30	0.016250286891

The errors presented in Table II are larger than the errors from Table I. This is due to the fact that we have differential equation with deviated argument in Example 5.1 and we approximate the unknown function at the points

$$(t^{(r)}, 0.5x^{(m_1)}, 0.5y^{(m_2)}) \text{ and } (0.5t^{(r)}, x^{(m_1)}, y^{(m_2)}).$$

In Example 5.2 we have differential integral equation and the function z is approximated on the set

$$\{\, t^{(r)}\} \times \left[-x^{(m_1)}, x^{(m_1)}\right] \times \left[-y^{(m_2)}, y^{(m_2)}\right].$$

Difference methods considered in the paper have the following property: a large number of previous values $z^{(i,m)}$ must be preserved because they are needed to compute an approximate solution with $t = t^{(r+1)}$.

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