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On maximal ideals of pseudo MV-algebras

Abstract. We investigate maximal ideals of pseudo MV-algebras and give some characterizations of them. Some properties of a family of maximal ideals of a pseudo MV-algebra generating this algebra are shown as well. Finally, we are interested in finding an example of a pseudo MV-algebra generated by its maximal ideal.

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1. Preliminaries.

Pseudo MV-algebras, introduced by G. Georgescu and A. Iorgulescu in [5] and independently by J. Rachůnek in [7] (he uses the name generalized MV-algebra or, in short, GMV-algebra), are a non-commutative generalization of MV-algebras.

Let $A = (A, \oplus, ^-, ^\sim, 0, 1)$ be an algebra of type $(2, 1, 1, 0, 0)$. Set $x \cdot y = (y^- \oplus x^-)^\sim$ for any $x, y \in A$. We consider that the operation $\cdot$ has priority to the operation $\oplus$, i.e., we will write $x \oplus y \cdot z$ instead of $x \oplus (y \cdot z)$. The algebra A is called a *pseudo MV-algebra* if for any $x, y, z \in A$ the following conditions are satisfied:

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(A2) $x \oplus 0 = 0 \oplus x = x$;
(A3) $x \oplus 1 = 1 \oplus x = 1$;
(A4) $1^\sim = 0; 1^- = 0$;
(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$;
(A6) $x \oplus x^\sim \cdot y = y \oplus y^\sim \cdot x = x \cdot y^- \oplus y = y \cdot x^- \oplus x$;
(A7) $x \cdot (x^- \oplus y) = (x \oplus y^\sim) \cdot y$;
(A8) $(x^-)^\sim = x$.

If the addition $\oplus$ is commutative, then both unary operations $^-$ and $^\sim$ coincide and A can be considered as an MV-algebra.

Throughout this paper A will denote a pseudo MV-algebra. We will write $x^=$ instead of $(x^-)^-$ and $x^\approx$ instead of $(x^\sim)^\sim$. For any $x \in A$ and $n = 0, 1, 2, \ldots$ we put

$$\begin{aligned} 0x &= 0 \text{ and } (n+1)\,x = nx \oplus x; \\ x^0 &= 1 \text{ and } x^{n+1} = x^n \cdot x. \end{aligned}$$

Proposition 1.1 (Georgescu and Iorgulescu [5]) *The following properties hold for any $x, y \in A$:*
(a) $(x^\sim)^- = x$;
(b) $(x^-)^\approx = x^\sim$;
(c) $0^\sim = 0^- = 1$;
(d) $1^\approx = 1$;
(e) $x^\sim \cdot x = 0$;
(f) $x \oplus x^\sim = 1; x^- \oplus x = 1$;
(g) $(x \oplus y)^- = y^- \cdot x^-; (x \oplus y)^\sim = y^\sim \cdot x^\sim$;
(h) $(x \cdot y)^- = y^- \oplus x^-; (x \cdot y)^\sim = y^\sim \oplus x^\sim$;
(i) $x \oplus y = (y^\sim \cdot x^\sim)^-$;
(j) $(x \oplus y)^\approx = x^\approx \oplus y^\approx$.

Proposition 1.2 (Georgescu and Iorgulescu [5]) *The following properties are equivalent for any $x, y \in A$:*
(a) $x^- \oplus y = 1$;
(b) $y^\sim \cdot x = 0$;
(c) $y \oplus x^\sim = 1$.

We define

$$x \leqslant y \Longleftrightarrow x^- \oplus y = 1. \tag{1}$$

As it is shown in [5], $(A, \leqslant)$ is a lattice in which the join $x \vee y$ and the meet $x \wedge y$ of any two elements x and y are given by:

$$\begin{aligned} x \vee y &= x \oplus x^\sim \cdot y = x \cdot y^- \oplus y; \\ x \wedge y &= x \cdot \left(x^- \oplus y\right) = (x \oplus y^\sim) \cdot y. \end{aligned}$$

For every pseudo MV-algebra A we set $\mathcal{L}(A) = (A, \vee, \wedge, 0, 1)$.

Proposition 1.3 (Georgescu and Iorgulescu [5]) *$\mathcal{L}(A)$ is a bounded distributive lattice.*

Proposition 1.4 (Georgescu and Iorgulescu [5]) *Let $a, x, y \in A$. Then the following properties hold:*

(a) $x \leqslant y \Longleftrightarrow y^- \leqslant x^- \Longleftrightarrow y^\sim \leqslant x^\sim$;
(b) $x \leqslant y \Longrightarrow x \cdot a \leqslant y \cdot a$;
(c) $x \cdot y \leqslant x; x \cdot y \leqslant y; x \leqslant x \oplus y; y \leqslant x \oplus y$;
(d) $(x \wedge y)^\sim = x^\sim \vee y^\sim; (x \wedge y)^- = x^- \vee y^-$.

Let $\mathrm{Inf}(A) = \{x \wedge x^- : x \in A\}$. Observe that $\mathrm{Inf}(A) = \{x \wedge x^\sim : x \in A\}$. Indeed, let $y = x \wedge x^-$. Then $y = x^- \wedge (x^-)^\sim = z \wedge z^\sim$, where $z = x^-$. On the other hand, if $y = x \wedge x^\sim$, then $y = x^\sim \wedge (x^\sim)^- = z \wedge z^-$, where $z = x^\sim$.

Theorem 1.5 *For every $x \in A$, the following conditions are equivalent:*
(a) $x \in \mathrm{Inf}(A)$;
(b) $x \leqslant x^-$;
(c) $x \leqslant x^\sim$;
(d) $x^2 = 0$;
(e) $2x^- = 1$;
(f) $2x^\sim = 1$.

Proof (a) $\Rightarrow$ (b): Let $x \in \mathrm{Inf}(A)$. Then $x = z \wedge z^- \leqslant z^- \leqslant z^- \vee (z^-)^- = (z \wedge z^-)^- = x^-$ by Proposition 1.4(d).

(b) $\Rightarrow$ (c): Suppose that $x \leqslant x^-$. By Axiom (A8) and Proposition 1.4(a), $x = (x^-)^\sim \leqslant x^\sim$.

(c) $\Rightarrow$ (d): Let $x \leqslant x^\sim$. We conclude from Proposition 1.4(b) that $x \cdot x \leqslant x^\sim \cdot x$, hence that $x^2 \leqslant 0$, and finally that $x^2 = 0$.

(d) $\Rightarrow$ (e): Let $x^2 = 0$. Then $\left(x^2\right)^- = 0^-$. Applying Proposition 1.1(h, c) we have $2x^- = 1$.

(e) $\Rightarrow$ (f): Suppose that $2x^- = 1$. Hence $(2x^-)^\approx = 1^\approx$. From Proposition 1.1(d, j) we deduce that $2\,(x^-)^\approx = 1$. By Proposition 1.1(b), $2x^\sim = 1$.

(f) $\Rightarrow$ (a): Let $2x^\sim = 1$. Then $(2x^\sim)^- = 1^-$. Applying Proposition 1.1(g) and Axiom (A4) we obtain $[(x^\sim)^-]^2 = 0$. Since $(x^\sim)^- = x = (x^-)^\sim$, we have $(x^-)^\sim \cdot x = 0$. Therefore $x \leqslant x^-$ by Proposition 1.2. Consequently, $x = x \wedge x^- \in \mathrm{Inf}(A)$. ∎

2. Ideals.

Definition 2.1 An *ideal* of A is a subset I of A satisfying the following conditions:
(I1) $0 \in I$;
(I2) If $x, y \in I$, then $x \oplus y \in I$;
(I3) If $x \in I$, $y \in A$ and $y \leqslant x$, then $y \in I$.

Under this definition, $\{0\}$ and A are the simplest examples of ideals.

Proposition 2.2 (Walendziak [9]) *Let I be a nonvoid subset of A. I is an ideal of A if and only if I satisfies conditions* (I2) *and*
(I3') *If $x \in I$, $y \in A$, then $x \wedge y \in I$.*

Denote by $\mathrm{Id}(A)$ the set of ideals of A and note that $\mathrm{Id}(A)$ ordered by set inclusion is a complete lattice.

Remark 2.3 Let $I \in \mathrm{Id}(A)$.
(a) If $x, y \in I$, then $x \cdot y, x \wedge y, x \vee y \in I$.
(b) I is an ideal of the lattice $\mathcal{L}(A)$.

Definition 2.4 Let I be a proper ideal of A (i.e., $I \neq A$).
(a) I is called *prime* if, for all $I_1, I_2 \in \mathrm{Id}(A)$, $I = I_1 \cap I_2$ implies $I = I_1$ or $I = I_2$.
(b) I is *maximal* iff whenever J is an ideal such that $I \subseteq J \subseteq A$, then either $J = I$ or $J = A$.

Next lemma is obvious and its proof will be omitted.

Lemma 2.5 *Every proper ideal of A can be extended to a maximal ideal.*

Proposition 2.6 (Walendziak [9]) *If $I \in \mathrm{Id}(A)$ is maximal, then I is prime.*

Proposition 2.7 (Georgescu and Iorgulescu [5]) *For $I \in \mathrm{Id}(A)$, the following are equivalent:*
(a) *I is prime;*
(b) *If $x \wedge y \in I$, then $x \in I$ or $y \in I$.*

Definition 2.8 An ideal I of A is called *regular* if I has the unique cover I^* in the lattice $\mathrm{Id}(A)$.

Definition 2.9 A pseudo *MV*-algebra A is called *normal-valued* if for any regular ideal I of A and any $x \in I^*$, $x \oplus I = I \oplus x$.

An element $x \neq 0$ of a pseudo MV-algebra A is called *infinitesimal* (see [8]) if x satisfies condition

$$nx \leqslant x^- \text{ for each } n \in \mathbb{N}.$$

Let us denote by $\mathrm{Infinit}(A)$ the set of all infinitesimal elements in A and by $\mathrm{Rad}(A)$ the intersection of all maximal ideals of A.

Proposition 2.10 (Rachůnek [8]) *Let A be a pseudo MV-algebra. Then:*
(a) $\mathrm{Rad}(A) \subseteq \mathrm{Infinit}(A)$;
(b) *If A is normal-valued, then* $\mathrm{Rad}(A) = \mathrm{Infinit}(A)$.

Proposition 2.11 *The following statements hold:*
(a) *If $x \in \mathrm{Inf}(A)$ and $y \leqslant x$, then $y \in \mathrm{Inf}(A)$;*
(b) $\mathrm{Infinit}(A) \subseteq \mathrm{Inf}(A)$.

Proof (a) Let $x \in \mathrm{Inf}(A)$ and $y \leqslant x$. We have $y \leqslant x \leqslant x^- \leqslant y^-$ by Proposition 1.4(a). Hence $y \in \mathrm{Inf}(A)$.

(b) Let $x \in \mathrm{Infinit}(A)$. Since $x \leqslant nx \leqslant x^-$ for any $n \in \mathbb{N}$, $x \in \mathrm{Inf}(A)$. Thus $\mathrm{Infinit}(A) \subseteq \mathrm{Inf}(A)$. ■

Proposition 2.12 *If* $\mathrm{Inf}(A)$ *is an ideal of* A*, then* $\mathrm{Inf}(A) = \mathrm{Infinit}(A)$.

Proof Let $\mathrm{Inf}(A)$ be an ideal of A and $x \in \mathrm{Inf}(A)$. Then, for any $n \in \mathbb{N}$, $nx \in \mathrm{Inf}(A)$ and hence $nx \leqslant (nx)^- \leqslant x^-$ by Proposition 1.4(a). Therefore $x \in \mathrm{Infinit}(A)$. Thus $\mathrm{Inf}(A) \subseteq \mathrm{Infinit}(A)$. Proposition 2.11 now gives $\mathrm{Inf}(A) = \mathrm{Infinit}(A)$. ■

By Propositions 2.10 and 2.12 we have the following corollary.

Corollary 2.13 *Let* A *be a normal-valued pseudo* MV*-algebra. Then* $\mathrm{Inf}(A)$ *is an ideal of* A *if and only if* $\mathrm{Inf}(A) = \mathrm{Rad}(A)$.

Definition 2.14 An ideal I of A is called *implicative* if for any $x, y, z \in A$ it satisfies the following condition:
(Im) $(x \cdot y \cdot z \in I$ and $z^\sim \cdot y \in I) \Longrightarrow x \cdot y \in I$.

Proposition 2.15 (Walendziak [9]) *The implication* (Im) *is equivalent to*
(Im') *For all* $x, y, z \in A$*, if* $x \cdot y \cdot z^- \in I$ *and* $z \cdot y \in I$*, then* $x \cdot y \in I$.

Proposition 2.16 (Walendziak [9]) *Let* $I \in \mathrm{Id}(A)$*. Then the following conditions are equivalent:*
(a) I *is implicative;*
(b) $\mathrm{Inf}(A) \subseteq I$.

For a nonvoid subset B of a pseudo MV-algebra A we put:

$$B^\sim = \{x^\sim : x \in B\} \text{ and } B^- = \{x^- : x \in B\}.$$

Proposition 2.17 *Let* I *be a proper ideal of* A*. Then the following statements are equivalent:*
(a) $A = I \cup I^\sim$;
(b) $A = I \cup I^-$;
(c) $I^\sim = I^- = A - I$.

Proof (a) $\Rightarrow$ (b): Let $A = I \cup I^\sim$ and suppose that $x \in A - I$. Observe that $x^\sim \in I$. Indeed, if $x^\sim \notin I$, then $x^\sim \in I^\sim$, and hence $x \in I$, a contradiction. Therefore $x = (x^\sim)^- \in I^-$. Consequently, $A = I \cup I^-$.

Similarly (b) $\Rightarrow$ (a). The proof is completed by showing that $I \cap I^- = I \cap I^\sim = \varnothing$. Let $x \in I$ and $x = y^-$, where $y \in I$. By Proposition 1.1(f), $x \oplus y = y^- \oplus y = 1$. Since I is an ideal, $x \oplus y \in I$, and consequently $1 \in I$. Thus $I = A$, which is impossible. Therefore $I \cap I^- = \varnothing$ and similarly $I \cap I^\sim = \varnothing$. ■

Definition 2.18 An ideal I of A is called *normal* if it satisfies the condition:
(N) For all $x, y \in I$, $x \cdot y^- \in I \Longleftrightarrow y^\sim \cdot x \in I$.

Lemma 2.19 (Georgescu and Iorgulescu [5]) *Let I be a normal ideal of A. Then for all $x \in A$:*

$$x \in I \Longleftrightarrow x^{\approx} \in I.$$

For every subset $W \subseteq A$, the smallest ideal of A which contains W, i.e., the intersection of all ideals $I \supseteq W$, is said to be the ideal *generated* by W, and will be denoted by $(W]$.

Proposition 2.20 (Georgescu and Iorgulescu [5]) *Let I be a normal ideal of A and $x \in A$. Then*

$$(I \cup \{x\}] = \{t \in A : t \leqslant y \oplus nx \text{ for some } y \in I \text{ and } n \in \mathbb{N}\}.$$

Following [5], for any normal ideal I of A, we define the congruence on A:

$$x \sim_I y \Longleftrightarrow x \cdot y^- \vee y \cdot x^- \in I.$$

We denote by x/I the congruence class of an element $x \in A$ and on the set $A/I = \{x/I : x \in A\}$ we define the operations:

$$x/I \oplus y/I = (x \oplus y)/I,\ (x/I)^- = \left(x^-\right)/I,\ (x/I)^\sim = (x^\sim)/I.$$

The resulting quotient algebra $A/I = (A/I, \oplus, ^-, ^\sim, 0/I, 1/I)$ becomes a pseudo MV-algebra, called *the quotient algebra of A by the normal ideal I*. Observe that for all $x, y \in A$,

$$\begin{aligned} x/I \cdot y/I &= (x \cdot y)/I; \\ x/I \vee y/I &= (x \vee y)/I; \\ x/I \wedge y/I &= (x \wedge y)/I. \end{aligned}$$

It is clear that:

(2) $\quad x/I = y/I \Longleftrightarrow x \cdot y^- \vee y \cdot x^- \in I \Longleftrightarrow x^\sim \cdot y \vee y^\sim \cdot x \in I;$

(3) $\quad x/I = 0/I \Longleftrightarrow x \in I;$

(4) $\quad x/I = 1/I \Longleftrightarrow x^- \in I \Longleftrightarrow x^\sim \in I.$

3. Maximal ideals.

In this section we generalize some facts concerning maximal ideals of MV-algebras. First, we give a generalization of Theorem 4.7 of [2]. Recall that a pseudo MV-algebra A is *locally finite* if and only if for any $x \neq 0$ there exists $n \in \mathbb{N}$ such that $nx = 1$. Recall also that a pseudo MV-algebra A is *simple* if and only if there is no non-trivial proper ideal of A.

PROPOSITION 3.1 (DVUREČENSKIJ [4]) *A normal ideal I of a pseudo MV-algebra A is maximal if and only if A/I is a simple pseudo MV-algebra.*

THEOREM 3.2 *Let M be a normal ideal of A, then the following conditions are equivalent:*
(a) *M is maximal;*
(b) $(\forall x \notin M)(\exists n \in \mathbb{N})\,(x^-)^n \in M$;
(c) $(\forall x \notin M)(\exists n \in \mathbb{N})\,(x^\sim)^n \in M$;
(d) *A/M is a locally finite pseudo MV-algebra;*
(e) *A/M is a simple pseudo MV-algebra.*

PROOF (a) $\Rightarrow$ (b): Let $x \notin M$. Take $I = (M \cup \{x\}]$. By Proposition 2.20, $I = \{t \in A : t \leqslant y \oplus nx$ for some $y \in M$ and $n \in \mathbb{N}\}$. Note that $M \subset I$ because if $y \in M$, then by Proposition 1.4(c), $y \leqslant y \oplus x$ and hence $y \in I$. Since $x \in I - M$ and M is maximal, we obtain $I = A$. Thus $1 \in I$, i.e.,

$$1 = y \oplus nx \text{ for some } y \in M \text{ and } n \in \mathbb{N}.$$

From Axiom (A8) we have $y \oplus \Big((nx)^-\Big)^\sim = 1$ and hence $(nx)^- \leqslant y$ (by (1) and Proposition 1.2). Therefore $(x^-)^n \leqslant y$ by Proposition 1.1(g). Finally, since $y \in M$ and $(x^-)^n \leqslant y$, we have $(x^-)^n \in M$.

(b) $\Leftrightarrow$ (c): Applying Proposition 1.1(g, b) we get

$$(x^\sim)^n = (nx)^\sim = \Big((nx)^-\Big)^\approx = \big((x^-)^n\big)^\approx. \tag{5}$$

Since M is a normal ideal of A, we have

$$(x^-)^n \in M \Leftrightarrow ((x^-)^n)^\approx \in M \tag{6}$$

by Lemma 2.19. Combining (5) with (6) yields

$$(x^-)^n \in M \Leftrightarrow (x^\sim)^n \in M.$$

Consequently, (b) $\Leftrightarrow$ (c).

(c) $\Rightarrow$ (d): Let $x/M \neq 0/M$. From (3) it follows that $x \notin M$. Since $(x^\sim)^n \in M$ for some $n \in \mathbb{N}$, we have $0/M = (x^\sim)^n/M = (x^\sim/M)^n = ((x/M)^\sim)^n$. Hence

$$1/M = \big(0^-/M\big) = (0/M)^- = \big[((x/M)^\sim)^n\big]^- = n\,(x/M)$$

by Proposition 1.1(i). Thus A/M is locally finite.

(d) $\Rightarrow$ (e): Assume that A/M is a locally finite pseudo MV-algebra and there is a non-trivial proper ideal I of A/M. Take $x/M \in I$ such that $x/M \neq 0/M$. Since A/M is locally finite, $n(x/M) = 1/M$ for some $n \in \mathbb{N}$. But $n(x/M) \in I$ because I is an ideal. Therefore $1/M \in I$ and hence $I = A/M$ which contradicts the assumption that I is proper.

(e) $\Rightarrow$ (a): See Proposition 3.1. ∎

THEOREM 3.3 *Let M be a proper ideal of A. Then the following conditions are equivalent:*
(a) *M is maximal and implicative;*
(b) *M is prime and implicative;*
(c) *M is prime and* $\mathrm{Inf}(A) \subseteq M$;
(d) $A = M \cup M^{\sim}$;
(e) $A = M \cup M^{-}$.

PROOF (a) $\Rightarrow$ (b): See Proposition 2.6.

(b) $\Rightarrow$ (c): By Proposition 2.16.

(c) $\Rightarrow$ (d): Let $x \in A - M$. From (c) it follows that $x \wedge x^{-} \in M$. By Proposition 2.7, $x \in M$ or $x^{-} \in M$. Since $x \notin M$, we have $x^{-} \in M$. Then $x = (x^{-})^{\sim} \in M^{\sim}$, and consequently (d) holds.

(d) $\Rightarrow$ (e): See Proposition 2.17.

(e) $\Rightarrow$ (a): Let $A = M \cup M^{-}$. It is easy to see that M is maximal. Now let $x \in A$ and suppose that $x \wedge x^{-} \notin M$. Hence $x \wedge x^{-} \in M^{-}$, and therefore $(x \wedge x^{-})^{\sim} \in M$. By Proposition 1.4(d), $(x \wedge x^{-})^{\sim} = x^{\sim} \vee (x^{-})^{\sim} = x^{\sim} \vee x$. Consequently, $x \vee x^{\sim} \in M$. From this we deduce that $x, x^{\sim} \in M$ and hence $1 = x \oplus x^{\sim} \in M$, a contradiction. Thus $\mathrm{Inf}(A) \subseteq M$ and by Proposition 2.16, M is implicative. ∎

COROLLARY 3.4 *Let M be a proper ideal of A. If $A = M \cup M^{\sim}$ $(= M \cup M^{-})$, then M is a maximal ideal generating A.*

Next two theorems generalize Theorems 4.10 and 4.12 of [3], respectively. First, for any pseudo MV-algebra A we define

$$\mathcal{M}_1 = \mathcal{M}_1(A) = \{M \in \mathrm{Id}(A) : M \text{ is prime and implicative}\}.$$

By Theorem 3.3,

$$\begin{aligned} \mathcal{M}_1 &= \{M \in \mathrm{Id}(A) : M \text{ is prime and } \mathrm{Inf}(A) \subseteq M\} \\ &= \{M \in \mathrm{Id}(A) - \{A\} : A = M \cup M^{\sim}\} \\ &= \{M \in \mathrm{Id}(A) - \{A\} : A = M \cup M^{-}\}. \end{aligned}$$

From Corollary 3.4 it is easy to see that

$$\mathcal{M}_1 \subseteq \mathcal{M}_0(A) = \{M : M \text{ is a maximal ideal of } A \text{ generating } A\}.$$

Let I be a proper ideal of an MV-algebra A. C. S. Hoo [6] describes the subalgebra A_I generated by I. He proved that $A_I = I \cup I^{-}$. Therefore, if M is a maximal ideal of A generating A, then $A = M \cup M^{-}$. Hence $\mathcal{M}_0(A) = \mathcal{M}_1(A)$.

THEOREM 3.5 $\mathcal{M}_1 = \varnothing$ *if and only if* $(\mathrm{Inf}(A)] = A$.

PROOF First, assume that $\mathcal{M}_1 = \varnothing$ and $(\mathrm{Inf}(A)]$ is a proper ideal of A (i.e., $(\mathrm{Inf}(A)] \neq A$). By Lemma 2.5, there exists a maximal ideal M containing $(\mathrm{Inf}(A)]$. We can write $\mathrm{Inf}(A) \subseteq (\mathrm{Inf}(A)] \subseteq M$. From Theorem 3.3 it follows that $A = M \cup M^{\sim}$ and we obtain a contradiction. For the converse, take a prime ideal M such that $\mathrm{Inf}(A) \subseteq M$. Then $A = (\mathrm{Inf}(A)] \subseteq M$, which contradicts the assumption that M is proper. ∎

THEOREM 3.6 $\mathcal{M}_1 = \{(\mathrm{Inf}(A)]\}$ *if and only if* $(\mathrm{Inf}(A)]$ *is maximal.*

PROOF If $\mathcal{M}_1 = \{(\mathrm{Inf}(A)]\}$, then, of course, $(\mathrm{Inf}(A)]$ is maximal. Now, assume that $(\mathrm{Inf}(A)]$ is maximal. Since $\mathrm{Inf}(A) \subseteq (\mathrm{Inf}(A)]$, we deduce from Theorem 3.3 that $(\mathrm{Inf}(A)] \in \mathcal{M}_1$. Let $M \in \mathcal{M}_1$. Hence $\mathrm{Inf}(A) \subseteq M$. Consequently, $(\mathrm{Inf}(A)] \subseteq M$. We have $(\mathrm{Inf}(A)] = M$, because $(\mathrm{Inf}(A)]$ is maximal. Therefore $\mathcal{M}_1 = \{(\mathrm{Inf}(A)]\}$. ■

Finally, we give an example of a pseudo MV-algebra generated by its maximal ideal.

EXAMPLE 3.7 Let $A = \{(1, y) \in \mathbb{R}^2 : y \geqslant 0\} \cup \{(2, y) \in \mathbb{R}^2 : y \leqslant 0\}$, $\mathbf{0} = (1, 0)$, $\mathbf{1} = (2, 0)$. For any $(a, b), (c, d) \in A$, we define operations $\oplus, ^-, ^\sim$ as follows:

$$\begin{aligned} (a,b) \oplus (c,d) &= \begin{cases} (1, b+d) & \text{if } a = c = 1, \\ (2, ad+b) & \text{if } ac = 2 \text{ and } ad + b \leqslant 0, \\ (2,0) & \text{in other cases,} \end{cases} \\ (a,b)^- &= \left(\frac{2}{a}, -\frac{2b}{a}\right), \\ (a,b)^\sim &= \left(\frac{2}{a}, -\frac{b}{a}\right). \end{aligned}$$

Then $A = (A, \oplus, ^-, ^\sim, \mathbf{0}, \mathbf{1})$ is a pseudo MV-algebra. Let $M = \{(1, y) \in \mathbb{R}^2 : y \geqslant 0\}$. Then M is a maximal ideal of A and $A = M \cup M^-$ is generated by M.

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