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On new multiple extension of Hilbert's integral inequality (2)

Abstract. Via Laplace transformation some new generalization of Hardy-Hilbert's integral inequality is given.

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1. Introduction. Let $f, g \geq 0$ satisfy

$$0 < \int_0^\infty f^2(t)\,dt < \infty \text{ and } 0 < \int_0^\infty g^2(t)\,dt < \infty.$$

Then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dxdy < \pi \left(\int_0^\infty f^2(t)\,dt \int_0^\infty g^2(t)\,dt \right)^{\frac{1}{2}}, \tag{1}$$

where the constant factor π is the best possible (cf. Hardy et al. [2]). Inequality (1) is well known as Hilbert's integral inequality. This inequality had been extended by Hardy ([1]) as follows.

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $f, g \geq 0$ satisfy

$$0 < \int_0^\infty f^p(t)\,dt < \infty \text{ and } \int_0^\infty g^q(t)\,dt < \infty,$$

then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}\,dxdy < \frac{\pi}{sin(\pi/p)}\left(\int_0^\infty f^p(t)\,dt\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(t)\,dt\right)^{\frac{1}{q}}, \tag{2}$$

where the constant factor $\frac{\pi}{sin(\pi/p)}$ is the best possible. Inequality (2) is called Hardy-Hilbert's integral inequality and is important in analysis and application (cf. Mitrinovic et al. [4]).

B. Yang gave the following extensions of (2) as follows :

THEOREM 1.1 ([6]) *If $\lambda > 2 - min\{p, g\}$ and $f, g \geq 0$ satisfy*

$$0 < \int_0^\infty t^{1-\lambda} f^p(t)\,dt < \infty \text{ and } \int_0^\infty t^{1-\lambda} g^q(t)\,dt < \infty,$$

then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}\,dxdy < k_\lambda(p)\left(\int_0^\infty t^{1-\lambda} f^p(t)\,dt\right)^{\frac{1}{p}}\left(\int_0^\infty t^{1-\lambda} g^q(t)\,dt\right)^{\frac{1}{q}}, \tag{3}$$

where the constant factor $k_\lambda(p) = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$ is the best possible (B is the beta function).

The object of this paper is to give the following generalizations :

2. Main Results. The following is a generalization of [3, Proposition 319]

THEOREM 2.1 *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f, g, \phi, \psi \geq 0$, $f(0) = g(0) = 0$, $f(\infty) = g(\infty) = \infty$, ϕ^p and ψ^p are in $P(0,\infty)$, $K(f, g)$ is positive in $(0,\infty)\times(0,\infty)$ and homogeneous of degree -1 with*

$$\int_0^\infty f^{-1/p} K(f, 1) f^{'}\,du = k. \tag{4}$$

Then

$$\int_0^\infty\int_0^\infty K(f, g)\phi(f)\psi(g)\,df\,dg < k\|\phi\|_p\|\psi\|_q \tag{5}$$

and

$$\left\|\int_0^\infty K(f, g)\phi(f)\,df\right\|_p < k\|\phi\|_p. \tag{6}$$

The constant k is the best possible.

Proof

$$\begin{aligned}
\int_0^\infty\int_0^\infty K(f,g)\phi(f)\psi(g)\,df\,dg &= \int_0^\infty\int_0^\infty K(f,g)\phi(f)\psi(g)f^{'}g^{'}\,du\,d\nu \\
&= \int_0^\infty\int_0^\infty K^{\frac{1}{p}}(f,g)\phi(f)(f^{'}g^{'})^{\frac{1}{p}}\left(\frac{f}{g}\right)^{\frac{1}{pq}} K^{\frac{1}{q}}(f,g)\psi(g)(f^{'}g^{'})^{\frac{1}{p}}\left(\frac{f}{g}\right)^{-\frac{1}{pq}}\,du\,d\nu \\
&\le \left(\int_0^\infty\int_0^\infty K(f,g)\phi^p(f)f^{'}g^{'}\left(\frac{f}{g}\right)^{\frac{1}{q}}\,du\,d\nu\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_0^\infty\int_0^\infty K(f,g)\psi^q(g)f^{'}g^{'}\left(\frac{f}{g}\right)^{\frac{1}{p}}\,du\,d\nu\right)^{\frac{1}{q}} \\
&= M^{\frac{1}{p}}N^{\frac{1}{q}}, say.
\end{aligned}$$

First, we consider

$$Q = \int_0^\infty \psi^q(g)g^{'}\,d\nu \int_0^\infty \left(\frac{f}{g}\right)^{-\frac{1}{p}} K(f,g)f^{'}\,du.$$

By putting $f/g = h$, we have

$$\begin{aligned}
Q &= \int_0^\infty \psi^q(g)g^{'}\,d\nu \int_0^\infty h^{-\frac{1}{p}}K(gh,g)gh^{'}\,du \\
&= \int_0^\infty \psi^q(g)g^{'}\,d\nu \int_0^\infty h^{-\frac{1}{p}}g^{-1}K(h,1)gh^{'}\,du \\
&= \int_0^\infty \psi^q(g)g^{'}\,d\nu \int_0^\infty h^{-\frac{1}{p}}K(h,1)h^{'}\,du \\
&= k\|\psi\|_q.
\end{aligned}$$

$$\begin{aligned}
P &= \int_0^\infty \phi^p(f) f' \, du \int_0^\infty \left(\frac{f}{g}\right)^{\frac{1}{q}} K(f,g) g' \, d\nu \\
&= \int_0^\infty \phi^p(f) f' \, du \int_0^\infty h^{\frac{1}{q}} K(gh,g) f h^{-2} h' \, d\nu \\
&= \int_0^\infty \phi^p(f) f' \, du \int_0^\infty h^{\frac{1}{q}} g^{-1} K(h,1) g h^{-1} \, d\nu \\
&= \int_0^\infty \phi^p(f) f' \, du \int_0^\infty h^{-\frac{1}{p}} K(h,1) h' \, d\nu \\
&= k\|\phi\|_p.
\end{aligned}$$

Concerning the other part, by virtue of the first part, we have

$$\begin{aligned}
&\int_0^\infty \left(\int_0^\infty K(f,g)\phi(f)\, df \right)^p dg \\
= &\int_0^\infty \int_0^\infty K(f,g)\phi(f) \left(\int_0^\infty K(f,g)\phi(f) \right)^{p-1} df dg \\
< &\; k \left(\int_0^\infty \phi^p(f)\, df \right)^{1/p} \left(\int_0^\infty \left(\int_0^\infty K(f,g)\phi(f)\, df \right)^{q(p-1)} dg \right)^{1/q} \\
= &\left(\int_0^\infty \phi^p(f)\, df \right)^{1/p} \left(\int_0^\infty \left(\int_0^\infty K(f,g)\phi(f)\, df \right)^{p} dg \right)^{1/q}
\end{aligned}$$

which implies

$$\left(\int_0^\infty \left(\int_0^\infty K(f,g)\phi(f)\, df \right)^p dg \right)^{1/q} < k \left(\int_0^\infty \phi^p(f)\, df \right)^{1/p}$$

This completes the proof of the theorem. ■

Corollary 2.2 *Let* $p > 1$, $\alpha + 1/p > 0, f \geq 0$ $, f(0) = 0$, $f(\infty) = \infty$, $f^{p-p\alpha-2}\phi^p(f)$ *is* $P(0,\infty)$ Φ *is the Laplace transform of* ϕ*, then*

$$\|h^\alpha \Phi(h)\|_p < \Gamma(\alpha + 1/p)\|f^{1-\alpha-2/p}\phi(f)\|_p. \tag{7}$$

PROOF In Theorem 2.1 , by putting

$$K(f,g) = e^{-f/g} f^{\alpha-1+2/p} g^{-\alpha-2/p}$$

and replacing $\phi(f)$ by $f^{-\alpha+1-2/p}\phi(f)$, we obtain

$$\|g^{-\alpha-2/p}\Phi(1/g)\|_p = \Gamma(\alpha+1/p)\|f^{1-\alpha-2/p}\phi(f)\|_p,$$

then writing $g = 1/h$, to get

$$\|h^{\alpha}\Phi(h)\|_p < \Gamma(\alpha+1/p)\|f^{1-\alpha-2/p}\phi(f)\|_p.$$

Now we are coming to the main result :

THEOREM 2.3 *Let* $\sum_{i=1}^{n} 1/p_i = 1$, $p_i > 1$, $\lambda > 1 + 1/p_i$, $f_i \geq 0$, $f_i(0) = 0$, $f_i(\infty) = \infty$, $f^{p_i-\lambda-1}\phi_i^{p_i}(f)$ *are in* $P_i(0,\infty)$, $(i = 1, 2, \ldots, n)$. *Then we have*

$$\int_0^\infty \cdots \int_0^\infty \frac{\phi_1(f_1)\ldots\phi_n(f_n}{(f_1+\cdots+f_n)^\lambda} df_1 \ldots df_n < \prod_{i=1}^{n} \frac{\Gamma(\lambda-1-1/p_i)}{\Gamma\lambda} \left\| f^{1-\frac{\lambda+1}{p_i}} \phi_i(f) \right\|_{p_i}. \tag{8}$$

PROOF We first consider

$$\begin{aligned}
&\int_0^\infty f^\alpha \phi_1(f)\ldots\phi_n(f)\, df \\
=\ &\int_0^\infty f^\alpha \int_0^\infty e^{-ff_1}\Phi_1(f_1)\, df_1 \ldots \int_0^\infty e^{-ff_n}\Phi_n(f_n)\, df_n\, df \\
=\ &\int_0^\infty \cdots \int_0^\infty \phi_1(f_1)\ldots\phi_n(f_n)\, df_1 \ldots df_n \int_0^\infty f^\alpha e^{-f(f_1+\cdots+f_n)}\, df \\
=\ &\int_0^\infty \cdots \int_0^\infty \phi_1(f_1)\ldots\phi_n(f_n)\, df_1 \ldots df_n \int_0^\infty [f(f_1+\cdots+f_n)]^\alpha \\
&\times e^{-f(f_1+\cdots+f_n)}(f_1+\cdots+f_n)\, df \\
=\ &\Gamma(\alpha+1)\int_0^\infty \cdots \int_0^\infty \frac{\phi_1(f_1)\ldots\phi_n(f_n)}{(f_1+\cdots+f_n)^{\alpha+1}}\, df_1 \ldots df_n.
\end{aligned}$$

Therefore, by virtue of Corollary 2.2, we have

$$\begin{aligned}
&\int_0^\infty \cdots \int_0^\infty \frac{\phi_1(f_1)\dots\phi_n(f_n)}{(f_1+\cdots+f_n)^{\alpha+1}}\,df_1\dots df_n \\
= &\ \frac{1}{\Gamma(\alpha+1)}\int_0^\infty f^\alpha \Phi_1(f)\dots\Phi_n(f)\,df \\
= &\ \frac{1}{\Gamma(\alpha+1)}\int_0^\infty f^{\alpha/p_1}\Phi_1(f)\dots f^{\alpha/p_n}\Phi_n(f)\,df \\
\le &\ \frac{1}{\Gamma(\alpha+1)}\left\|f^{\alpha/p_1}\Phi_1(f)\right\|\times\cdots\times\left\|f^{\alpha/p_n}\Phi_n(f)\right\| \\
= &\ \frac{1}{\Gamma(\alpha+1)}\prod_{i=1}^n\left\|f^{\alpha/p_i}\Phi_i(f)\right\|_{p_i} \\
< &\ \frac{1}{\Gamma(\alpha+1)}\prod_{i=1}^n\frac{\Gamma(\alpha+1/p_i)}{\Gamma(\lambda)}\left\|f^{1-\frac{\lambda+1}{p_i}}\Phi_i(f)\right\|_{p_i}
\end{aligned}$$

consequently , we have

$$\begin{aligned}
&\int_0^\infty \cdots \int_0^\infty \frac{\phi_1(f_1)\dots\phi_n(f_n)}{(f_1+\cdots+f_n)^{\lambda}}\,df_1\dots df_n \\
< &\ \frac{1}{\Gamma(\alpha+1)}\prod_{i=1}^n\frac{\Gamma(\lambda-1-1/p_i)}{\Gamma(\lambda)}\left\|f^{1-\frac{\lambda+1}{p_i}}\phi_i(f)\right\|_{p_i}. \qquad \blacksquare
\end{aligned}$$

Corollary 2.4 *On putting* $f_i(x) = x$, $\phi_i = f_i$, $i = 1,\dots,n$ *in Theorem* 2.3 *we obtain*

$$\begin{aligned}
&\int_0^\infty \cdots \int_0^\infty \frac{f_1(x_1)\dots f_n(x_n)}{(x_1+\cdots+x_n)^{\lambda}}\,dx_1\dots dx_n \\
< &\ \frac{1}{\Gamma(\alpha+1)}\prod_{i=1}^n\frac{\Gamma(\lambda-1-1/p_i)}{\Gamma(\lambda)}\left\|x^{1-\frac{\lambda+1}{p_i}}f_i(x)\right\|_{p_i}.
\end{aligned}$$

References

[1] G. H. Hardy, *Note on a theorem of Hilbert concerning series of positive terms*, Proc. Math. Soc. **23(2)** (1925), Records of Proc. XLV-XLVI.

[2] G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities*, Cambridge University Press, Cambidge 1952.

[3] G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities (Second Edition)*, Cambridge University Press, Cambidge 1952.

[4] D. S. Mitrinovic, J. E. Pecaric and A. M. Fink, *Inequalities involving functions and their integrals and derivatives*, Kluwer Academic Publishers, Boston 1991.

[5] T. C. Peachey, *Some integral inequalities related to Hilbert's*, Journal of Inequalities in Pure and Applied Mathematics, Volume **4**, Issue **1**, Article **19**, 2003.

[6] B. Yang, *On Hardy-Hilbert's integral inequality*, J. Math. Anal. Appl. **261** (2001), 295-306.

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