ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE
XLVI (2) (2006), 263-273

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On $C^{(n)}$- Almost Periodic Solutions to Some Nonautonomous Differential Equations in Banach Spaces

Abstract. In this paper we prove the existence and uniqueness of $C^{(n)}$-almost periodic solutions to the nonautonomous ordinary differential equation $x'(t) = A(t)x(t) + f(t)$, $t \in \mathbb{R}$, where $A(t)$ generates an exponentially stable family of operators $(U(t,s))_{t \geq s}$ and f is a $C^{(n)}$-almost periodic function with values in a Banach space $\mathbb{X}$. We also study a Volterra-like equation with a $C^{(n)}$-almost periodic solution.

1991 Mathematics Subject Classification: 34C37;43A60;34G20.

Key words and phrases: $C^{(n)}$-almost periodic function, family of bounded operators, exponentially stable, Acquistapace-Terreni conditions, uniform spectrum of bounded functions.

1. Introduction. Harald Bohr's interest in which functions could be represented by a Dirichlet series, i.e. of the form $\sum_{n=1}^{+\infty} a_n e^{-\lambda_n z}$, where $a_n, z \in \mathbb{C}$ and $(\lambda_n)_{n\in\mathbb{N}}$ is a monotone increasing sequence of real numbers (series which play an important role in complex analysis and analytic number theory), led him to devise a theory of almost periodic real (and complex) functions, founding this theory between the years 1923 and 1926. Several generalizations and classes of almost periodic functions have been introduced in the literature, including pseudo-almost periodic functions ([9], [10], [11], [27]), almost automorphic functions ([19], [20]), p-almost automorphic functions ([8]), etc...

*The authors are grateful to the referee for his/her valuable comments. This research was conducted while the third author was visiting the Laboratoire Marin Mersenne-University of Paris 1 Panthéon-Sorbonne in June 2006. He wishes to express his gratitude for the invitation.

$C^{(n)}$-almost periodic functions $\mathbb{R} \to \mathbb{R}$ were studied initially in [3] and [4]. Actually these are functions which are almost periodic up to their n-th derivatives. In [6], D. Bugajewski and G. M. N'Guérékata have extended the study to functions $\mathbb{R} \to X$, where X is a Banach space. They also introduced the concept of $C^{(n)}$-asymptotically almost periodic functions and discussed some applications to ordinary and partial differential equations. More results were obtained also in [16]. In particular the equation

$$x'(t) = A(t)x(t) + f(t), \quad t \in \mathbb{R} \tag{1}$$

where $A(t) : \mathbb{R} \to \mathbb{C}^n$ is τ-periodic and $f : \mathbb{R} \to \mathbb{C}^n$ is $C^{(n)}$-almost periodic was investigated.

In the present paper we study the same equation in an infinite dimensional space $\mathbb{X}$ and we assume that $A(t)$ is not necessarily periodic (see Theorem 3.6 in Section 3 below) but generates a family of exponentially stable bounded operators $(U(t,s))_{t\geq s}$ with the so-called "Acquistapace-Terreni" conditions.

The following notations will be used in the whole paper: $BC(\mathbb{R}, X)$, $BUC(\mathbb{R}, X)$, $\rho(D)$, $R(\lambda, D)$, $sp(f)$ will denote respectively the space of all bounded continuous functions $f : \mathbb{R} \to X$, the space of all bounded uniformly continuous functions $f : \mathbb{R} \to X$, the resolvent set of the operator D ([22], page 234), and the Carleman spectrum of $f \in L^{+\infty}(\mathbb{R}, X)$ (see for instance [12] for definition).

We begin with some elementary properties of the so-called $C^{(n)}$-almost periodic functions with values in a Banach space and present some properties of the uniform spectrum of bounded functions (see [12], [17]) in the context of $C^{(n)}$-almost periodicity with an application to a Volterra-type equation.

2. $C^{(n)}$-Almost Periodic Functions. Let $\mathbb{X} = (\mathbb{X}, \| \bullet \|)$ be a (complex) Banach space and $f_\tau(x) := f(x+\tau)$, where $f : \mathbb{R} \to \mathbb{X}$, and $x, \ \tau \in \mathbb{R}$.

Denote by $C^{(n)}(\mathbb{R}, \mathbb{X})$ (briefly $C^{(n)}(\mathbb{X})$) the space of all functions $\mathbb{R} \to \mathbb{X}$ which have a continuous n-th derivative on $\mathbb{R}$. Let $C_b^{(n)}(\mathbb{R}, \mathbb{X})$ (briefly $C_b^{(n)}(\mathbb{X})$) be the subspace of $C^{(n)}(\mathbb{R}, \mathbb{X})$ consisting of such functions satisfying

$$\sup_{t\in\mathbb{R}} \sum_{i=0}^{n} \|f^{(i)}(t)\| < +\infty$$

where $f^{(i)}$ denote the i-th derivative of f and $f^{(0)} := f$. Clearly $C^{(n)}(\mathbb{X})$ turns out to be a Banach space with the norm

$$\|f\|_n = \sup_{t\in\mathbb{R}} \sum_{i=0}^{n} \|f^{(i)}(t)\|.$$

Definition 2.1 Let $\epsilon > 0$. A number $\tau \in \mathbb{R}$ is said to be a $(\|\bullet\|_n, \epsilon)$-almost period of a function $f \in C^{(n)}(\mathbb{X})$, if $\|f_\tau - f\|_n < \epsilon$.

The set of all $(\| \bullet \|_n, \epsilon)$-almost periods of a function f will be denoted by $E^{(n)}(\epsilon, f)$.

Definition 2.2 A function $f \in C^{(n)}(\mathbb{X})$ is said to be $C^{(n)}$-almost periodic (briefly $C^{(n)} - a.p.$) if for every $\epsilon > 0$, the set $E^{(n)}(\epsilon, f)$ is relatively dense in $\mathbb{R}$. The set of all $C^{(n)} - a.p$ functions $f : \mathbb{R} \to \mathbb{X}$ will be denoted by $AP^{(n)}(\mathbb{R}, \mathbb{X})$, (briefly $AP^{(n)}(\mathbb{X})$). $AP^{(0)}(\mathbb{X}) = AP(\mathbb{X})$, the classical Banach space of all almost periodic functions in Bohr's sense.

Equipped with the $\| \bullet \|_n$ norm above, $AP^{(n)}(\mathbb{X})$ turns out to be a Banach space (cf. [6, Corollary 2.12]).

Example 2.3 Let $g(t) = cos(\alpha t) + cos(\beta t),\ t \in \mathbb{R}$, where α and β are incommensurate real numbers. Then the function $f(t) = e^{g(t)}$ is $C^{(n)}$-almost periodic for any $n = 1, 2, ...$ The proof is straightforward from [6, Theorem 4.3].

We recall that $AP^{(n+1)}(\mathbb{X}) \subset AP^{(n)}(\mathbb{X}) \subset C_b^{(n)}(\mathbb{X})$, for all $n = 0, 1, 2, ...$ All the inclusions are strict (cf. [6, Example 4.5]).

One can find more examples of $C^{(n)}$-almost periodic functions in [3] and [6].

The uniform limit of $C^{(n)}$-almost periodic functions in $AP^n(\mathbb{X})$ is in $AP^n(\mathbb{X})$, too (see [6, Theorem 2.11]).

We also have the following (cf. [6, Theorem 3.4]).

Theorem 2.4 *Let $F(t) := \int_0^t f(s)ds$ where $f \in AP^{(n)}(\mathbb{X})$, $t \in \mathbb{R}$. Then $F \in AP^{(n+1)}(\mathbb{X})$ if $\mathcal{R}_F$, the range of F, is relatively compact in $\mathbb{X}$.*

Let us recall that for $f \in AP(\mathbb{X})$ where $\mathbb{X}$ is a uniformly convex Banach space, the primitive $F(t) = \int_0^t f(s)ds$ is a.p. iff $\mathcal{R}_F$ is bounded in $\mathbb{X}$. This is known as the Bohl-Bohr theorem (see for instance [7, Theorem 6.20]).

This result can be extended to $AP^{(n)}(\mathbb{X})$ as follows.

Theorem 2.5 *Let $\mathbb{X}$ be a Banach space which does not contain a subspace isomorphic to c_0 and $f \in AP^{(n)}(\mathbb{X})$. Then the function $F(t) = \int_0^t f(s)ds \in AP^{(n+1)}(\mathbb{X})$ iff $\mathcal{R}_F$ is bounded in $\mathbb{X}$.*

Proof We have just to prove the only if part. It comes by induction. The case $n = 0$ is known as Kadets' Theorem (see for instance [15]). Assume now that f is in $AP^{(n)}(\mathbb{X})$, and that the theorem is true for $n - 1$; then $F \in AP^{(n)}(\mathbb{X})$. But we have $F' = f$ and so $F' \in AP^{(n)}(\mathbb{X})$, from which we conclude that $F \in AP^{(n+1)}(\mathbb{X})$. ■

We recall that Banach spaces $\mathbb{X}$ which do not contain subspaces isomorphic to c_0 (also called sometimes perfect Banach spaces, [19]) include uniformly convex Banach spaces and finite dimensional spaces.

We now recall some properties of uniform spectrum of bounded functions. This concept was recently introduced in [12]. See also [17].

2.1. Uniform spectrum of a function in $BC(\mathbb{R}, \mathbb{X})$. Let us consider the following simple ordinary differential equation in a complex Banach space $\mathbb{X}$

$$x'(t) - \lambda x = f(t), \tag{2}$$

where $f \in BC(\mathbb{X})$. If $Re\lambda \neq 0$, the homogeneous equation associated with this has an exponential dichotomy; so, (2) has a unique bounded solution which we denote by $x_{f,\lambda}(\cdot)$. Moreover, from the theory of ordinary differential equations, it follows that for every fixed $\xi \in \mathbb{R}$,

$$x_{f,\lambda}(\xi) := \begin{cases} \int_{-\infty}^{\xi} e^{\lambda(\xi-t)} f(t)dt & (\text{if } Re\lambda < 0) \\ -\int_{\xi}^{+\infty} e^{\lambda(\xi-t)} f(t)dt & (\text{if } Re\lambda > 0). \end{cases} \tag{3}$$

$$= \begin{cases} \int_{-\infty}^{0} e^{-\lambda\eta} f(\xi+\eta)d\eta & (\text{if } Re\lambda < 0) \\ -\int_{0}^{+\infty} e^{-\lambda\eta} f(\xi+\eta)d\eta & (\text{if } Re\lambda > 0). \end{cases} \tag{4}$$

As is well known, the differentiation operator $\mathcal{D}$ is a closed operator on $BC(\mathbb{R}, \mathbb{X})$. The above argument shows that $\rho(\mathcal{D}) \supset \mathbb{C}\backslash i\mathbb{R}$ and $x_{f,\lambda} = (\mathcal{D} - \lambda)^{-1} f$ for every $\lambda \in \mathbb{C}\backslash i\mathbb{R}$ and $f \in BC(\mathbb{R}, \mathbb{X})$.

Hence, for every $\lambda \in \mathbb{C}$ with $Re\lambda \neq 0$ and $f \in BC(\mathbb{R}, \mathbb{X})$ the function $[(\lambda - \mathcal{D})^{-1} f](t) = \widehat{S(t)f}(\lambda) \in BC(\mathbb{R}, \mathbb{X})$. Moreover, $(\lambda - \mathcal{D})^{-1} f$ is analytic on $\mathbb{C}\backslash i\mathbb{R}$.

Definition 2.6 Let f be in $BC(\mathbb{R}, \mathbb{X})$. Then,

1. $\alpha \in \mathbb{R}$ is said to be *uniformly regular* with respect to f if there exists a neighborhood $\mathcal{U}$ of $i\alpha$ in $\mathbb{C}$ such that the function $(\lambda - \mathcal{D})^{-1} f$, as a complex function of λ with $Re\lambda \neq 0$, has an analytic continuation into $\mathcal{U}$.

2. The set of $\xi \in \mathbb{R}$ such that ξ is not uniformly regular with respect to $f \in BC(\mathbb{R}, \mathbb{X})$ is called *uniform spectrum* of f and is denoted by $sp_u(f)$.

Observe that, if $f \in BUC(\mathbb{R}, \mathbb{X})$, then $\alpha \in \mathbb{R}$ is uniformly regular if and only if it is regular with respect to f (cf. [17]).

We now list some properties of uniform spectra of functions in $BC(\mathbb{R}, \mathbb{X})$.

Proposition 2.7 *Let $g, f, f_n \in BC(\mathbb{R}, \mathbb{X})$ such that $f_n \to f$ as $n \to +\infty$ and let $\Lambda \subset \mathbb{R}$ be a closed subset satisfying $sp_u(f_n) \subset \Lambda$ for all $n \in \mathbb{N}$. Then the following assertions hold:*

1. $sp_u(f) = sp_u(f(h + \cdot))$;

2. $sp_u(\alpha f(\cdot)) \subset sp_u(f)$, $\alpha \in \mathbb{C}$;

3. $sp(f) \subset sp_u(f)$;

4. $sp_u(Bf(\cdot)) \subset sp_u(f)$, $B \in L(\mathbb{X})$;

5. $sp_u(f + g) \subset sp_u(f) \cup sp_u(g)$;

6. $sp_u(f) \subset \Lambda$.

We also recall the following important result (see [17] for the proof).

PROPOSITION 2.8 *Let $f \in BC(\mathbb{R}, \mathbb{X})$. Then*

$$sp_u(f) = sp_c(f),$$

where $sp_c(f)$ denotes the Carleman spectrum of f.

From the above properties, we obtain:

PROPOSITION 2.9 *Let $f \in C_b^{(n)}(\mathbb{X})$. Then*

$$sp_u(f^{(i)}) \subset sp_u(f^{(i-1)}),\ for\ every\ i = 1, 2, ..., n.$$

PROOF We just check $sp_u(f') \subset sp_u(f)$. First note that for each $n = 1, 2, ...$, $sp_u[n(f(t + \frac{1}{n}) - f(t))] \subset sp_u(f)$. This can be proved by using Proposition 2.7 (*1*, *2* and *5*).
Now $(n(f(t + \frac{1}{n}) - f(t))) \to f'(t)$ as $n \to +\infty$. So by Proposition 2.7 (*6*) we obtain $sp_u(f') \subset sp_u(f)$. ■

LEMMA 2.10 *Let $f \in AP^{(n)}(\mathbb{X})$ and $\phi \in L^1(\mathbb{R})$ whose Fourier transform has compact support $supp(\phi)$. Then $g := \phi * f \in AP^{(n)}(\mathbb{X})$ and $sp_u(g) \subset sp_u(f) \cap supp(\phi)$.*

PROOF The property is known for $n = 0$ (see for instance [12]), Also we know that g is C^n with derivatives: $g^{(k)} = \phi * f^{(k)}$ (if $k \leq n$). So, for each $k \leq n$, $g^{(k)} \in AP(\mathbb{R}, \mathbb{X})$, and the lemma follows. ■

EXAMPLE 2.11 Let $\phi \in L^1_{loc}(\mathbb{R})$. Then the function

$$f(t) := \int_{\mathbb{R}} \phi(t - s)[\sin(\alpha s) + \sin(\beta s)]ds,$$

where α and β are incommensurate numbers, is $C^{(n)}$-almost periodic for any $n = 1, 2, ...$

2.2. An Application. A Volterra-like Equation. Consider the equation

$$x(t) = g(t) + \int_{-\infty}^{+\infty} a(t - \sigma)x(\sigma)d\sigma,\ t \in \mathbb{R}, \tag{5}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a continuous function and $a \in L^1(\mathbb{R})$ with compact support.

PROPOSITION 2.12 *Suppose $g \in AP^{(n)}(\mathbb{R})$ and $\|a\|_{L^1} < 1$. Then Eq. (4) above has a unique $C^{(n)}$-almost periodic solution.*

Proof It is clear by Lemma 2.10 above that

$$x \in AP^{(n)}(\mathbb{R}) \mapsto \int_{-\infty}^{+\infty} a(t-\sigma)x(\sigma)d\sigma \in AP^{(n)}(\mathbb{R})$$

is well-defined. Now consider the application $\Gamma : AP^{(n)}(\mathbb{R}) \to AP^{(n)}(\mathbb{R})$ defined by

$$(\Gamma x)(t) := g(t) + \int_{-\infty}^{+\infty} a(t-\sigma)x(\sigma)d\sigma,\ t \in \mathbb{R}.$$

We can easily check that

$$\|(\Gamma x) - (\Gamma y)\|_n \leq \|a\|_{L^1}\|x-y\|_n.$$

The conclusion follows by the principle of contraction. ■

3. Main Results.

3.1. Linear Equations. Consider in a (complex) Banach space $\mathbb{X}$ the linear equation

$$x'(t) = Ax(t) + f(t),\ \ t \in \mathbb{R}, \tag{6}$$

where $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is a linear operator, and $f \in C(\mathbb{R}, \mathbb{X})$.

In what follows, we will use the notation: $\Pi := \{z \in \mathbb{C}\ :\ \ Re z \neq 0\}$.

Definition 3.1 A linear operator $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ where $\mathbb{X}$ is a complex Banach space is said to be of simplest type if $A \in \mathcal{L}(\mathbb{X})$ and $A = \sum_{k=1}^{n} \lambda_k P_k$, where $\lambda_k \in \mathbb{C},\ k = 1, ...n$, and $(P_k)_{1 \leq k \leq n}$ forms a complex system $\sum_{k=1}^{n} P_k = I$ of mutually disjoint operators on $\mathbb{X}$, that is $P_j P_k = \delta_{jk} P_k$, where $\delta_{jk} = I$,(the identity operator on $\mathbb{X}$), if $j = k$, and $\delta_{jk} = 0$, otherwise.

We shall use the following result which is an extension of Lemma 4.1 [16].

Lemma 3.2 *Suppose $\mathbb{X}$ is a Banach space which does not contain a subspace isomorphic to c_0, and consider in $\mathbb{X}$ the differential equation*

$$x'(t) = \lambda x(t) + f(t),\ \ t \in \mathbb{R}, \tag{7}$$

where $\lambda \in \mathbb{C}$ and $f \in AP^{(n)}(\mathbb{X})$. Then every bounded solution x of Eq. (7) satisfies $x \in AP^{(n+1)}(\mathbb{X})$, if $\lambda \notin \Pi$ and $x \in AP^{(n)}(\mathbb{X})$ if $\lambda \in \Pi$.

Proof The proof follows the one of [16], Lemma 4.1 and uses Theorem 2.5 above. ■

Theorem 3.3 *Assume that $f \in AP^n(\mathbb{X})$, where $\mathbb{X}$ does not contain a subspace isomorphic to c_0, and that A is of simplest type.*
Then every bounded solution x to Eq. (6) satisfies $x \in AP^{n+1}(\mathbb{X})$, if $\lambda_k \notin \Pi,\ k = i, ..., n$, and $x \in AP^n(\mathbb{X})$, if $\lambda_k \in \Pi$ for some $k \in \{i, ..., n\}$.

Proof Let's apply the projection P_j to Eq. (6). We get

$$P_j x'(t) = \frac{d}{dt}(P_j x)(t) = P_j(\sum_{k=1}^{n} \lambda_k P_k)x(t) + P_j f(t)$$

$$= \lambda_j (P_j x)(t) + (P_j f)(t).$$

It is clear that $P_j f \in AP^n(\mathbb{X})$,since $P_j \in \mathcal{L}(\mathbb{X})$ (cf. [6]). Thus by Lemma 3.2 above, $P_j x \in AP^n(\mathbb{X})$. We conclude that

$$x(t) = \sum_{j=1}^{n} (P_j x)(t) \in AP^{(n)}(\mathbb{X}),$$

and the theorem is proved.

3.2. Nonlinear Case. Now consider the nonautonomous equation (1), i.e.:

$$x'(t) = A(t)x(t) + f(t), \quad t \in \mathbb{R}. \tag{8}$$

We also assume that $A(t)$, $t \in \mathbb{R}$, satisfy the 'Acquistapace-Terreni' conditions introduced in [2]; namely, there exist constants $\lambda_0 \geq 0$, $\theta \in (\frac{\pi}{2}, \pi)$, $L, K \geq 0$, and $\alpha, \beta \in (0, 1]$ with $\alpha + \beta > 1$ such that

$$\Sigma_\theta \cup \{0\} \subset \rho(A(t) - \lambda_0), \quad \|R(\lambda, A(t) - \lambda_0)\| \leq \frac{K}{1 + |\lambda|} \tag{9}$$

and

$$\|(A(t) - \lambda_0)R(\lambda, A(t) - \lambda_0)[R(\lambda_0, A(t)) - R(\lambda_0, A(s))]\| \leq L|t - s|^\alpha |\lambda|^\beta$$

for $t, s \in \mathbb{R}$, $\lambda \in \Sigma_\theta := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| \leq \theta\}$. Then there exists a unique evolution family $\{U(t, s)\}_{-\infty < s \leq t < +\infty}$ on X, which governs the linear version of (1). This follows from [1, Theorem 2.3]; see also [2, 23, 24].

The family $(U(t, s))_{t \geq s}$, will satisfy the following properties:

- (i) $U(t, t) = I$ for all $t \in \mathbb{R}$,
- (ii) $U(t, s)U(s, r) = U(t, r)$ for all $t \geq s \geq r$,
- (iii) The map $(t, s) \mapsto U(t, s)x$ is continuous for every $x \in \mathbb{X}$.

We will assume in this paper that $(U(t, s))_{t \geq s}$ is exponentially stable, that is there exist some positive constants N, ω independent of $t \geq s$ such that $\|U(t, s)\| \leq Ne^{-\omega(t-s)}$.

Now we point out the following result which is an immediate consequence of Proposition 4.4 [18]:

LEMMA 3.4 *Suppose $A(t)$ satisfy the 'Acquistapace-Terreni' conditions, $U(t,s)$ is exponentially stable and $R(\lambda_0, A(\cdot)) \in AP(\mathbb{R}, L(\mathbb{X}))$. Let $f \in AP(\mathbb{X})$ and $h > 0$. Then, for any $\varepsilon > 0$, there exists $l(\varepsilon) > 0$ such that every interval I of length $l(\varepsilon)$ contains a number τ with the property that*

$$\|U(t+\tau, s+\tau) - U(t,s)\| \leq \varepsilon e^{-\frac{\omega}{2}(t-s)}$$

for all $t - s \geq h$ and

$$\|f(t+\tau) - f(t)\| < \eta, \quad t \in \mathbb{R},$$

where $\eta = \eta(\varepsilon, h) \to 0$ as $\varepsilon \to 0$.

DEFINITION 3.5 Under the above assumptions, a mild solution of Eq. (1) is a continuous function $x : \mathbb{R} \to \mathbb{X}$ satisfying the formula

$$x(t) = U(t,s)x(s) + \int_s^t U(t,\sigma)f(\sigma)d\sigma, \;\; t \geq s \in \mathbb{R}.$$

Now we state and prove.

THEOREM 3.6 *Assume that the family $(U(t,s))_{t\geq s}$ is exponentially stable and $f \in AP^n(\mathbb{X})$. Then the above equation Eq. (1) possesses a unique mild solution in $AP^{(n)}(\mathbb{X})$.*

PROOF Consider a mild solution of Eq.(1):

$$x(t) = U(t,s)x(s) + \int_s^t U(t,\sigma)f(\sigma)d\sigma, \;\; t \geq s \in \mathbb{R}.$$

Now let

$$y(t) = \int_{-\infty}^t U(t,\sigma)f(\sigma)d\sigma, \;\; t \in \mathbb{R},$$

defined as

$$\lim_{r \searrow -\infty} \int_r^t U(t,\sigma)f(\sigma)d\sigma.$$

It is clear that for each $r < t$, the integral $\int_r^t U(t,\sigma)f(\sigma)d\sigma$ exists. Moreover $\| \int_r^t U(t,\sigma)f(\sigma)d\sigma\| \leq \frac{N}{|\omega|}\|f\|_\infty$; thus $\int_{-\infty}^t U(t,\sigma)f(\sigma)d\sigma$ is absolutely convergent.

Now we prove that $y \in AP^{(n)}(\mathbb{X})$. First, it is easy to show that $y(t) \in C^{(n)}(\mathbb{X})$. Further, in view of Lemma 3.4, given $\varepsilon > 0$, we can find $l(\varepsilon) > 0$ such that every interval I of length $l(\varepsilon)$ contains a number τ with the property that

$$\|U(t+\tau, s+\tau) - U(t,s)\| \leq \varepsilon e^{-\frac{\omega}{2}(t-s)}$$

for all $t - s \geq \varepsilon$ and

$$\|f^i(t+\tau) - f^i(t)\| < \eta \quad \text{for all} \quad t \in \mathbb{R}, \; i = 0, 1, ..., n,$$

where $\eta = \eta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Therefore,

$$\begin{aligned}
\|y(t+\tau) - y(t)\|_n &= \left\| \int_{-\infty}^{t+\tau} U(t+\tau, s) f(s) ds - \int_{-\infty}^{t} U(t,s) f(s) ds \right\|_n \\
&= \left\| \int_0^{+\infty} U(t+\tau, t+\tau-s) f(t+\tau-s) ds \right. \\
&\quad \left. - \int_0^{+\infty} U(t, t-s) f(t-s) ds \right\|_n \\
&\le \left\| \int_0^{+\infty} U(t+\tau, t+\tau-s)[f(t+\tau-s) - f(t-s)] ds \right\|_n + \\
&\quad \left\| \left(\int_\varepsilon^{+\infty} + \int_0^\varepsilon \right) [U(t+\tau, t+\tau-s) - U(t, t-s)] f(t-s) ds \right\|_n \\
&\le \int_0^{+\infty} N(n+1)\eta e^{-\omega s} ds + \int_\varepsilon^{+\infty} \varepsilon e^{-\frac{\omega}{2}s} \|f\|_n ds + 2N\varepsilon \|f\|_n \\
&= \frac{N(n+1)}{\omega}\eta + \frac{2\varepsilon\|f\|_n}{\omega} + 2N\varepsilon\|f\|_n,
\end{aligned}$$

which gives that $y(t) \in AP^{(n)}(\mathbb{X})$.

Now let

$$y(s) = \int_{-\infty}^{s} U(s,\sigma) f(\sigma) d\sigma.$$

Then

$$U(t,s)y(s) = \int_{-\infty}^{s} U(t,\sigma) f(\sigma) d\sigma.$$

If we let $t \ge s$, then

$$\begin{aligned}
\int_s^t U(t,\sigma) f(\sigma) d\sigma &= \int_{-\infty}^{t} U(t,\sigma) f(\sigma) d\sigma - \int_{-\infty}^{s} U(t,\sigma) f(\sigma) d\sigma \\
&= y(t) - U(t,s)y(s),
\end{aligned}$$

therefore

$$y(t) = U(t,s)y(s) + \int_s^t U(t,\sigma) f(\sigma) d\sigma.$$

If we fix $x(s) = y(s)$, then $x(t) = y(t)$, that is $x \in AP^{(n)}(\mathbb{X})$. Uniqueness can be proved as follows.

Suppose x_1, x_2 are two solutions be two solutions to Eq. (1) in $AP^{(n)}(\mathbb{X})$. Let $z = x_1 - x_2$. Then

$$z'(t) = A(t)z(t), \ t \in \mathbb{R},$$

and

$$z(t) = U(t,s)z(s), \ t \ge s.$$

We also have

$$\|z(t)\| \le N e^{-\omega(t-s)}.$$

Take a sequence of real numbers (s_n), such that $s_n \to -\infty$. For any fixed $t \in \mathbb{R}$, we can find a subsequence $(s_{n_k}) \subset (s_n)$ such that $s_{n_k} < t$ for all $k = 1, 2, ...$ Using the fact that $\omega > 0$, we obtain $z = 0$. This completes the proof. ∎

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(Received: 13.07.2006)