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On new multiple extension of Hilbert's integral inequality

Abstract. This paper gives a new multiple extension of Hilbert's integral inequality generalizing many known results

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1. Introduction. Let $f, g \geq 0$ satisfy

$$0 < \int_0^\infty f^2(t)\,dt < \infty \text{ and } 0 < \int_0^\infty f^2(t)\,dt < \infty.$$

Then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dxdy < \pi \left(\int_0^\infty f^2(t)\,dt \int_0^\infty g^2(t)\,dt \right)^{\frac{1}{2}}, \tag{1}$$

where the constant factor π is the best possible (cf. Hardy et al. [2]). Inequality (1) is well known as Hilbert's integral inequality. This inequality had been extended by Hardy ([1]) as follows.

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $f, g \geq 0$ satisfy

$$0 < \int_0^\infty f^p(t)\,dt < \infty \text{ and } \int_0^\infty g^q(t)\,dt < \infty,$$

then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}\,dxdy < \frac{\pi}{sin(\pi/p)}\left(\int_0^\infty f^p(t)\,dt\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(t)\,dt\right)^{\frac{1}{q}}, \tag{2}$$

where the constant factor $\frac{\pi}{sin(\pi/p)}$ is the best possible. Inequality (2) is called Hardy-Hilbert's integral inequality and is important in analysis and application (cf. Mitrinovic et al. [3]).

Gradually B. Yang gave the following extensions of (2) as follows :

Theorem 1.1 ([4]) *If $\lambda > 2 - min\{p, g\}$ and $f, g \geq 0$ satisfy*

$$0 < \int_0^\infty t^{1-\lambda} f^p(t)\,dt < \infty \text{ and } \int_0^\infty t^{1-\lambda} g^q(t)\,dt < \infty,$$

then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}\,dxdy < k_\lambda(p)\left(\int_0^\infty f^{1-\lambda}(t)\,dt\right)^{\frac{1}{p}}\left(\int_0^\infty g^{1-\lambda}(t)\,dt\right)^{\frac{1}{q}}, \tag{3}$$

where the constant factor $k_\lambda(p) = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$ is the best possible (B is the beta function).

Theorem 1.2 ([5]) *If $n \in \mathbb{N} - \{1\}$, $p_i > 1$, $\sum_{i=1}^n \frac{1}{p_i} = 1$, $\lambda > n - min_{1\leq i\leq n}\{p_i\}$, $f_i \geq 0$ satisfy*

$$0 < \int_0^\infty t^{n-1-\lambda} f_i^{p_i}(t)\,dt < \infty \qquad (i = 1, \dots, n),$$

then

$$\int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n x_j\right)^\lambda} \prod_{i=1}^n f_i(x_i)\,dx_1 \dots dx_n \tag{4}$$

$$< \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{p_i+\lambda-n}{p_i}\right)\left(\int_0^\infty t^{n-1-\lambda} f_i^{p_i}(t)\,dt\right)^{\frac{1}{p_i}},$$

where the constant factor $\frac{1}{\Gamma(\lambda)}\prod_{i=1}^n \Gamma\left(\frac{p_i+\lambda-n}{p_i}\right)$ is the best possible.

Inequality (4) is a multiple extension of inequalities (1), (2) and (3).

THEOREM 1.3 ([6]) *If* $n \in \mathbb{N} - \{1\}$, $p_i > 1$, $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, $\lambda > 0$, $f_i \geq 0$ *satisfy*

$$0 < \int_0^\infty t^{p_i - 1 - \lambda} f_i^{p_i}(t)\, dt < \infty \qquad (i = 1, \dots, n),$$

then

$$\int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^{n} x_j\right)^\lambda} \prod_{i=1}^{n} f_i(x_i)\, dx_1 \dots dx_n \tag{5}$$
$$< \frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \Gamma\left(\frac{\lambda}{p_i}\right) \left(\int_0^\infty t^{p_i - 1 - \lambda} f_i^{p_i}(t)\, dt\right)^{\frac{1}{p_i}},$$

where the constant factor $\frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \Gamma\left(\frac{\lambda}{p_i}\right)$ *is the best possible.*

The object of this paper is to give a genetralizaton of Hardy-Hilbert's integral inequality which includes all the previous results. First we need some lemmas

2. Auxiliary lemmas.

LEMMA 2.1 *If* $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, *then we have*

$$\prod_{j=1}^{n} \left[f_j'(x_j)\right]^{\frac{1}{p_j} - 1} \prod_{\substack{i=1 \\ i \neq j}}^{n} \left[f_i'(x_i)\right]^{\frac{1}{p_j}} = 1.$$

PROOF

$$\begin{aligned}
1 &= \prod_{j=1}^{n} \left[f_j'(x_j)\right]^{\sum_{i=1}^{n} \frac{1}{p_i} - 1} = \prod_{j=1}^{n} \left[f_j'(x_j)\right]^{-1} \prod_{j=1}^{n} \left[f_j'(x_j)\right]^{\sum_{i=1}^{n} \frac{1}{p_i}} \\
&= \prod_{j=1}^{n} \left[f_j'(x_j)\right]^{-1} \prod_{j=1}^{n} \prod_{i=1}^{n} \left[f_j'(x_j)\right]^{\frac{1}{p_i}} = \prod_{j=1}^{n} \left[f_j'(x_j)\right]^{-1} \prod_{i=1}^{n} \prod_{j=1}^{n} \left[f_i'(x_i)\right]^{\frac{1}{p_j}} \\
&= \prod_{j=1}^{n} \left[f_j'(x_j)\right]^{-1} \prod_{j=1}^{n} \prod_{i=1}^{n} \left[f_i'(x_i)\right]^{\frac{1}{p_j}} = \prod_{j=1}^{n} \left[f_j'(x_j)\right]^{-1} \prod_{i=1}^{n} \left[f_i'(x_i)\right]^{\frac{1}{p_j}} \\
&= \prod_{j=1}^{n} \left[f_j'(x_j)\right]^{\frac{1}{p_j} - 1} \prod_{j \neq i=1}^{n} \left[f_i'(x_i)\right]^{\frac{1}{p_j}}.
\end{aligned}$$

■

LEMMA 2.2 *If* $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, *then we have*

$$\prod_{j=1}^{n} \left[f_j(x_j)\right]^{(a_j - 1)\left(\frac{1}{p_j} - 1\right)} \prod_{\substack{i=1 \\ i \neq j}}^{n} \left[f_i(x_i)\right]^{\frac{a_i - 1}{p_j}} = 1.$$

Proof

$$
\begin{aligned}
1 &= \prod_{j=1}^{n} [f_j(x_j)]^{(1-a_j)+\sum_{i=1}^{n} \frac{a_j-1}{p_i}} \\
&= \prod_{j=1}^{n} [f_j(x_j)]^{(1-a_j)} [f_j(x_j)]^{\sum_{i=1}^{n} \frac{a_j-1}{p_i}} \\
&= \prod_{j=1}^{n} [f_j(x_j)]^{(1-a_j)} \prod_{i=1}^{n} [f_j(x_j)]^{\frac{a_j-1}{p_i}} \\
&= \left(\prod_{j=1}^{n} [f_j(x_j)]^{(1-a_j)}\right) \left(\prod_{j=1}^{n}\prod_{i=1}^{n} [f_j(x_j)]^{\frac{a_j-1}{p_i}}\right) \\
&= \left(\prod_{j=1}^{n} [f_j(x_j)]^{(1-a_j)}\right) \left(\prod_{i=1}^{n}\prod_{j=1}^{n} [f_i(x_i)]^{\frac{a_i-1}{p_j}}\right) \\
&= \left(\prod_{j=1}^{n} [f_j(x_j)]^{(1-a_j)}\right) \left(\prod_{j=1}^{n}\prod_{i=1}^{n} [f_i(x_i)]^{\frac{a_i-1}{p_j}}\right) \\
&= \prod_{j=1}^{n} [f_j(x_j)]^{(1-a_j)} \prod_{i=1}^{n} [f_i(x_i)]^{\frac{a_i-1}{p_j}} \\
&= \prod_{j=1}^{n} [f_j(x_j)]^{(1-a_j)+\frac{a_j-1}{p_j}} \prod_{\substack{i=1\\ i\neq j}}^{n} [f_i(x_i)]^{\frac{a_i-1}{p_j}} \\
&= \prod_{j=1}^{n} [f_j(x_j)]^{(a_j-1)\left(\frac{1}{p_j}-1\right)} \prod_{\substack{i=1\\ i\neq j}}^{n} [f_i(x_i)]^{\frac{a_i-1}{p_j}} .
\end{aligned}
$$

■

Lemma 2.3 *If $a_1 + a_0 = 0$, then we have*

(i) $$\prod_{i=1}^{n} \frac{\Gamma(a_i)\Gamma\left(\lambda-\sum_{k=1}^{i} a_k\right)}{\Gamma\left(\lambda-\sum_{k=1}^{i-1} a_k\right)} = \frac{\Gamma\left(\lambda-\sum_{k=1}^{n} a_k\right)}{\Gamma(\lambda)} \prod_{i=1}^{n} \Gamma(a_i)$$

(ii) $$\prod_{i=1}^{n} B\left(a_i, \lambda - \sum_{j\neq k=1}^{i} a_k\right) = \prod_{i=1}^{n} \frac{\Gamma(a_i)\Gamma\left(\lambda-\sum_{j\neq k=1}^{i} a_k\right)}{\Gamma\left(\lambda-\sum_{j\neq k=1}^{i-1} a_k\right)} = \frac{\Gamma\left(\lambda-\sum_{j\neq k=1}^{n} a_k\right)}{\Gamma(\lambda)\Gamma(a_j)} \prod_{i=1}^{n} \Gamma(a_i),$$

where the empty sum is to be understood to be nil.

Proof The case (i) follows just by opening the product.

(ii). We have

$$\prod_{i=1}^{n}\frac{\Gamma(a_i)\Gamma\left(\lambda-\sum\limits_{j\neq k=1}^{i}a_k\right)}{\Gamma\left(\lambda-\sum\limits_{j\neq k=1}^{i-1}a_k\right)}=\prod_{i=1}^{n}\frac{\Gamma(a_i)\Gamma\left(\lambda-\sum\limits_{k=1}^{i}a_k\right)}{\Gamma\left(\lambda-\sum\limits_{k=1}^{i-1}a_k\right)}$$

$$\times\prod_{i=j+1}^{n}\frac{\Gamma(a_i)\Gamma\left(\lambda-\sum\limits_{j\neq k=1}^{i}a_k\right)}{\Gamma\left(\lambda-\sum\limits_{j\neq k=1}^{i-1}a_k\right)}\times\prod_{i=j-1}^{n-1}\frac{\Gamma\left(\lambda-\sum\limits_{k=1}^{i}a_k\right)}{\Gamma(a_{i+1})\Gamma\left(\lambda-\sum\limits_{k=1}^{i+1}a_k\right)}$$

$$=\frac{\Gamma\left(\lambda-\sum\limits_{k=1}^{n}a_k\right)}{\Gamma(\lambda)}\prod_{i=1}^{n}\Gamma(a_i)$$

$$\times\frac{\Gamma\left(\lambda-\sum\limits_{j\neq k=1}^{n}a_k\right)}{\Gamma\left(\lambda-\sum\limits_{k=1}^{i-1}a_k\right)}\prod_{i=j+1}^{n}\Gamma(a_i)$$

$$\times\frac{\Gamma\left(\lambda-\sum\limits_{k=1}^{j-1}a_k\right)}{\Gamma\left(\lambda-\sum\limits_{k=1}^{n}a_k\right)}\frac{1}{\prod\limits_{i=j}^{n}\Gamma(a_i)}=\frac{\Gamma\left(\lambda-\sum\limits_{j\neq k=1}^{n}a_k\right)}{\Gamma(\lambda)\Gamma(a_j)}\prod_{i=1}^{n}\Gamma(a_i).$$

■

3. Main result. We state and prove the following

THEOREM 3.1 *If* $n\in\mathbb{N}-\{1\}$, $p_i>1$, $\sum_{i=1}^{n}\frac{1}{p_i}=1$, $a_i>0$, $\lambda>\sum_{i=1}^{n}a_i$, $F_i,f_i\geq 0$, $f_i^{'}>0$, $f_i(0)=0$, $f_i(\infty)=\infty$ *satisfy*

$$0<\int_0^\infty x_j^{p_j(1-a_j)+\sum_{k=1}^{n}a_k-\lambda-1}\left[f_i^{'}(x_j)\right]^{1-p_j}F_j^{p_j}(x_j)\,dx_j<\infty\qquad(j=1,\ldots,n),$$

then

$$\text{(6)}\qquad\int_0^\infty\cdots\int_0^\infty\frac{F_1(x_1)\ldots F_n(x_n)}{\left(\sum_{j=1}^{n}f_j(x_j)\right)^\lambda}\,dx_1\ldots dx_n$$

$$<\prod_{j=1}^{n}\left(K_j\int_0^\infty[f_j(x_j)]^{p_j(1-a_j)+\sum_{i=1}^{n}a_i-\lambda-1}\left[f_j^{'}(x_j)\right]^{1-p_j}F_j^{p_j}(x_j)\,dx_j\right)^{\frac{1}{p_j}},$$

where $K_j = \dfrac{\Gamma\left(\lambda - \sum\limits_{j\neq k=1}^{n} a_k\right)}{\Gamma(\lambda)\Gamma(a_j)} \prod\limits_{i=1}^{n} a_i.$

Proof By virtue of Lemmas 2.1 and 2.2, we have

$$\int_0^\infty \cdots \int_0^\infty \frac{F_1(x_1)\dots F_n(x_n)}{\left(\sum_{j=1}^n f_j(x_j)\right)^\lambda} dx_1 \dots dx_n$$

$$= \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^n \frac{[f_j(x_j)]^{(a_j-1)\left(\frac{1}{p_j}-1\right)} \left[f_j'(x_j)\right]^{\frac{1}{p_j}-1} F_j(x_j) \prod\limits_{j\neq i=1}^n [f_i(x_i)]^{\frac{a_i-1}{p_j}} [f_i'(x_i)]^{\frac{1}{p_j}}}{\left(\sum_{j=1}^n f_j(x_j)\right)^{\lambda/p_j}} dx_j$$

$$\leq \prod_{j=1}^n \left(\int_0^\infty \cdots \int_0^\infty \frac{[f_j(x_j)]^{(a_j-1)(1-p_j)} \left[f_j'(x_j)\right]^{1-p_j} F_j^{p_j}(x_j) \prod\limits_{j\neq i=1}^n [f_i(x_i)]^{a_i-1} f_i'(x_i)}{\left(\sum_{j=1}^n f_j(x_j)\right)^\lambda} dx_j \right)^{\frac{1}{p_j}}$$

$$= \prod_{j=1}^n P_j^{1/p_j},$$

say, where

$$P_j = \int_0^\infty [f_j(x_j)]^{(a_j-1)(1-p_j)} \left[f_j'(x_j)\right]^{1-p_j} F_j^{p_j}(x_j)\, dx_1 \dots dx_n \int_0^\infty \cdots \int_0^\infty \frac{\prod\limits_{j\neq i=1}^n [f_i(x_i)]^{a_i-1} f_i'(x_i)}{\left(\sum_{j=1}^n f_j(x_j)\right)^\lambda} dx_j$$

Now we consider

$$\int_0^\infty \cdots \int_0^\infty \frac{\prod\limits_{j\neq i=1}^n [f_i(x_i)]^{a_i-1} f_i'(x_i)}{\left(\sum_{j=1}^n f_j(x_j)\right)^\lambda} dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_n$$

$$= \int_0^\infty \cdots \int_0^\infty \prod_{j\neq i=1}^n [f_i(x_i)]^{a_i-1} f_i'(x_i)\, dx_2 \dots dx_{j-1} dx_{j+1} \dots dx_n \int_0^\infty \frac{[f_1(x_1)]^{a_1-1} f_1'(x_1)}{\left(f_j(x_j) + \sum_{j\neq k=1}^n f_k(x_k)\right)^\lambda} dx_1$$

$$= \int_0^\infty \cdots \int_0^\infty \frac{\prod\limits_{j\neq i=2}^n [f_i(x_i)]^{a_i-1} f_i'(x_i)}{\left(f_j(x_j) + \sum_{j\neq k=1}^n f_k(x_k)\right)^{\lambda-a_i}} dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_n$$

$$\times \int_0^\infty \frac{\left(\frac{f_1(x_1)}{f_j(x_j+f_2(x_2)+\cdots+f_n(x_n))}\right)^{a_1-1} \left(\frac{f_1'(x_1)}{f_j(x_j+f_2(x_2)+\cdots+f_n(x_n))}\right)}{\left(1 + \frac{f_1(x_1)}{f_j(x_j+f_2(x_2)+\cdots+f_n(x_n))}\right)^\lambda} dx_1$$

Since the last integral is equal to $\int\limits_0^\infty \frac{t^{a_1-1}}{(1+t)^\lambda} dt$, we have successively

$$= B(a_1, \lambda - a_1) B(a_2, \lambda - a_1 - a_2) \dots B(a_{j-1}, \lambda - a_1 - \cdots - a_{j-1})$$

$$\times \int_0^\infty \cdots \int_0^\infty \frac{\prod\limits_{j\neq i=j+1}^n [f_i(x_i)]^{a_i-1} f_i'(x_i)}{f_j(x_j) + f_{j+1}(x_{j+1}) + \cdots + f_n(x_n))^{\lambda - a_1 - \cdots - a_{j-1}}} dx_1 \dots dx_{j+1} dx_{j+1} \dots dx_n$$

$$= \dots\dots$$

$$B(a_1, \lambda - a_1) B(a_2, \lambda - a_1 - a_2) \dots B(a_{j-1}, \lambda - a_1 - \cdots - a_{j-1})$$

$$\times B(a_{j+1}, \lambda - a_1 - \cdots - a_{j-1} - a_{j+1}) B(a_{j+2}, \lambda - a_1 - \cdots - a_{j-1} - a_{j+1} - a_{j+2})$$

$$\cdots \times B(a_n, \lambda - a_1 - \cdots - a_{j-1} - a_{j+1} - \cdots a_n) \frac{1}{[f_j(x_j)]^{\lambda - \sum_{j\neq i=1}^n a_i}}.$$

We obtain , via Lemma 2.3,

$$P_j = \prod_{j=1}^{n} B\left(a_j, \lambda - \sum_{j\neq k=1}^{n} a_k\right) \int_0^\infty [f_j(x_j)]^{p_j(1-a_j)+\sum_{i=1}^n a_i-\lambda-1} \left[f_j'(x_j)\right]^{1-p_j} F_j^{p_j}(x_j)\, dx_j$$

$$= \left(\frac{\Gamma\left(\lambda - \sum_{j\neq k=1}^{n} a_k\right)}{\Gamma(\lambda)\Gamma(a_j)} \prod_{j=1}^{n} \Gamma(a_j) \int_0^\infty [f_j(x_j)]^{p_j(1-a_j)+\sum_{i=1}^n a_i-\lambda-1} \left[f_j'(x_j)\right]^{1-p_j} F_j^{p_j}(x_j)\, dx_j\right)$$

This completes the proof of the theorem. ■

Remark 3.2 It is not difficult to check that :

(i) Inequality (4) follows from inequality (6) by putting $a_i = (p_i + \lambda - n)/p_i$, $f_i(x) = x$.

(ii) Inequality (5) follows from inequality (6) by putting $a_i = \lambda/p_i$, $f_i(x) = x$.

As inequalities (1), (2) and (3) follows from (4), then inequality (6) covers all the previous inequalities.

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