ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE
XLVI (2) (2006), 245-254

Jan Milewski*

Holomorphons and the standard almost complex structure on S^6

Abstract. We consider Euler–Lagrange equations of families of nonnegative functionals defined on tensor fields of the type $(1,1)$, which are equal to zero only for complex structures tensor fields. As a solution of the equations we define the notion of holomorphon to distinguish a new class of tensor fields on Riemannian manifolds. Next, as our main result, we construct a holomorphon on the 6–dimensional sphere S^6.

1991 Mathematics Subject Classification: 53C15, 53C21, 58E15, 58E30.

Key words and phrases: Riemannian geometry, almost complex structure, Nijenhuis tensor, variational principle.

1. Introduction. Let M be a Riemannian manifold and let $d\mu$ be the Riemannian volume form on M. In the paper we consider the following integrals

$$F_k[t] = \int_M (||E + t^2||^2 + ||N_t||^{2k}) d\mu \tag{1}$$

and

$$F_{k,l}[t] = \int_M (||E + t^2||^2 + ||N_t||^{2k} + ||\nabla t^2||^{2l}) d\mu \ , \tag{2}$$

where $k, l \in N$, E is the identity tensor field on M, t is a tensor field of the type $(1,1)$, N_t is the Nijenhuis tensor of t and ∇ denotes the Riemannian covariant derivative.

*The work has been supported by projects: grant KBN No 1 P03A 039 29 and PB-43-047/06 BW of Poznań University of Technology.

The Newlander–Nirenberg theorem [1] implies that above integrals vanish iff t is a complex structure on M.

It is very well known that S^2 admits integrable complex structures and S^6 – almost complex structures [2]. Moreover S^6 does not admit any orthogonal integrable complex strucutre [3]. Other spheres do not possess any almost complex structure [4], but one can observe that for $k, l < n$ the infimum of the functional (2) on S^{2n} equals zero (it is easy to construct a minimizing sequence of tensor fields (t_n) such that $\lim_{n\longrightarrow\infty} F_{kl}[t_n] = 0$).

The question of the existence of any integrable complex structure on S^6 is an open one [5, 6], however the study of our functional may put some light on the problem. The aim of the paper is to present the Euler–Lagrange equations for the integral (2). By study of some differential properties of the standard almost complex structure on S^6, as our main result (Theorem 5.4), we show that a solution of the Euler–Lagrange equation is expressed by the standard almost complex structure multiplied by a constant depending on k.

The paper is organised as follows. In Section 2 we give some properties of the Nijenhuis tensor. In Section 3 we define holomorphons as solutions of the Euler–Lagrange equation of the functional (2). In Section 4 we deal with the vector product algebras in R^6 and R^7 and the Cayley algebra R^8, what is very usefull in the study of the Nijenhuis tensor of the standard almost complex structure on S^6 and its covariant derivative in the next section. Finally we give a solution of the Euler–Lagrange equation of the integral (2) for S^6.

2. Nijenhuis tensor. Let M be a differential manifold, and t be a tensor field of the type $(1, 1)$ on M. The Nijenhuis tensor of t is a tensor field of the type $(1, 2)$ given by the formula

$$N_t(A, B) = t^2[A, B] + [tA, tB] - t[tA, B] - t[A, tB] \ . \tag{3}$$

For a symmetric connection we have

$$N_t(A, B) = (\nabla_{tA}t)B - (\nabla_{tB}t)A - t\left\{(\nabla_A t)B - (\nabla_B t)A\right\} \ . \tag{4}$$

So the components of the Nijenhuis tensor are given by

$$N^k{}_{ij} = t^k{}_l(t^l{}_{i;j} - t^l{}_{j;i}) + t^l{}_i t^k{}_{j;l} - t^l{}_j t^k{}_{i;l} \tag{5}$$

(in this paper we use the Einstein summing convention).

Lemma 2.1 (An algebraic interpretation of the Nijenhuis tensor.) *Let A be a real algebra and let the tensor t be given by means of multiplication in the algebra:*

$$t_p(X) = pX \ \textit{for}\ p \in A, X \in T_pA = A \ . \tag{6}$$

Then the Nijenhuis tensor is equal to the difference of associators:

$$N_t(X, Y) = [(pX)Y - p(XY)] - [(pY)X - p(YX)] \ . \tag{7}$$

Proof The proof is a consequence of (3) or (4) (c.f. [7]). ∎

Definition 2.2 Let P be a submanifold of M, and let T be a tensor field on M of the type $(1,k)$. We say that T is restrictible to P if for any $x \in P$ and any $X_1,\ldots,X_k \in T_xP$ the vector $T(X_1,\ldots,X_k) \in T_xP$.

When the tensor field T is restrictible then the restriction of T to $\bigotimes^k TP$ is a tensor field on P which will be denoted by $T||_P$. From the definition above and the definition of Nijenhuis tensor we get the following

Lemma 2.3 *If P is a submanifold of M and t is a restrictible tensor field of the type $(1,1)$, then the Nijenhuis tensor N_t of t is also restrictible and*

$$(N_t)||_P = N_{t||_P} \ . \tag{8}$$

3. Holomorphons.. Except for the functional (2), we consider also the following functional

$$G[t] = \int_M ||N_t||^2 d\mu \ . \tag{9}$$

Theorem 3.1 *The Euler–Lagrange derivatives of above functionals are:*

$$\frac{\delta G}{\delta t^q{}_r} = 4\left\{(N_q{}^{ij}t^r{}_{i;j} + N_l{}^{ri}t^l{}_{i;q}) - (N_i{}^{rs}t^i{}_q + N_q{}^{ir}t^s{}_i)_{;s}\right\} \tag{10}$$

and

$$\frac{\delta F_{k,l}}{\delta t^q{}_r} = 2\left\{(E+t^2)_q{}^s t^r{}_s + (E+t^2)_s{}^r t^s{}_q\right\} + k||N_t||^{2k-2}\frac{\delta G}{\delta t^q{}_r} - \tag{11}$$

$$2l||\nabla t^2||^{2l-2}(\Delta t^2{}_q{}^s t^r{}_s + \Delta t^2{}_s{}^r t^s{}_q) \ .$$

Proof Our approach will be analogical to derivation of the field equations from principles of least action in mathematical physics or field theory [9, 10]. Henceforth, we use the same notation for intergrands and the corresponding functionals. It is very well known that

$$\frac{\delta G}{\delta t^q{}_r} = \frac{\partial G}{\partial t^q{}_r} - \left(\frac{\partial G}{\partial t^q{}_{r;s}}\right)_{;s} \ . \tag{12}$$

Further from the equation (5) we have

$$\frac{\partial G}{\partial t^q{}_r} = 2N_k{}^{ij}\frac{\partial N^k{}_{ij}}{\partial t^q{}_r} = \tag{13}$$

$$2N_k{}^{ij}[\delta^k_q\delta^r_l(t^l{}_{i;j} - t^l{}_{j;i}) + \delta^l_q\delta^r_i t^k{}_{j;l} - \delta^l_q\delta^r_j t^k{}_{i;l}] =$$

$$4({N_q}^{ij}{t^r}_{i;j} + {N_k}^{ri}{t^k}_{i;q})\ .$$

Analogously

$$\frac{\partial G}{\partial {t^q}_{r;s}} = 2{N_k}^{ij}\frac{\partial {N^k}_{ij}}{\partial {t^q}_{r;s}} = \tag{14}$$

$$2{N_k}^{ij}[{t^k}_l(\delta^l_q\delta^r_i\delta^s_j - \delta^l_q\delta^r_j\delta^s_i) + {t^l}_i\delta^k_q\delta^r_j\delta^s_l - {t^l}_j\delta^k_q\delta^r_j\delta^s_l] = 4({N_i}^{rs}{t^i}_q + {N_q}^{ir}{t^s}_i)\ .$$

So we have proved (10). The formula (11) is a result of the equalities

$$\frac{\partial ||E+t^2||^2}{\partial {t^q}_r} = 2\left\{{(E+t^2)_q}^k{t^r}_k + {(E+t^2)_k}^r{t^k}_q\right\} \tag{15}$$

and

$$\frac{\delta ||\nabla s||^2}{\delta {s^q}_r} = -2\Delta {s_q}^r \tag{16}$$

for any $(1,1)$ tensor field s. ∎

Definition 3.2 A *holomorphon* is a solution of the Euler–Lagrange equations' system of the functional (2).

A holomorphon is a critical point of the functional $F_{k,l}$. On a complex manifold any complex structure J is a holomorphon for any k, l because J is an absolute minimum of $F_{k,l}$.

4. Vector products in $\mathbf{R}^6$ and $\mathbf{R}^7$. Cayley numbers.. Let us define the vector product $\times_6$ and the complex structure e in $\mathbf{R}^6$ in the following way

$$(X_1, X_2)\times_6(A_1, A_2) = (X_1\times A_1 - X_2\times A_2, -X_1\times A_2 - X_2\times A_1) \tag{17}$$

and

$$e(A_1, A_2) = (-A_2, A_1)\ , \tag{18}$$

for $X_1, X_2, A_1, A_2 \in \mathbf{R}^3$.

We define also vector product $\times_7$ in $\mathbf{R}^7$ by the formula

$$X\times_7 Y = (\bar{X}\times_6\bar{Y} + e(x\bar{Y} - y\bar{X}), - < \bar{X}, e\bar{Y} >)\ , \tag{19}$$

where $X := (\bar{X}, x), Y := (\bar{Y}, y) \in \mathbf{R}^7$, $\bar{X}, \bar{Y} \in \mathbf{R}^6$ and $x, y \in \mathbf{R}$. Here and in the sequel $<,>$ denotes the standard scalar product in $\mathbf{R}^n$.

The vector product $\times_6$ is antibilinear with respect to the complex structure:

$$X\times_6(eY) = (eX)\times_6 Y = -e(X\times_6 Y)\ , \tag{20}$$

for $X, Y \in \mathbf{R}^6$ and vectors

$$X, eX, X\times_6 Y, e(X\times_6 Y) \tag{21}$$

are pairwise perpendicular.

The mixed product satisfies the following relation

$$< X, Y \times_n Z >=< X \times_n Y, Z > \tag{22}$$

for $X, Y, Z \in \mathbf{R}^n$, $n = 3, 6, 7$. We denote the mixed product also by $< X, Y, Z >$.

The double vector products for $\times_6$ and $\times_7$ satisfy the identities:

$$X \times_6 (Y \times_6 Z) =< X, Z > Y - < X, Y > Z$$

$$- < X, eZ > eY + < X, eY > eZ \ . \tag{23}$$

for $X, Y, Z \in \mathbf{R}^6$, and

$$X \times_7 (Y \times_7 Z) =< X, Z > Y - < X, Y > Z +$$

$$(e(x\bar{Y} \times_6 \bar{Z} + < \bar{X}, e\bar{Y} > \bar{Z})_{\mathrm{cykl(X,Y,Z)}}, < e\bar{X}, \bar{Y}, \bar{Z} >) \ , \tag{24}$$

for $X, Y, Z \in \mathbf{R}^7$. In particular for $X = Y$ we have

$$X \times_7 (X \times_7 Z) =< X, Z > X - ||X||^2 Z \ . \tag{25}$$

Lemma 4.1 *Let $n = 3,\ 6,\ 7$ and let (e_i) be an orthonormal basis in $\mathbf{R}^n$. Then the following formula*

$$< X \times_n e_i, Y \times_n e_i >= a_n < X, Y > \tag{26}$$

holds for any $X, Y \in \mathbf{R}^n$, where $a_3 = 2$, $a_6 = 4$, $a_7 = 6$.

Proof We deal with the case $n = 6$.

$$< X \times_6 e_i, Y \times_6 e_i >=< X, e_i \times_6 (Y \times_6 e_i) >$$

$$=< X, (||e_i||^2 Y - < Y, e_i > e_i + < e_i, eY > ee_i) >$$

$$= 6 < X, Y > - < X, Y > + < X, ee_i >< eY, e_i >= 4 < X, Y > \ .$$

For $n = 3,\ 7$ the proof is analogical. ∎

The Cayley algebra or octonion algebra can be defined as an eight–dimensional algebra equipped with the following multiplication:

$$(x, X)(y, Y) = (xy - < X, Y >, xY + yX + X \times_7 Y) \ , \tag{27}$$

where $x, y \in \mathbf{R}$ and $X, Y \in \mathbf{R}^7$. The real and the imaginary part of an octonion is given by

$$\mathrm{Re}(x, X) = x \ \ , \ \ \mathrm{Im}(x, X) = X \ . \tag{28}$$

Let $(e_0, e_1, \ldots, e_7)$ be the standard basis in $\mathbf{R}^8$. In Cayley algebra the same basis we denote as

$$(1, i, j, k, ei, ej, ek, e_7) \ . \tag{29}$$

We see that for example $ei = e_7 i$, where e denotes the complex structure in R^6 and $e_7 i$ is the multiplication of octonions. Our notation is different from the standard numeration of the basis of the Cayley algebra, what is a consequence that we defined the multiplication of octonions by means of the vector product $\times_7$.

The Cayley algebra is a nonassociative algebra but it is very well known that it is an alternative one [8] (it means that any subalgebra generated by two elements is associative) what one can check using the property (25).

The standard almost complex structure on S^6 is given by means of the vector product in $\mathbf{R}^7$:

$$J_p(X) = p \times_7 X \ \text{ for } \ p \in S^6, X \in T_p S^6 \ . \tag{30}$$

We define the vector product in $T_p S^6$ in the way:

$$X \times_p Y = X \times_7 Y - < p, X, Y > p \ . \tag{31}$$

Let a, b be unit orthogonal vectors in $T_p S^6$, and let b be orhogonal to Ja. Let $c = a \times_p b$. Then the table of multiplication of the Cayley algebra in the orthonormal basis

$$(1, a, b, c, Ja, Jb, Jc, p) \tag{32}$$

is the same as in (29). So the vector product $\times_p$ in $T_p S^6$ satisfies analogical formulas to the vector product $\times_6$ in $\mathbf{R}^6$. For $X, Y, Z \in T_p S^6$ the following formula

$$X \times_p (JY) = (JX) \times_p Y = -J(X \times_p Y) \tag{33}$$

is fulfilled. Vectors

$$X, JX, X \times_p Y, J(X \times_p Y) \tag{34}$$

are pairwise perpendicular. The mixed product satisfied the following relation

$$< X, Y \times_p Z > = < X \times_p Y, Z > \ . \tag{35}$$

The double vector product satisfies the identity:

$$\begin{aligned} X \times_p (Y \times_p Z) = & < X, Z > Y - < X, Y > Z \\ & - < X, JZ > JY + < X, JY > JZ \ . \end{aligned} \tag{36}$$

Moreover for any orthonormal basis (e_i) in $T_p S^6$ we get

$$< X \times_p e_i, Y \times_p e_i > = 4 < X, Y > \ . \tag{37}$$

Further we have

$$X \times_7 (Y \times_7 Z) =< X, Z > Y - < X, Y > Z + \quad (38)$$
$$(< X, JY > JZ)_{\text{cykl}(X,Y,Z)} + < JX, Y, Z >_p p$$

for $X, Y, Z \in T_pS^6$, where $<, \ ,>_p$ is the mixed product in T_pS^6.

5. A nontrivial holomorphon on S^6.. The Gauss form for the embending $S^6 \subset R^7$ is

$$\alpha(X, Y) = - < X, Y > p \ , \ \ p \in S^6, \ X, Y \in T_pS^6. \quad (39)$$

Hence we can calculate the covariant derivative of the structure J in the following way

$$(\nabla_X J)Y = X \times_7 Y + < X, p \times_7 Y > p - p \times_7 (< X, Y > p). \quad (40)$$

The last summand equals zero and we obtain

$$(\nabla_X J)Y = X \times_p Y \ . \quad (41)$$

Further from (4, 33) we get the Nijenhuis tensor of J

$$N_p(A, B) = -4p \times_7 (A \times_p B) \ . \quad (42)$$

The last result one can obtain also using (7, 8). Now from (25) and (26) we get the following

Lemma 5.1 *The square of the norm of the Nijenhuis tensor N of the standard almost complex structure J on S^6 is constant and its value is*

$$||N||^2 = 4^3 \cdot 6 \ . \quad (43)$$

Proof For any ortonormal basis in T_pS^6 at any point $p \in S^6$ we have

$$||N_p||^2 = 16 \sum_{i,j} ||p \times_7 (e_i \times_p e_j)||^2 =$$

$$= 16 < e_i \times_p e_j, e_i \times_p e_j >= 4^3 \cdot 6.$$

This finishes the proof. ∎

The covariant derivative of the Nijenhuis tensor is

$$(\nabla_X N)(A, B)_p = -4\{X \times_7 (A \times_7 B) - < p, X \times_7 (A \times_7 B) > p \quad (44)$$
$$- < X, B > A + < X, A > B\} \ ,$$

and using (38) we get

$$(\nabla_X N)(A, B) = -4(< A, JB > JX)_{\text{cykl}(A,B,X)} \ . \quad (45)$$

Remark 5.2 By straightforward verification, the tensors $J_{ij}, J_{ij;k}, N_{ijk}, N_{ijk;l}$ are antisymmetric.

Now we formulate and proof two theorems.

Theorem 5.3 *The Euler–Lagrange derivative of the functional (9) for $t = J$ is given by the following formula*

$$\frac{\delta G}{\delta t^q{}_r} = -4^4 J^r{}_q \ . \tag{46}$$

Proof We calculate succesive summands in the formula (10) in an orthonormal basis (e_i) in T_pS^6 for $t = J$. From (eq. 37) we get

$$\begin{aligned} N_{qij}J_{ri;j} &= -4 < e_q \times_7 p, e_i \times_p e_j >< e_r, e_j \times_p e_i > \\ &= -16 < e_r, p, e_q >= -16 J_{rq}. \end{aligned} \tag{47}$$

and by the Remark 5.2 we obtain

$$N_{qij}J_{ri;j} = N_{lri}J_{li;q} = -N_{irs}J_{iq;s} \ . \tag{48}$$

Further from (45) we have

$$\begin{aligned} -N_{qir;s}J_{si} &= 4(J_{qi}J_{rs} + J_{qr}J_{si} + J_{qs}J_{ir})J_{si} \\ &= 4(-J_{qi}\delta_{ri} + 6J_{q,r} - \delta_{qi}J_{ir}) = -16 J_{rq}. \end{aligned} \tag{49}$$

This finishes the proof. ∎

Theorem 5.4 *The tensor field*

$$t = uJ \ , \tag{50}$$

is a holomorphon on S^6 for the functional (2), where u is a constant satisfying the equality

$$Au^{8k-6} + u^2 - 1 = 0 \ , \ A := 4^{6k-3}6^{2k-2}k \ . \tag{51}$$

Proof From (46) we get

$$\frac{\delta F_{kl}}{\delta t^q{}_r} = 4u(1-u^2)J^r{}_q - k||N_t||^{2k-2}4^4u^3J^r{}_q \tag{52}$$

for $t = uJ$ because $||N_t||^2 = 4^3 6 u^4$ is a constant. So for u satisying (eq. 51) we have

$$\frac{\delta F_{kl}}{\delta t^q{}_r} = 0, \tag{53}$$

This finishes the proof. ∎

Finally we can formulate the following proposition [11]

Proposition 5.5 *The tensor field* $t = \frac{1}{\sqrt{65}} J$ *is a holomorphon for functionals* $F_{1,l}$. *The value of the functional* $F_{1,l}$ *on the holomorphon equals*

$$F[t] = \frac{64}{65} F[0] \; . \tag{54}$$

We omit the proof as a very elementary one.

6. Concluding remarks.. We have defined a new class of tensor fields which may be useful in deciding whether a manifold admits a complex structure. Moreover, the variational rule for holomorphons can have more general application. The Euler–Lagrange equations of the functional (2) can be considered on manifolds of arbitrary (not necesary even) dimension. These equations are second order partial differential equations. One can treat the Euler–Lagrange equations as mathematical physics equations of a kind of field theory.

We showed that the Euler–Lagrange derivative for the functional G for the standard almost complex structure J on S^6 equals $-4^4 J$, which is a very interesting property of the structure J. As a consequence we get that uJ is a holomorphon for a suitable constant u. We observe that the last summand $||\nabla t^2||^{2l}$ equals zero on the holomorphon uJ. Consequently this summand has no influence on our result. The summand was added for weakening a singularity of the functional (1). But in general resuls for the functionals (1) and (2) are not the same (for a holomorphon t such that $\nabla t^2 \neq 0$).

References

[1] A. Newlander and L. Nirenberg, *Complex analytic coordinates in almost complex manifolds*, Ann. Mat. **65** (1957), 391-404.

[2] S. Kobayashi and K. Nomizu, *Foundations of differential geometry I, II*, Interscience Publications, New York 1963, 1969.

[3] C. LeBrun, *Orthogonal complex structure on* S^6, Proc. Amer. Math. Soc. **101** (1987), 136-138.

[4] M. Karoubi, *K-Theory an introduction*, Springer-Verlag 1978.

[5] A. Gray, *A property of a Hypothetical Complex Structure on the Six Sphere*, Boll. Un. Mat. It. **B7(11)(2)** (1997), 251-255.

[6] A. Marshakov, A. I. Niemi *Yang–Mills, Complex Structures and Chern's Last Theorem*, Mod. Phys. Lett. **A20** (2005), 2583-2600.

[7] A. Winterhalder, *Linear Nijenhuis Tensors and the Construction of Integrable Systems*, Freiburg Preprint THEP 97/16.

[8] A. G. Kurosz, *Lectures in General Algebra*, GIFML, Moscow 1962 (in Russian).

[9] L. D. Landau and E. M. Lifshitz *Field Theory*, Nauka, Moscow 1973 (in Russian).

[10] B. S. DeWitt, *Dynamical Theory of Groups and Fields*, Gordon and Breach Science Publishers, New York 1965.

[11] J. Milewski, *Holomorphons*, Grant PB-43-035/04 BW, Poznań University of Technology 2004.

Jan Milewski
Institute of Mathematics, Poznań University of Technology
ul. Piotrowo 3A, 60-965 Poznań, Poland
E-mail: jsmilew@wp.pl

(Received: 14.06.2006)