ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE
XLVI (2) (2006), 215-231

Mohamed Morsli, Mannal Smaali

Characterization of the uniform convexity of the Besicovitch-Musielak-Orlicz spaces of almost periodic functions.

Abstract. We introduce the new class of Besicovitch-Musielak-Orlicz spaces of almost periodic functions $B^{\varphi}_{a.p.}$.The uniform convexity of this space is characterized in terms of it's generating functional φ.

2000 Mathematics Subject Classification: 46B20, 42A75.

Key words and phrases: Besicovitch-Orlicz space, Musielak-Orlicz space, almost periodic functions, uniform convexity .

1. Preliminaries. In the sequel the notation φ stands for a function $\varphi : \mathbb{R}\times [0, +\infty[\to [0, +\infty[$ satisfying the following conditions:

- $\varphi(t, u)$ is convex on $[0, +\infty[$ with respect to u.
- $\varphi(t, u)$ is periodic with respect to $t \in \mathbb{R}$.
- $\varphi(t, u)$ is continuous on $\mathbb{R}\times [0, +\infty[$.
- $\lim\limits_{u\to+\infty} \varphi(t, u) = +\infty$; $\lim\limits_{u\to+\infty} \frac{\varphi(t,u)}{u} = +\infty$ and $\lim\limits_{u\to 0} \frac{\varphi(t,u)}{u} = 0$.

Moreover it is supposed that $\inf\{\varphi(t, \alpha), t \in \mathbb{R}\} = C(\alpha) > 0, \forall\alpha > 0$ and $\varphi(t, 0) = 0, \forall t \in \mathbb{R}$.

We denote by $M(\mathbb{R})$ the space of all Lebesgue measurable functions on $\mathbb{R}$ and by $L^{\varphi}_{loc}(\mathbb{R})$ it's subspace of $\varphi-$locally integrable functions, i.e. the subspace of functions $f \in M(\mathbb{R})$ such that for each compact $K \subset \mathbb{R}$ there exists a $\lambda_K > 0$ for

which $\int\limits_K \varphi(t, \lambda_K |f(t)|)\, dt < +\infty$.
The functional

$$\begin{array}{llll} \rho_\varphi : & L^{\varphi}_{loc}(\mathbb{R}) & \to & [0, +\infty] \\ & f & \mapsto & \rho_\varphi(f) = \overline{\lim}_{T\to+\infty} \frac{1}{2T} \int\limits_{-T}^{+T} \phi(t, |f(t)|)\, dt \end{array}$$

is a convex pseudomodular on $L^{\varphi}_{loc}(\mathbb{R})$.

The linear space associated to this pseudomodular,

$$\begin{aligned} B^{\varphi}(\mathbb{R}) &= \left\{ f \in L^{\varphi}_{loc}(\mathbb{R}) : \lim_{\alpha\to 0} \rho_\varphi(\alpha f) = 0 \right\} \\ &= \left\{ f \in L^{\varphi}_{loc}(\mathbb{R}) : \rho_\varphi(\alpha f) < +\infty, \ \texttt{for some } \alpha > 0 \right\}, \end{aligned}$$

is called the Besicovitch-Musielak-Orlicz space and is endowed with the usual Luxemburg's pseudonorm

$$\|f\|_\varphi = \inf\left\{ k > 0/\rho_\varphi\left(\frac{f}{k}\right) \le 1 \right\}, \qquad f \in B^{\varphi}(\mathbb{R}).$$

Let A be the linear set of all generalized trigonometric polynomials, i.e.:

$$A = \left\{ P_n(t) = \sum_{j=1}^{n} a_j e^{i\lambda_j t}, a_j \in \mathbb{C}, \lambda_j \in \mathbb{R}, n \in \mathbb{N} \right\}.$$

We denote by $\tilde{B}^{\varphi}_{a.p.}(\mathbb{R})$ (resp. $B^{\varphi}_{a.p.}(\mathbb{R})$) the closure of A with respect to the pseudomodular ρ_φ (resp. with respect to the pseudonorm $\|.\|_\varphi$), more precisely:

$$\begin{aligned} \tilde{B}^{\varphi}_{a.p.}(\mathbb{R}) &= \left\{ f \in B^{\varphi}(\mathbb{R}) : \exists f_n \in A, \exists k_0 > 0 \ \texttt{s.t.} \lim_{n\to+\infty} \rho_\varphi(k_0(f_n - f)) = 0 \right\} \\ B^{\varphi}_{a.p.}(\mathbb{R}) &= \left\{ f \in B^{\varphi}(\mathbb{R}) : \exists f_n \in A, \ \texttt{s.t.} \forall k > 0 \ \lim_{n\to+\infty} \rho_\varphi(k(f_n - f)) = 0 \right\} \\ &= \left\{ f \in B^{\varphi}(\mathbb{R}) : \exists f_n \in A \ \texttt{s.t.} \lim_{n\to+\infty} \|f_n - f\|_\varphi = 0 \right\}. \end{aligned}$$

$\tilde{B}^{\varphi}_{a.p.}(\mathbb{R})$ and $B^{\varphi}_{a.p.}(\mathbb{R})$ will be called Besicovitch-Musielak-Orlicz spaces of almost periodic functions.

As usual $\{u.a.p.\}$ denotes the algebra of Bohr's almost periodic functions i.e., the closure of the set A in the uniform norm of $C_b(\mathbb{R})$ (the space of continuous and bounded functions on $\mathbb{R}$).

The following inclusions hold :

$$B^{\varphi}_{a.p.}(\mathbb{R}) \subseteq \tilde{B}^{\varphi}_{a.p.}(\mathbb{R}) \subseteq B^{\varphi}(\mathbb{R}).$$

The function $\varphi(t, u)$ is called uniformly convex if : $\forall \varepsilon \in]0, 1[, \exists f_\varepsilon \in M(\mathbb{R})$ with $\rho_\varphi(f_\varepsilon) = \varepsilon$ and $\exists p(\varepsilon) \in]0, 1[$ s.t. $\forall x, y \in [0, +\infty[$: if $\max(x, y) \ge |f_\varepsilon(t)|$ and

$|x-y| \geq \varepsilon \max(x,y)$ then

$$\varphi\left(t, \frac{x+y}{2}\right) \leq \frac{1-p(\varepsilon)}{2}\left(\varphi(t,x)+\varphi(t,y)\right), \text{ for almost all } t \in \mathbb{R}.$$

We say that $\varphi(t,u)$ satisfies the condition Δ_2 $(\varphi \in \Delta_2)$ if there exist $k>1$ and a measurable nonnegative function h with $\rho_\varphi(h) < +\infty$ such that $\varphi(t,2u) \leq k\varphi(t,u)$ for almost all $t \in \mathbb{R}$ and all $u \geq h(t)$.

Note that this definition of condition Δ_2 is similar to that for classical Musielak-Orlicz spaces, the condition of integrability of h being replaced by $\rho_\varphi(h) < +\infty$ (*see* [2]).

2. Auxiliary results. The structure of the class $B^\varphi_{a.p.}(\mathbb{R})$ is not adapted to the methods of Lebesgue measure theory. In fact, the usual convergence results are not valid in the $B^\varphi_{a.p.}(\mathbb{R})$ spaces (see [9]).

To handle $B^\varphi_{a.p.}(\mathbb{R})$ spaces as $L^\varphi(\mathbb{R})$ ones, we introduce the set function $\overline{\mu}$.

Let $\Sigma(\mathbb{R})$ denotes the $\sigma-$algebra of Lebesgue measurable subsets of $\mathbb{R}$. We denote by $\overline{\mu}$ the set function defined on $\Sigma(\mathbb{R})$ by:

$\bar{\mu}(A) = \overline{\lim}_{T\to+\infty} \frac{1}{2T} \int_{-T}^{+T} \chi_A(t)\,dt = \overline{\lim}_{T\to+\infty} \frac{1}{2T} \mu(A \cap [-T,+T])$

where μ is the Lebesgue's measure on $\mathbb{R}$.

As usual a sequence $\{f_n\}$ from $B^\varphi(\mathbb{R})$ is said to be $\bar{\mu}-$convergent to some $f \in B^\varphi(\mathbb{R})$ (in symbol $f_n \xrightarrow{\overline{\mu}} f$) when, $\forall \alpha > 0$

$$\lim_{n\to+\infty} \bar{\mu}\{x \in \mathbb{R}, |f_n(x) - f(x)| > \alpha\} = 0.$$

We give here some technical results that are the key arguments in the proof of the main theorem.

Lemma 2.1 *Let* $\{f_n\}_{n\geq1} \subset B^\varphi_{a.p.}(\mathbb{R})$. *If* $\lim_{n\to+\infty} \rho_\varphi(f_n - f) = 0$ *for some* $f \in B^\varphi_{a.p.}(\mathbb{R})$, *then* $f_n \xrightarrow{\overline{\mu}} f$.

Proof Put $A_n^\alpha = \{t \in \mathbb{R}, |f_n(t) - f(t)| \geq \alpha\}$, then

$$\begin{aligned} \rho_\varphi(f_n - f) &= \overline{\lim}_{T\to+\infty} \frac{1}{2T} \int_{-T}^{+T} \varphi(t, |f_n(t) - f(t)|)\,dt \\ &\geq \overline{\lim}_{T\to+\infty} \frac{1}{2T} \int_{-T}^{+T} \varphi\left(t, \alpha\chi_{A_n^\alpha}(t)\right)dt \\ &\geq C(\alpha)\,\bar{\mu}(A_n^\alpha). \end{aligned}$$

It follows directly that $\lim_{n\to+\infty} \bar{\mu}(A_n^\alpha) = 0$, i.e. $f_n \xrightarrow{\overline{\mu}} f$. ■

Remark 2.2 The condition $\inf\{\varphi(t,\alpha), t \in \mathbb{R}\} = C(\alpha) > 0, \forall \alpha > 0$ is necessary in Lemma 2.1, as it is shown by the following example:

We define $\varphi : \mathbb{R}\times [0,+\infty[\to [0,+\infty[$ by $\varphi(t,u) = f(t).u^2$ where $f : \mathbb{R} \to [0,+\infty[$ is a continuous and periodic function (with period $T = 1$) defined by:

$$f(t) = \begin{cases} 0 & \texttt{if} \quad t \in \left[0, \frac{3}{8}\right] \\ 8t-3 & \texttt{if} \quad t \in \left[\frac{3}{8}, \frac{1}{2}\right] \\ -8t+5 & \texttt{if} \quad t \in \left[\frac{1}{2}, \frac{5}{8}\right] \\ 0 & \texttt{if} \quad t \in \left[\frac{5}{8}, 1\right] \end{cases}$$

Consider now the continuous and periodic function (with period $T = 1$) defined as follows

$$h(t) = \begin{cases} 1 & \texttt{if} \quad t \in \left[0, \frac{1}{4}\right[\\ -8t+3 & \texttt{if} \quad t \in \left[\frac{1}{4}, \frac{3}{8}\right] \\ 0 & \texttt{if} \quad t \in \left[\frac{3}{8}, \frac{5}{8}\right] \\ 8t-5 & \texttt{if} \quad t \in \left]\frac{5}{8}, \frac{3}{4}\right[\\ 1 & \texttt{if} \quad t \in \left[\frac{3}{4}, 1\right] \end{cases}$$

It is easily seen that $\varphi(t,|h(t)|) = f(t)\ h^2(t) = 0\ \forall t \in \mathbb{R}$ and then $\rho_\varphi(h) = 0$. But clearly $\bar{\mu}\left\{t \in \mathbb{R}, |h(t)| \geq \frac{1}{2}\right\} \geq \frac{1}{2}$.

Lemma 2.3 *Let $h \in B^{\varphi}(\mathbb{R})$ with $\rho_\varphi(h) = a > 0$. Then*

$$(1) \qquad \forall \theta \in (0,1), \exists \beta > 0 \ \textit{and}\ G = \{t \in \mathbb{R}, |h(t)| \leq \beta\} \ \textit{such that}\ \bar{\mu}(G) \geq \theta.$$

Proof For $\theta \in (0,1)$ take $\beta > 0$ such that $(1-\theta)\, C(\beta) > 2\rho_\varphi(h) = 2a$ and suppose that (1) is false. Then, for some sequence $\{T_n\}$ increasing to infinity we will have $\frac{\mu\{G\cap[-T_n,+T_n]\}}{2T_n} < \theta$. Hence denoting by G' the complementary of G in $\mathbb{R}$ we get

$$\begin{aligned} \frac{1}{2T_n}\int_{-T_n}^{+T_n} \varphi(t,|h(t)|)\, dt &\geq \frac{1}{2T_n}\int_{G'\cap[-T_n,+T_n]} \varphi(t,|h(t)|)\, dt \\ &\geq C(\beta)\frac{1}{2T_n}\mu\{G'\cap[-T_n,+T_n]\} \\ &\geq C(\beta)(1-\theta) > 2a, \end{aligned}$$

and letting n tends to infinity, it follows $\rho_\varphi(h) > 2a$, a contradiction. ∎

Lemma 2.4 *Let $g \in B^{\varphi}_{a.p}(\mathbb{R})$, then*

$$\forall \varepsilon > 0\ \ \exists \delta > 0\ \ \textit{s.t.} \forall Q \in \sum(\mathbb{R}),\ \ \bar{\mu}(Q) \leq \delta \Rightarrow \rho_\varphi(g\chi_Q) \leq \varepsilon.$$

PROOF We may suppose $\rho_\varphi(g) > 0$.

For $\varepsilon > 0$ take $P_\varepsilon \in A$ such that $\rho_\varphi(2(g - P_\varepsilon)) < \frac{\varepsilon}{2}$ and put $M_\varepsilon = \sup_{t\in\mathbb{R}} \varphi(t, 2|P_\varepsilon(t)|)$. Let $\theta \in (0,1)$ satisfying $M_\varepsilon(1-\theta) < \frac{\varepsilon}{2}$. If β and G are as in Lemma 2.3 we will have

$$
\begin{aligned}
(2) \qquad & \overline{\lim}_{T\to+\infty} \frac{1}{2T} \int_{G'\cap[-T,+T]} \varphi(t, |g(t)|)\, dt \\
\leq\ & \frac{1}{2} \overline{\lim}_{T\to+\infty} \frac{1}{2T} \int_{G'\cap[-T,+T]} [\varphi(t, 2|g(t) - P_\varepsilon(t)|) + \varphi(t, 2|P_\varepsilon(t)|)]\, dt \\
\leq\ & \frac{\varepsilon}{4} + \frac{1}{2} M_\varepsilon(1-\theta) \leq \frac{\varepsilon}{2}
\end{aligned}
$$

Let $\delta = \frac{\varepsilon}{2 \sup_{t\in\mathbb{R}} \varphi(t,\beta)}$ and $Q \in \sum(\mathbb{R})$ with $\bar{\mu}(Q) \leq \delta$. Putting $Q_1 = Q \cap G$ and $Q_2 = Q \cap G'$ it follows

$$
\begin{aligned}
\overline{\lim}_{T\to+\infty} \frac{1}{2T} \int_{Q_1\cap[-T,T]} \varphi(t, |g(t)|)\, dt \ &\leq\ \overline{\lim}_{T\to+\infty} \frac{1}{2T} \int_{Q_1\cap[-T,T]} \varphi(t, \beta)\, dt \\
&\leq\ \sup_{t\in\mathbb{R}} \varphi(t,\beta)\, \bar{\mu}(Q_1) \\
&\leq\ \delta \sup_{t\in\mathbb{R}} \varphi(t,\beta) \leq \frac{\varepsilon}{2}.
\end{aligned}
$$

Similarly, in view of (2) we have

$$
\overline{\lim}_{T\to+\infty} \frac{1}{2T} \int_{Q_2\cap[-T,T]} \varphi(t, |g(t)|)\, dt \leq \overline{\lim}_{T\to+\infty} \frac{1}{2T} \int_{G'\cap[-T,+T]} \varphi(t, |g(t)|)\, dt \leq \frac{\varepsilon}{2}.
$$

Finally,

$$
\overline{\lim}_{T\to+\infty} \frac{1}{2T} \int_{Q\cap[-T,+T]} \varphi(t, |g(t)|)\, dt \leq \varepsilon,
$$

which means that $\rho_\varphi(g\chi_Q) \leq \varepsilon$. ∎

PROPOSITION 2.5 *Let $f \in B^{\varphi}_{a.p}(\mathbb{R})$. Then $\varphi(t, |f(t)|) \in B^1_{a.p}(\mathbb{R})$ and consequently $\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{+T} \varphi(t, |f(t)|)\, dt$ exists (and is finite).*

PROOF Let $\{f_n\}$ be a sequence of trigonometric polynomials such that $\|f_n - f\|_\varphi \to 0$, then using Lemma 2.1 we have also $f_n \xrightarrow{\bar{\mu}} f$.

Let $\theta \in (0,1)$, in view of Lemma 2.3 there exist $\beta > 0$ and a set $G = \{t \in \mathbb{R}, |f(t)| \leq \beta\}$ for witch $\bar{\mu}(G) \geq \theta$.

Take $\alpha > 0$ and $A^\alpha_n = \{t \in \mathbb{R}, |f_n(t) - f(t)| > \alpha\}$, then it is easily seen that $|f_n(t)| \leq \beta + \alpha, \forall t \in G \cap (A^\alpha_n)'$ and since φ is continuous on $\mathbb{R}\times [0, +\infty[$, periodic with respect to $t \in \mathbb{R}$, using the fact that $|f_n(t)|, |f(t)| \in [0, \beta + \alpha]$ for $t \in G \cap (A_n^{\alpha})'$, we claim that

$$\forall \eta > 0 \;\; \exists \alpha_\eta > 0 \;\; \forall t \in G \cap (A_n^{\alpha})', |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \eta \Longrightarrow |f_n(t) - f(t)| > \alpha_\eta$$

then, since $f_n \xrightarrow{\bar{\mu}} f$ we get also

$$\lim_{n \to +\infty} \bar{\mu} \left\{ t \in G \cap (A_n^{\alpha})', |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \eta \right\} = 0.$$

Consequently,

$$\begin{aligned}
& \bar{\mu} \{ t \in \mathbb{R}, |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \eta \} \\
\leq \; & \bar{\mu} \left\{ t \in G \cap (A_n^{\alpha})', |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \eta \right\} \\
& + \bar{\mu} \{ t \in G', |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \eta \} \\
& + \bar{\mu} \{ t \in A_n^{\alpha}, |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \eta \} \\
\leq \; & \bar{\mu} \left\{ t \in G \cap (A_n^{\alpha})', |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \eta \right\} \\
& + \bar{\mu}(G') + \bar{\mu}(A_n^{\alpha}) \\
\leq \; & \bar{\mu} \left\{ t \in G \cap (A_n^{\alpha})', |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \eta \right\} \\
& + (1 - \theta) + \bar{\mu}(A_n^{\alpha}),
\end{aligned}$$

hence, letting n tends to infinity we will have

$$\overline{\lim}_{n \to +\infty} \bar{\mu} \{ t \in \mathbb{R}, |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \eta \} \leq (1 - \theta).$$

Finally, since $\theta \in (0, 1)$ is arbitrary, we deduce the following

$$\forall \eta > 0, \lim_{n \to +\infty} \bar{\mu} \{ t \in \mathbb{R}, |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \eta \} = 0. \tag{3}$$

Let now $\varepsilon > 0$, then there exists $\delta = \delta(f, \varepsilon) > 0$ such that for all $Q \in \sum(\mathbb{R})$ with $\bar{\mu}(Q) \leq \delta$ we have $\rho_\varphi(f_n \chi_Q) \leq \varepsilon$ for sufficiently large n. Indeed, since $\rho_\varphi(f_n \chi_Q) \leq \frac{1}{2} [\rho_\varphi(2(f - f_n)\chi_Q + \rho_\varphi(f \chi_Q))]$, using lemma 2.4 the result is immediate.

On the other hand from (3) there is $n_0 \in \mathbb{N}$ such that for $n \geq n_0$

$$\bar{\mu} \{ t \in \mathbb{R}, |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \varepsilon \} \leq \delta.$$

Let $A_n^{\varepsilon} = \{ t \in \mathbb{R}, |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \varepsilon \}$. Then

$$\begin{aligned}
& \overline{\lim}_{T \to +\infty} \frac{1}{2T} \int_{-T}^{+T} |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \, dt \\
\leq \; & \overline{\lim}_{T \to +\infty} \frac{1}{2T} \int_{A_\varepsilon^n \cap [-T, T]} |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \, dt \\
& + \overline{\lim}_{T \to +\infty} \frac{1}{2T} \int_{(A_\varepsilon^n)' \cap [-T, T]} |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \, dt,
\end{aligned}$$

and since $\bar{\mu}\left(A_{\varepsilon}^{n}\right) \leq \delta$ for some $\delta > 0$ and $n \geq n_0$ we get

$$\begin{aligned} & \overline{\lim}_{T\to+\infty} \frac{1}{2T} \int\limits_{A_{\varepsilon}^{n}\cap[-T,T]} \left|\varphi\left(t,\left|f_n\left(t\right)\right|\right) - \varphi\left(t,\left|f\left(t\right)\right|\right)\right| dt \\ \leq \; & \overline{\lim}_{T\to+\infty} \frac{1}{2T} \int\limits_{A_{\varepsilon}^{n}\cap[-T,T]} \varphi\left(t,\left|f_n\left(t\right)\right|\right) dt + \overline{\lim}_{T\to+\infty} \frac{1}{2T} \int\limits_{A_{\varepsilon}^{n}\cap[-T,T]} \varphi\left(t,\left|f\left(t\right)\right|\right) dt \\ \leq \; & 2\varepsilon. \end{aligned}$$

On the other hand we have also

$$\overline{\lim}_{T\to+\infty} \frac{1}{2T} \int\limits_{(A_{\varepsilon}^{n})'\cap[-T,T]} \left|\varphi\left(t,\left|f_n\left(t\right)\right|\right) - \varphi\left(t,\left|f\left(t\right)\right|\right)\right| dt \leq \varepsilon$$

then, since $\varepsilon > 0$ is arbitrary, we get

$$\lim_{n\to+\infty} \overline{\lim}_{T\to+\infty} \frac{1}{2T} \int\limits_{-T}^{+T} \left|\varphi\left(t,\left|f_n\left(t\right)\right|\right) - \varphi\left(t,\left|f\left(t\right)\right|\right)\right| dt = 0. \tag{4}$$

Now, the continuous function $\varphi : \mathbb{R}\times [0,+\infty[\; \to [0,+\infty[$ being periodic with respect to $t \in \mathbb{R}$ and since for each $n \in \mathbb{N}$, $f_n \in \{u.a.p\}$, we deduce that $\varphi\left(t,\left|f_n\left(t\right)\right|\right) \in \{u.a.p\}$ (see[3], p.61).

Finally from (4) it follows that $\varphi\left(t,\left|f\left(t\right)\right|\right) \in B_{a.p}^{1}\left(\mathbb{R}\right)$ and by a classical result (see[1]) the limit $\lim\limits_{T\to+\infty} \frac{1}{2T} \int\limits_{-T}^{+T} \varphi\left(t,\left|f\left(t\right)\right|\right) dt$ exists (and is finite). ∎

Lemma 2.6 *Let* $\{f_n\} \subset B_{a.p}^{1}\left(\mathbb{R}\right)$ *be such that* $f_n \xrightarrow{\bar{\mu}} f \in B_{a.p}^{1}\left(\mathbb{R}\right)$. *Suppose there exists* $g \in B_{a.p}^{1}\left(\mathbb{R}\right)$ *for which* $\max\left(\left|f_n\left(t\right)\right|,\left|f\left(t\right)\right|\right) \leq g\left(t\right), \forall t \in \mathbb{R}$, *then* $\rho_1\left(f_n\right) \to \rho_1\left(f\right)$.

Remark 2.7 The notation $B_{a.p}^{1}\left(\mathbb{R}\right)$ is used for the space corresponding to the function $\varphi\left(t,x\right) = |x|$ and ρ_1 is the associated modular.

Proof Take $\varepsilon > 0$ and put $E_n^{\varepsilon} = \left\{t \in \mathbb{R}/\left|f_n\left(t\right) - f\left(t\right)\right| \geq \frac{\varepsilon}{2}\right\}$. Let $\delta > 0$ be associated to the function $2g$ and $\frac{\varepsilon}{2}$ as in Lemma 2.4. Since $f_n \xrightarrow{\bar{\mu}} f$ it follows that $\bar{\mu}\left(E_n^{\varepsilon}\right) \leq \delta$ for $n \geq n_0$ and then by Lemma 2.4

$$\rho_1\left(\left(f_n\left(t\right) - f\left(t\right)\right)\chi_{E_n^{\varepsilon}}\right) \leq \rho_1\left(2g\chi_{E_n^{\varepsilon}}\right) \leq \frac{\varepsilon}{2}.$$

Finally we get for $n \geq n_0$

$$\begin{aligned} |\rho_1(f_n(t)) - \rho_1(f(t))| &\leq \rho_1((f_n(t) - f(t))) \\ &\leq \rho_1\left((f_n(t) - f(t))\chi_{E_n^{\varepsilon}}\right) + \rho_1\left((f_n(t) - f(t))\chi_{(E_n^{\varepsilon})'}\right) \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

i.e. $\lim\limits_{n\to+\infty} \rho_1(f_n) = \rho_1(f)$ ■

LEMMA 2.8 *Let $f \in B^{\varphi}_{a.p}(\mathbb{R})$, then the functional $\lambda \mapsto \rho_{\varphi}\left(\frac{f}{\lambda}\right)$ is continuous on* $]0, +\infty[$.

PROOF Let $\lambda_0 \in]0, +\infty[$ and$\{\lambda_n\}$ be a sequence of real numbers which converges to λ_0.We have

$$\rho_{\varphi}\left(\frac{f}{\lambda_n} - \frac{f}{\lambda_0}\right) \leq \left|\frac{1}{\lambda_n} - \frac{1}{\lambda_0}\right| \rho_{\varphi}(f), \forall n \geq n_0$$

and then $\lim\limits_{n\to+\infty} \rho_{\varphi}\left(\frac{f}{\lambda_n} - \frac{f}{\lambda_0}\right) = 0.$

From Lemma 2.1, it follows $\frac{f}{\lambda_n} \xrightarrow{\overline{\mu}} \frac{f}{\lambda_0}$ and then $\varphi\left(t, \frac{|f(t)|}{\lambda_n}\right) \xrightarrow{\overline{\mu}} \varphi\left(t, \frac{|f(t)|}{\lambda_0}\right)$ (see (3)) . Furthermore, $\max\left(\varphi\left(t, \frac{|f(t)|}{\lambda_n}\right), \varphi\left(t, \frac{|f(t)|}{\lambda_0}\right)\right) \leq \varphi\left(t, \frac{2}{\lambda_0}|f(t)|\right) \in B^1_{a.p}(\mathbb{R})$(see prop. 2.5), consequently, using Lemma 2.6 we deduce

$$\rho_{\varphi}\left(\frac{f}{\lambda_n}\right) \to \rho_{\varphi}\left(\frac{f}{\lambda_0}\right)$$

this means that $\lambda \mapsto \rho_{\varphi}\left(\frac{f}{\lambda}\right)$ is continuous at λ_0. ■

COROLLARY 2.9 *Let $f \in B^{\varphi}_{a.p}(\mathbb{R})$, then*

1. $\|f\|_{\varphi} \leq 1$ *if and only if* $\rho_{\varphi}(f) \leq 1$.
2. $\|f\|_{\varphi} = 1$ *if and only if* $\rho_{\varphi}(f) = 1$.
3. $\forall \varepsilon \in]0, 1[, \exists \delta \in]0, 1[$ *such that* $\rho_{\varphi}(f) \leq \delta$ *implies* $\|f\|_{\varphi} \leq \varepsilon$.

PROOF This follows from Lemma 2.8 and usual arguments of Orlicz spaces theory.

Note that a similar result holds in classical Musielak-Orlicz spaces with the additional condition Δ_2 on the function φ. In fact, this condition is necessary for the continuity of the function $\lambda \mapsto \rho_{\varphi}\left(\frac{f}{\lambda}\right)$.

Here the continuity holds without condition Δ_2, but with the restriction $f \in B^{\varphi}_{a.p}(\mathbb{R})$. ■

LEMMA 2.10 *Let* $\varphi(t,u) \in \Delta_2$ *then*
$\forall \theta \in (0,1)\, \forall \varepsilon > 0 \exists h_\varepsilon \in B^\varphi(\mathbb{R})$ *and* $k' > 1$ *such that:*

$$\varphi(t,2u) \leq k'\varphi(t,u) \qquad \forall u \geq h_\varepsilon(t), \forall t \in G \tag{5}$$

with $\bar{\mu}(G') < \theta$ *and* $\rho_\varphi(h_\varepsilon) < \varepsilon$.

PROOF Since $\varphi(t,u) \in \Delta_2$, by definition there exist $k > 1$ and a nonnegative function $h \in B^\varphi(\mathbb{R})$ such that $\varphi(t,2u) \leq k\varphi(t,u)$ for almost all $t \in \mathbb{R}$ and all $u \geq h(t)$.

Let us remark that the later holds if we replace $h(t)$ by the function
$h_1(t) = \begin{cases} \delta & \texttt{if} \quad 0 \leq h(t) \leq \delta \\ h(t) & \texttt{if} \quad h(t) \geq \delta \end{cases}$.

Then we may assume that $h(t) \geq \delta$, $\forall t \in \mathbb{R}$ for some $\delta > 0$. Moreover for $\varepsilon > 0$ there exists $\eta_0 > 0$ such that $\rho_\varphi\left(\frac{h}{\eta_0}\right) < \varepsilon$.

We take $\theta \in (0,1)$ and $\beta > 0$ as in Lemma 2.3 then $\bar{\mu}(G') \leq \theta$ where $G = \{t \in \mathbb{R}, |h(t)| \leq \beta\}$.

Now if we put $h_\varepsilon = \frac{h}{\eta_0}$ and $k' = \max(k, k_1)$ where

$$k_1 = \max\left(\frac{\varphi(t,2u)}{\varphi(t,u)},\ u \in \left[\frac{\delta}{\eta_0}, \beta\right],\ t \in \mathbb{R}\right),$$

we obtain (5). Note that $k_1 < +\infty$ due to the periodicity of $\varphi(.,u)$. ■

LEMMA 2.11 *Let* $f \in \tilde{B}^\varphi_{a.p}(\mathbb{R})$ *and* $\varphi \in \Delta_2$, *then*
$\forall \varepsilon \in]0,1[\, \exists \delta(\varepsilon) \in]0,1[$ *s.t.* $\rho_\varphi(f) \leq 1 - \varepsilon \Rightarrow \|f\|_\varphi \leq 1 - \delta(\varepsilon)$

PROOF In view of Lemma 2.10 we have
$\Delta_2' : \forall \theta \in (0,1)\, \forall \varepsilon > 0, \exists h_\varepsilon \in B^\varphi(\mathbb{R})$ and $k' > 1$ such that

$$\varphi(t,2u) \leq k'\varphi(t,u) \qquad \forall u \geq h_\varepsilon(t), \forall t \in G$$

with $\bar{\mu}(G') < \theta$ and $\rho_\varphi(h_\varepsilon) < \varepsilon$.

On the other hand, if $p(t,u)$ is the right derivative of $\varphi(t,u)$ with respect to u, we have (see [2], [10])

$$up(t,u) \leq \varphi(t,2u) \leq 2up(t,2u) \quad \forall u \in \mathbb{R}_+, \forall t \in \mathbb{R}.$$

Let $\lambda \in \left(0, \frac{1}{2}\right)$, then

$$\frac{\varphi\left(t, \frac{u}{1-\lambda}\right)}{\varphi(t,u)} = 1 + \int_u^{\frac{u}{1-\lambda}} \frac{p(t,s)}{\varphi(t,u)} ds,$$

hence, since $p(t,.)$ is non-decreasing we obtain

$$\begin{aligned}\left|\frac{\varphi\left(t,\frac{u}{1-\lambda}\right)}{\varphi(t,u)}-1\right| &\leq \left(\left|\frac{u}{1-\lambda}-u\right|/\varphi(t,u)\right).p\left(t,\frac{u}{1-\lambda}\right)\\ &\leq \left|\frac{1}{1-\lambda}-1\right|u.\left(\frac{p(t,2u)}{\varphi(t,u)}\right)\leq\frac{1}{2}\frac{\lambda}{1-\lambda}.\frac{\varphi(t,4u)}{\varphi(t,u)}\\ &\leq \lambda\frac{\varphi(t,4u)}{\varphi(t,u)}. \qquad (6)\end{aligned}$$

Consider the function

$$f_{\lambda}(t,u)=\frac{\varphi\left(t,\frac{u}{1-\lambda}\right)}{\varphi(t,u)}.$$

From (6), we have

$$f_{\lambda}(t,u)\leq 1+\lambda\frac{\varphi(t,4u)}{\varphi(t,u)},\ \forall u\in\mathbb{R}_{+},\forall t\in\mathbb{R}$$

and using the condition Δ_2', we get

$$f_{\lambda}(t,u)\leq 1+(k')^2\lambda,\ \forall u\geq h_{\varepsilon}(t),\forall t\in G$$

with $\bar{\mu}(G')<\theta$ and $\rho_{\varphi}(h_{\varepsilon})<\varepsilon$.

Moreover it is easily seen that given $\varepsilon>0$ there exists $\delta_1(\varepsilon)>0$ s.t.

$$|\lambda|\leq\delta_1(\varepsilon)\Rightarrow f_{\lambda}(t,u)\leq 1+(k')^2\lambda\leq\frac{1}{1-\frac{\varepsilon}{2}}$$

(take for example $\delta_1(\varepsilon)=\frac{\varepsilon}{(2k'^2)\left(1-\frac{\varepsilon}{2}\right)}$), so we have $f_{\delta_1(\varepsilon)}(t,u)\leq\frac{1}{1-\frac{\varepsilon}{2}}$, i.e.

$$\left(1-\frac{\varepsilon}{2}\right)\varphi\left(t,\frac{u}{1-\delta_1(\varepsilon)}\right)\leq\varphi(t,u),\ \ \forall u\geq h_{\varepsilon}(t),\forall t\in G$$

with $\bar{\mu}(G')<\theta$ and $\rho_{\varphi}(h_{\varepsilon})<\varepsilon$. This inequality remains valid for each $\delta\leq\delta_1(\varepsilon)$ (we may assume $\delta_1(\varepsilon)\leq\frac{1}{2}$).

Suppose now that $\rho_{\varphi}(f)\leq 1-\varepsilon$ and put $E=\{t\in\mathbb{R},f(t)\geq h_{\varepsilon}(t)\}$, then

$$\begin{aligned}&\frac{1}{2T}\int_{-T}^{+T}\left(1-\frac{\varepsilon}{2}\right)\varphi\left(t,\frac{|f(t)|}{1-\delta_1(\varepsilon)}\right)dt\\ =&\ \frac{1}{2T}\int_{G\cap[-T,+T]}\left(1-\frac{\varepsilon}{2}\right)\varphi\left(t,\frac{|f(t)|}{1-\delta_1(\varepsilon)}\right)dt\\ &+\frac{1}{2T}\int_{G'\cap[-T,+T]}\left(1-\frac{\varepsilon}{2}\right)\varphi\left(t,\frac{|f(t)|}{1-\delta_1(\varepsilon)}\right)dt.\end{aligned}$$

In view of Lemma 2.4, we choose θ such that $\bar{\mu}(G') \leq \theta$ implies $\rho_\varphi(2f\chi_{G'}) \leq \frac{\varepsilon}{4}$, then

$$\begin{aligned} & \frac{1}{2T} \int\limits_{G' \cap [-T,+T]} \left(1-\frac{\varepsilon}{2}\right) \varphi\left(t, \frac{|f(t)|}{1-\delta_1(\varepsilon)}\right) dt \\ \leq \; & \frac{1}{2T} \int\limits_{G' \cap [-T,+T]} \left(1-\frac{\varepsilon}{2}\right) \varphi(t, 2|f(t)|)\, dt \\ \leq \; & \frac{\varepsilon}{4}. \end{aligned}$$

On the other hand

$$\begin{aligned} & \frac{1}{2T} \int\limits_{G \cap [-T,+T]} \left(1-\frac{\varepsilon}{2}\right) \varphi\left(t, \frac{|f(t)|}{1-\delta_1(\varepsilon)}\right) dt \\ = \; & \frac{1}{2T} \int\limits_{G \cap E \cap [-T,+T]} \left(1-\frac{\varepsilon}{2}\right) \varphi\left(t, \frac{|f(t)|}{1-\delta_1(\varepsilon)}\right) dt \\ & + \frac{1}{2T} \int\limits_{G \cap E' \cap [-T,+T]} \left(1-\frac{\varepsilon}{2}\right) \varphi\left(t, \frac{|f(t)|}{1-\delta_1(\varepsilon)}\right) dt \\ \leq \; & \frac{1}{2T} \int\limits_{G \cap E \cap [-T,+T]} \varphi(t, |f(t)|)\, dt \\ & + \frac{1}{2T}\left(1-\frac{\varepsilon}{2}\right) \int\limits_{G \cap E' \cap [-T,+T]} \varphi\left(t, \frac{|f(t)|}{1-\delta_1(\varepsilon)}\right) dt \\ \leq \; & \frac{1}{2T} \int\limits_{G \cap [-T,+T]} \varphi(t, |f(t)|)\, dt - \frac{1}{2T} \int\limits_{G \cap E' \cap [-T,+T]} \varphi(t, |f(t)|)\, dt \\ & + \frac{1}{2T}\left(1-\frac{\varepsilon}{2}\right) \int\limits_{G \cap E' \cap [-T,+T]} \varphi\left(t, \frac{|f(t)|}{1-\delta_1(\varepsilon)}\right) dt \\ \leq \; & \frac{1}{2T} \int\limits_{G \cap [-T,+T]} \varphi(t, |f(t)|)\, dt \\ & + \frac{1}{2T} \int\limits_{G \cap E' \cap [-T,+T]} \left[\varphi\left(t, \frac{|f(t)|}{1-\delta_1(\varepsilon)}\right) - \varphi(t, |f(t)|)\right] dt \\ & - \frac{\varepsilon}{2} \frac{1}{2T} \int\limits_{G \cap E' \cap [-T,+T]} \varphi\left(t, \frac{|f(t)|}{1-\delta_1(\varepsilon)}\right) dt \end{aligned}$$

$$\leq \frac{1}{2T}\int\limits_{G\cap[-T,+T]}\varphi\left(t,\left|f\left(t\right)\right|\right)dt$$
$$+\frac{1}{2T}\int\limits_{G\cap E'\cap[-T,+T]}\left[\varphi\left(t,\frac{\left|f\left(t\right)\right|}{1-\delta_{1}\left(\varepsilon\right)}\right)-\varphi\left(t,\left|f\left(t\right)\right|\right)\right]dt.$$

Let now $t\in E'\cap G$, then $\left|f\left(t\right)\right|\leq h_{\varepsilon}\left(t\right)\leq\beta$ and $\frac{\left|f(t)\right|}{1-\delta_{1}(\varepsilon)}\leq 2\beta$. Using the uniform continuity of φ (while taking $\delta_{1}\left(\varepsilon\right)$ small enough) and it's periodicity with respect to $t\in\mathbb{R}$, we obtain

$$\left|\varphi\left(t,\frac{\left|f\left(t\right)\right|}{1-\delta_{1}\left(\varepsilon\right)}\right)-\varphi\left(t,\left|f\left(t\right)\right|\right)\right|\leq\frac{\varepsilon}{4},$$

it follows

$$\frac{1}{2T}\int\limits_{-T}^{+T}\left(1-\frac{\varepsilon}{2}\right)\varphi\left(t,\frac{\left|f\left(t\right)\right|}{1-\delta_{1}\left(\varepsilon\right)}\right)dt\leq\frac{1}{2T}\int\limits_{-T}^{+T}\varphi\left(t,\left|f\left(t\right)\right|\right)dt+\frac{\varepsilon}{2},\qquad\forall T\geq T_{0}$$

then letting T tends to infinity, we obtain

$$\left(1-\frac{\varepsilon}{2}\right)\rho_{\varphi}\left(\frac{f}{1-\delta_{1}\left(\varepsilon\right)}\right)\leq\rho_{\varphi}\left(f\right)+\frac{\varepsilon}{2}\leq 1-\varepsilon+\frac{\varepsilon}{2}=1-\frac{\varepsilon}{2}$$

and finally $\left\|f\right\|_{\varphi}\leq 1-\delta_{1}\left(\varepsilon\right).$ ■

LEMMA 2.12 *Let $\left(a_{n}\right)_{n\geq 1}$ be a sequence of positive real numbers. For each n, we associate a measurable set A_{n} such that*

i. $A_{i}\cap A_{j}=\phi$, *for* $i\neq j$ *and* $\bigcup\limits_{n\geq 1}A_{n}\subset\left[0,\alpha\right[,\alpha<1$.

ii. $\sum\limits_{n\geq 0}\int\limits_{0}^{1}\varphi\left(t,a_{n}\chi_{A_{n}}\right)dt<+\infty$.

Consider the function $f=\sum\limits_{n\geq 1}a_{n}\chi_{A_{n}}$ *on* $[0,1]$ *and let* $\tilde{f}$ *be the periodic extension of* f *to the whole* $\mathbb{R}$ (*with period* $\tau=1$). *Then* $\tilde{f}\in\tilde{B}_{a.p.}^{\varphi}$.

PROOF Let us first remark that in view of Lemma 2.4, for each $n\geq 1$ there exists a set $A_{n}\subset\left[0,\alpha\right[$ for which $\int\limits_{0}^{1}\varphi\left(t,a_{n}\chi_{A_{n}}\right)dt<\frac{1}{n^{2}}$. It is also clear that we may choose the A_{n}'s so that the conditions of the Lemma are satisfied. Now, for an arbitrary $\varepsilon>0$, fix n_{0} such that $\sum\limits_{n\geq n_{0}}\int\limits_{0}^{1}\varphi\left(t,a_{n}\chi_{A_{n}}\right)dt\leq\frac{\varepsilon}{3}$ and put $f_{1}=\sum\limits_{i=1}^{n_{0}}a_{i}\chi_{A_{i}}$ on $\left[0,1\right[$. Let then $M=\max\limits_{i\leq n_{0}}\sup\limits_{t}\varphi\left(t,2a_{i}\right)$ and $\delta\leq\frac{\varepsilon}{3M}$. We can suppose $1-\alpha>\delta$.

If f_1^r is the restriction of f_1 to $[0,1-\delta]$, by Luzin's theorem there exists a continuous function g_ε^r on $[0,1-\delta]$ such that

$$\mu\left\{t\in[0,1-\delta]/\varphi\left(t,\left|f_1^r(t)-g_\varepsilon^r(t)\right|\right)>0\right\}\leq\frac{\varepsilon}{3M}.$$

Moreover since f_1 is bounded so is g_ε^r (with the same bound).

Let g_ε be a linear extension of g_ε^r to $[0,1]$, more precisely g_ε is such that $g_\varepsilon=g_\varepsilon^r$ on $[0,1-\delta]$, g_ε is linear between $1-\delta$ and 1 and satisfies $g_\varepsilon(1)=g_\varepsilon^r(0)$.

We then get

$$\begin{aligned}
&\int_0^1\varphi\left(t,\frac{|f(t)-g_\varepsilon(t)|}{2}\right)dt\\
\leq\ &\int_0^1\varphi\left(t,\frac{|f(t)-f_1(t)|+|f_1(t)-g_\varepsilon(t)|}{2}\right)dt\\
\leq\ &\frac{1}{2}\int_0^1\varphi\left(t,|f(t)-f_1(t)|\right)dt+\frac{1}{2}\int_0^1\varphi\left(t,|f_1(t)-g_\varepsilon(t)|\right)dt\\
\leq\ &\frac{1}{2}\int_0^1\varphi\left(t,\sum_{n\geq n_0}a_n\chi_{A_n}\right)dt+\frac{1}{2}\int_0^{1-\delta}\varphi\left(t,|f_1^r(t)-g_\varepsilon^r(t)|\right)dt\\
+\ &\frac{1}{2}\int_{1-\delta}^1\varphi\left(t,|f_1(t)-g_\varepsilon(t)|\right)dt\\
\leq\ &\frac{1}{2}\sum_{n\geq n_0}\int_0^1\varphi\left(t,a_n\chi_{A_n}\right)dt+\frac{1}{2}M\frac{\varepsilon}{3M}+\frac{1}{2}M\frac{\varepsilon}{3M}\\
\leq\ &\frac{\varepsilon}{2}.
\end{aligned}$$

Finally the continuous function $g_\varepsilon:[0,1]\to\mathbb{R}$ satisfies

$$g_\varepsilon(0)=g_\varepsilon(1)\ \ \texttt{and}\ \ \int_0^1\varphi\left(t,\frac{|f(t)-g_\varepsilon(t)|}{2}\right)dt\leq\frac{\varepsilon}{2}.$$

Let now $\tilde{f}$ be the periodic extension of f to the whole $\mathbb{R}$ and $\tilde{g}_\varepsilon$ be the periodic extension of g_ε. Clearly $\tilde{g}_\varepsilon$ is *u.a.p.* and then it is also in $B^\varphi_{a.p.}(\mathbb{R})$.

Consequently, there exists $P_\varepsilon\in A$ for which $\rho_\varphi\left(\frac{\tilde{g}_\varepsilon-P_\varepsilon}{2}\right)\leq\frac{\varepsilon}{2}$.

On the other hand $\tilde{f}$ and $\tilde{g}$ being periodic with period $T = 1$, we have

$$\rho_{\varphi}\left(\frac{\tilde{f}-\tilde{g}_{\varepsilon}}{2}\right) = \overline{\lim}_{T\to+\infty}\frac{1}{2T}\int_{-T}^{+T}\varphi\left(t, \frac{\left|\tilde{f}(t)-\tilde{g}_{\varepsilon}(t)\right|}{2}\right)dt$$

$$= \int_{0}^{1}\varphi\left(t, \frac{|f(t)-g_{\varepsilon}(t)|}{2}\right)dt \leq \frac{\varepsilon}{2}.$$

Finally,

$$\rho_{\varphi}\left(\frac{\tilde{f}-P_{\varepsilon}}{4}\right) \leq \frac{1}{2}\left[\rho_{\varphi}\left(\frac{\tilde{f}-\tilde{g}_{\varepsilon}}{2}\right)+\rho_{\varphi}\left(\frac{\tilde{g}_{\varepsilon}-P_{\varepsilon}}{2}\right)\right] \leq \varepsilon.$$

i.e. $\tilde{f} \in \tilde{B}^{\varphi}_{a.p.}$. ∎

3. Result.

THEOREM 3.1 *$\tilde{B}^{\varphi}_{a.p.}$ is uniformly convex if and only if φ is uniformly convex and satisfies the condition Δ_2.*

PROOF **sufficiency:**

Let $\varepsilon \in]0,1[$ and f,g in $\tilde{B}^{\varphi}_{a.p.}$ be such that $\|f\|_{\varphi} = \|g\|_{\varphi} = 1$ and $\left\|\frac{f-g}{2}\right\|_{\varphi} \geq \varepsilon$.

From corollary 2.9 we have also $\rho_{\varphi}(f) = \rho_{\varphi}(g) = 1$ and $\rho_{\varphi}\left(\frac{f-g}{2}\right) \geq \delta$ for some $\delta = \delta(\varepsilon) \in]0,1[$.

Let h_{δ} be a measurable function such that $\rho_{\varphi}(h_{\delta}) = \frac{\delta}{4}$, then from the uniform convexity of φ there exists $p(\delta) \in]0,1[$ for which the following implication holds

$$\left[|h_{\delta}(t)| \leq \max(|f(t)|, |g(t)|) \leq \frac{4}{\delta}|f(t)-g(t)|\right]$$

$$\Longrightarrow \left[\varphi\left(t, \frac{|f(t)+g(t)|}{2}\right) \leq \frac{1-p(\delta)}{2}\left[\varphi(t,|f(t)|)+\varphi(t,|g(t)|)\right]\right].$$

Put

$$B = \left\{t \in \mathbb{R}, |h_{\delta}(t)| \leq \max(|f(t)|, |g(t)|) \leq \frac{4}{\delta}|f(t)-g(t)|\right\},$$

then

$$\frac{1}{2T}\int_{-T}^{+T}\varphi\left(t, \frac{|f(t)+g(t)|}{2}\right)\chi_B dt$$

$$\leq \frac{1-p(\delta)}{2}\left[\frac{1}{2T}\int_{-T}^{+T}\varphi(t,|f(t)|)\chi_B dt + \frac{1}{2T}\int_{-T}^{+T}\varphi(t,|g(t)|)\chi_B dt\right].$$

It follows

$$\rho_\varphi\left(\frac{f+g}{2}\chi_B\right)\leq\frac{1-p(\delta)}{2}\left[\rho_\varphi(f\chi_B)+\rho_\varphi(g\chi_B)\right]$$

and then,

$$\begin{aligned}1-\rho_\varphi\left(\frac{f+g}{2}\right) &= \frac{1}{2}\left(\rho_\varphi(f)+\rho_\varphi(g)\right)-\rho_\varphi\left(\frac{f+g}{2}\right)\\ &\geq \frac{1}{2}\left(\rho_\varphi(f\chi_B)+\rho_\varphi(g\chi_B)\right)-\rho_\varphi\left(\frac{f+g}{2}\chi_B\right)\\ &\geq \frac{p(\delta)}{2}\left(\rho_\varphi(f\chi_B)+\rho_\varphi(g\chi_B)\right). \qquad (7)\end{aligned}$$

We define the sets

$$C=\{t\in B' \ \texttt{s.t.} \quad \max(|f(t)|,|g(t)|)<|h_\delta(t)|\}$$

$$D=\left\{t\in B' \ \texttt{s.t.} \quad |f(t)-g(t)|<\frac{\delta}{4}\max(|f(t)|,|g(t)|)\right\}.$$

Then $B'=CUD$ and

$$\rho_\varphi\left(\frac{f-g}{2}\chi_C\right)\leq\frac{1}{2}\left[\rho_\varphi(f\chi_C)+\rho_\varphi(g\chi_C)\right]\leq\frac{\delta}{4}$$

$$\rho_\varphi\left(\frac{f-g}{2}\chi_D\right)\leq\frac{\delta}{8}\left[\rho_\varphi(f\chi_D)+\rho_\varphi(g\chi_D)\right]\leq\frac{\delta}{4}$$

hence $\rho_\varphi\left(\frac{f-g}{2}\chi_{B'}\right)\leq\frac{\delta}{2}$ and consequently $\rho_\varphi\left(\frac{f-g}{2}\chi_B\right)\geq\frac{\delta}{2}$.
Now, from the inequality

$$\frac{\delta}{2}\leq\rho_\varphi\left(\frac{f-g}{2}\chi_B\right)\leq\frac{1}{2}\left(\rho_\varphi(f\chi_B)+\rho_\varphi(g\chi_B)\right)$$

it follows

$$\left(\rho_\varphi(f\chi_B)+\rho_\varphi(g\chi_B)\right)\geq\delta$$

and using (7) we deduce

$$\rho_\varphi\left(\frac{f+g}{2}\right)\leq 1-\frac{\delta p(\delta)}{2}.$$

Finally, since $\varphi\in\Delta_2$ and recalling the fact that $\delta=\delta(\varepsilon)$ depends on ε, by lemma 2.11 there exists $q(\varepsilon)\in\,]0,1[$ such that $\left\|\frac{f+g}{2}\right\|\leq 1-q(\varepsilon)$.

Necessity : We denote by $E^{\varphi}([0,1]) = E^{\varphi}$ the usual Musielak-Orlicz class $E^{\varphi}([0,1]) = \left\{ f \in L^{\varphi}_{loc}(\mathbb{R}) : \forall \lambda > 0, \int\limits_0^1 \varphi(t, \lambda |f(t)|)\, dt < +\infty \right\}$ and by $\|.\|_{E^{\varphi}}$ it's usual Luxemburg norm. We consider the injection map

$$i : E^{\varphi} \hookrightarrow \tilde{B}^{\varphi}_{a.p.}(\mathbb{R}), \qquad i(f) = \widetilde{f}$$

where $\widetilde{f}$ is the periodic extension (with period $T = 1$) of f to $\mathbb{R}$. We show first that $i(E^{\varphi}) \subset \tilde{B}^{\varphi}_{a.p.}$. Indeed, if $f \in E^{\varphi}$, then using the density of the step functions in E^{φ} (see [2],[10]) we may find $f_{\varepsilon} = \sum\limits_{i=1}^{n} \alpha_i \chi_{A_i}$ with $\|f - f_{\varepsilon}\|_{E^{\varphi}} \leq \varepsilon$.

Let $\widetilde{f_{\varepsilon}}$and $\widetilde{f}$ denote the respective periodic extensions of f and f_{ε} to $\mathbb{R}$ (with the same period $T = 1$). From Lemma 2.12 we know that $\widetilde{f_{\varepsilon}} \in \tilde{B}^{\varphi}_{a.p.}(\mathbb{R})$ and we have clearly $\left\| \widetilde{f} - \widetilde{f_{\varepsilon}} \right\|_{\varphi} = \|f - f_{\varepsilon}\|_{E^{\varphi}} \leq \varepsilon$, it follows that $\widetilde{f} \in \tilde{B}^{\varphi}_{a.p.}(\mathbb{R})$.

Now, the uniform convexity of φ being necessary for the uniform convexity of E^{φ}(see[5]), from the equality $\left\| \widetilde{f} \right\|_{\varphi} = \|f\|_{E^{\varphi}}$ we deduce it's necessity for the uniform convexity of $\tilde{B}^{\varphi}_{a.p.}(\mathbb{R})$.

The necessity of the Δ_2 condition follows from it's necessity for the strict convexity of E^{φ}(see[6]). ■

References

[1] A. S. Besicovitch, *Almost periodic functions*, Dover Publ. Inc. , New York 1954.

[2] S. Chen, *Geometry of Orlicz spaces*, Dissertationes Math. **356**, 1996.

[3] C. Corduneanu, N. Gheorghiu and V. Barbu, *Almost periodic function*, Chelsea Publishing Company 1989.

[4] T. R. Hillmann, *Besicovitch-Orlicz spaces of almost periodic functions*, Real and Stochastic Analysis. **164**, Wiley Ser. in probability and Math. Stat., Wiley., New York 1986.

[5] H. Hudzik, *Convexity in Musielak- Orlicz spaces*, Hokkaido Mathematical Journal, **14** (1985), 85-96.

[6] H. Hudzik, *Strict convexity of Musielak- Orlicz spaces with Luxemburg 's Norm*, Bull. Acad. Polon. Sci. Math. **39(5-6)** (1981), 235-247.

[7] M. Morsli and F. Bedouhene, *On the uniform convexity of the Besicovitch-Orlicz space of almost periodic functions with Orlicz norm*, Colloquium Mathematicum **102(1)** (2005), 97-111.

[8] M. Morsli, On some convexity properties of the Besicovitch-Orlicz spaces of almost periodic functions, Comment. Math. **34** (1994), 137-152.

[9] M. Morsli and F. Bedouhene, *On the strict convexity of the Besicovitch-Orlicz space of almost periodic functions*, Revista Matematica Complutense **16(2)** (2003), 399-415.

[10] J. Musielak, *Orlicz spaces and modular spaces*, Lecture notes in Math., Springer-Verlag **1034**, 1983.

Mohamed Morsli
Department of Mathematics, Faculty of Sciences, University of Tizi-Ouzou
Oued-Aissi, Algeria
E-mail: mdmorsli@yahoo.fr

Mannal Smaali
Department of Mathematics, Faculty of Sciences, University of Tizi-Ouzou
Oued-Aissi, Algeria
E-mail: smaali-manel@yahoo.fr

(Received: 15.02.2006)