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On the Fractional Pettis and Aumann-Pettis Integral for Multifunctions

Abstract. Let α be a positive real number. In the present paper we present the definition of the Aumann Pettis integral and the Pettis integral of order α for multifunctions. The properties of these integrals and the relations between them are studied extensively. In particular, a Strassen type theorem in this case and continuation property are proved. Also, we give a version for Fatou's lemma and dominated convergence theorem for the Aumann-Pettis integral of order α and for multifunctions.

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1. Introduction. It is know that in the theory of integration in infinite dimensional spaces, Pettis integrability is more general concept than Bochner integrability. There are many papers in the literature dealing with the Bochner integral for multifunctions, which is defined in terms of Bochner integral selection, and the Pettis integral for multifunctions, which is defined in terms of Pettis integral selections and the relation between them (see for example [1], [2], [3], [4-7], [10], [12], [17-19], [21] and [25-31]).

The study of measurable multifunctions has been developed extensively with applications to mathematical economics and optimal control theory by many authors. The natural approach, which derives from the study of integro differential inclusions, is due to Aumann in 1965 [4] and is based on the integration of measurable selctions. Unfortunately the Aumann integral does not satisfy all the usual properties of an integral. So it seems to be natural to investigate whether the Aumann integral can be regarded as a Bochner or Debreu integral [10]. The comparison between the

Aumann and Bochner integrals for measurable multifunctions has been studied in [7], [9] and [26]. Pettis integrability for multifunction has been introduced-using the two notions of integrability-and studied in [12].

Let $J = [a, b], a$ be a non negative real number, α be a positive real number, X be a separable Banach space and F be a multifunction defined on J into the family of all nonempty closed and convex subsets of X. In [19] the authors introduced the definition of the Aumann integral for F of order α.

In the present paper we introduce the definition of the Pettis and Aumann Pettis integral for F of order α over any subinterval $[a, t]$ of $J, I_a^{P,\alpha}F(t)$ and $I_a^{AP,\alpha}F(t)$, the two integrals are shown to coincide when the values of F are nonempty, closed, convex and weakly compact subsets of X. The relation between Aumann and Aumann Pettis integral for F of order α is obtained. In particular, a Strassen type theorem in this case is proved. A continuation property is obtained, i.e. when α tends to 1 in our results, we obtain the known results when $\alpha = 1$, (see Theorem 3.2.1 in this paper). So, our results can be considered as a generalization to many known results in the literature when α tends to 1, (see [12] and [35]). Also, we give a version for Fatou's lemma and dominated convergence theorem for the Aumann Pettis integral of order α and for multifunctions. For more details about the factional calculus of function or multifunctions and its applications we refer to [13-16], [19], [20], [23] and [32-34].

2. Notation and some auxilary facts. Let $(S, \mathcal{A}, \mu)$ be a measure space, X be a separable Banach space with dual space X^* and Borel σ-field $\mathcal{B}(\mathcal{X})$. We recall facts about measurability and integrability of functions, (see [3], [9], [11], [12] and [25]).

Definition 2.1 A function $f : S \to X$ is said to be *weakly measurable or scalary measurable (integrable)* if for each $y \in X^*$, the numerical function $s \longmapsto \langle y, f(s)\rangle$ is measurable (integrable).

Definition 2.2 The function f is said to be *strongly measurable or measurable* If there exists a sequence of finitely-valued functions strongly convergent to f a.e. on S.

Definition 2.3 A function f defined on $(S, \mathcal{A}, \mu)$ with values in X is said to be *Bochner integrable*, if there exists a sequence of finitely valued functions such that $\{f_n(s)\}$ strongly converges to $f(s)$ a.e. and

$$\lim_{n\to\infty} \int_S \|f(s) - f_n(s)\|\mu(ds) = 0.$$

In this case for any set $A \in \mathcal{A}$, the Bochner integral of $f(s)$ over A is defined by

$$\int_A f(s)\mu(ds) = \lim_{n\to\infty} \int_A \chi_A(s) f_n(s)\mu(ds),$$

where χ_A is the characteristic function of A.

It is known that a strongly measurable function f is Bochner integrable if and only if the function $s \longmapsto \|f(s)\|$ is integrable.

DEFINITION 2.4 A measurable and scalary integrable function $f : S \to X$ is said to be *Pettis integrable* if for every $A \in \mathcal{A}$, there exists $x_A \in X$ such that

$$\langle y, x_A \rangle = \int_A \langle y, f(s) \rangle \mu(ds), \; \forall y \in X^*.$$

x_A is called the Pettis *(weak) integral* of f over A and we write

$$x_A = w - \int_A f(s)\mu(ds).$$

REMARK 2.5 (1) For any function $f : S \to X$, we have the following implications: f is Bochner integrable $\Rightarrow$ f is Pettis integrable $\Rightarrow$ f is scalary integrable.

(2) When X is finite dimensional, the notions of Bochner integrability, Pettis integrability, and scalar integrability all coincide. When X is infinite-dimensional, there are Pettis integrable functions, which are not Bochner integrable. There also exist scalary integrable functions, which are not Pettis integrable.

THEOREM 2.6 *A scalary integrable function $f : S \to X$ is Pettis integrable if and only if the set*

$$\{\langle y, f \rangle : y \in B^*\}$$

is uniformly integrable, where B^* is the closed unit ball of X^*.

Now we present the previous concepts for multifunctions.

We will use the following notations:

$\mathcal{P}_{\mathcal{C}}(\mathcal{X})$*: The set of all nonempty and closed subsets of X.*

$\mathcal{P}_{\mathcal{CC}}(\mathcal{X})$*: The set of all nonempty, closed and convex subset of X.*

$\mathcal{P}_{\mathcal{CB}}(\mathcal{X})$*: The set of all nonempty, bounded, closed and convex subsets of X.*

$\mathcal{P}_{\mathcal{CK}}(\mathcal{X})$*: The set of all nonempty, compact and convex subsets of X.*

$\mathcal{P}_{\mathcal{CWK}}(\mathcal{X})$*: The set of all nonempty, weakly compact and convex subset of X.*

$\mathcal{P}_{\mathcal{LW}^{\#}}(\mathcal{X})$*: The set of all nonempty, closed, convex, weakly locally compact and do not contain any line (they may contain half only) subsets of X.*

DEFINITION 2.7 For every $C \in \mathcal{P}_{\mathcal{C}}(\mathcal{X})$, the *support function* of C is denoted by $\delta^*(., C)$ and defined on X^* by

$$\delta^*(y, C) = \sup \{< y, x >: x \in C\}, \; y \in X^*.$$

If $C = \phi$, then $\delta^*(y, C) = -\infty$.

Definition 2.8 A multifunction $F : S \to \mathcal{P}_{\mathcal{C}}(\mathcal{X})$ is said to be *scalary measurable (integrable)* if every $y \in X^*$, the map $\delta^*(y, F(.))$ is measurable (integrable).

Definition 2.9 A multifunction $F : S \to \mathcal{P}_{\mathcal{C}}(\mathcal{X})$ is said to be *Effros-measurable or measurable* if, $F^{-}U = \{s \in S : F(s) \cap U \neq \phi\} \in \mathcal{A}$ for any open subset U of X.

Effors measurability is stronger than the scalar measurability. On the other hand, it is known that, for multifunctions with values in $\mathcal{P}_{\mathcal{LW}^\#}$ both measurability concepts coincide.

Definition 2.10 A measurable multifunction $F : S \to \mathcal{P}_{\mathcal{C}}(\mathcal{X})$ is said to be *Aumann integrable* if it admits at least one Bochner integrable selection, that is to say $S_F^1 \neq \phi$, where S_F^1 is the set of all Bochner integrable selections of F. In this case, the Aumann integral $I_B^A F$ of F over a measurable set B is defined by

$$I_B^A F := \left\{ \int_B f(s)\mu(ds) : f \in S_F^1 \right\}$$

It is known that $S_F^1 \neq \phi$ if and only if the distance function

$$s \to d(0, F(s)) = \inf \{\|x\| : x \in F(s)\}$$

is integrable.

Definition 2.11 A multifunction $F : S \to \mathcal{P}_{\mathcal{C}}(\mathcal{X})$ is called *integrably bounded* if there exists an integrable non negative function $g : S \to [0, \infty[$ such that for a.e. $s \in S$, $\|F(s)\| \leq g(s)$, where

$$\|F(s)\| = \sup \{\|x\| : x \in F(s)\}$$

If $F : S \to \mathcal{P}_{\mathcal{C}}(\mathcal{X})$ is measurable and integrably bounded, then it is Aumann integrable.

Definition 2.12 A measurable and scalary integrable multifunction $F : S \to \mathcal{P}_{\mathcal{CC}}(\mathcal{X})$ is said to be *Pettis integrable* if, for every $B \in \mathcal{A}$ there exists a set $I_B^P F \in \mathcal{P}_{\mathcal{CC}}(\mathcal{X})$ such that

$$\delta^*(y, I_B^P F) = \int_B \delta^*(y, F(s))\mu(ds), \forall y \in X^*$$

The set $I_B^P F$ is called the *Pettis or weak integral* of F over B.

If $\mathcal{C}$ is a subspace of $\mathcal{P}_{\mathcal{CC}}(\mathcal{X})$, we say that the multifunction $F : S \to \mathcal{C}$ is Pettis integrable in $\mathcal{C}$ if $I_B^P F$ is a member of $\mathcal{C}$ for each $B \in \mathcal{A}$.

Proposition 2.13 ([12]]) *Let* $F : S \to \mathcal{P}_{\mathcal{CWK}}(\mathcal{X})$ *be measurable and scalary integrable multifunction. If we assume in addition that* F *is Pettis integrable in* $\mathcal{P}_{\mathcal{CWK}}(\mathcal{X})$*, then every measurable selection of* F *is scalary integrable and Pettis integrable.*

It is also possible to define an Aumann-type integral in terms of Pettis integrable selections as follows:

Definition 2.14 Given a measurable multifunction $F : S \to \mathcal{P}_{\mathcal{C}}(\mathcal{X})$ and $A \in \mathcal{A}$. F is *Aumann Pettis integrable* if it admits at least one Pettis integrable selection, i.e. If S_F^{Pe} is nonempty. In that case the *Aumann Pettis integrable* of F over B is denoted by $I_B^{AP}F$ and is defined by

$$I_B^{AP}F = \left\{ w - \int_B f(s)\mu(ds) : f \in S_F^{Pe} \right\},$$

where $w - \int_B f(s)\mu(ds)$ is the Pettis (weak) integral of f over B.

Clearly, $S_F^1 \subseteq S_F^{Pe}$. So, if F is Bochner integrable, then it is Aumann Pettis integrable.

We also use the following definitions and theorems:

Definition 2.15 A function $\phi : X^* \to (-\infty, +\infty]$ is *positively homogeneous* if $\phi(\alpha y) = \alpha\phi(y)$ for all $\alpha > 0$ and $y \in X^*$. It is *subadditive* if $\phi(y_1 + y_2) \le \phi(y_1) + \phi(y_2) \forall y_1, y_2 \in X^*$. It is *sublinear* if it is both positively homogeneous and subadditive.

Proposition 2.16 *Let* $\phi : X^* \to (-\infty, +\infty]$ *be a sublinear function. In order that there exist a nonempty* $C \in \mathcal{P}_{\mathcal{CK}}(\mathcal{X})$ *such that* $\phi = \delta^*(., C)$*, it is necessary and sufficient that* ϕ *be* w^**-lower semicontinuous.*

Definition 2.17 ([22]) A sequence $(\phi_n)_n$ of real-valued or vector-valued measureable functions is said to *Komlos-converge*, or, in short, to *K-converge*, to a function ϕ if, for every subsequence $(\phi_{n'})_{n'}$ of $(\phi_n)_n$, there exists a null set N (generally depending upon the subsequence) such that

$$\lim_{m\to\infty} \frac{1}{m} \sum_{n'=1}^{m} \phi_{n'}(s) = \phi(s) \qquad \forall s \in S \setminus N.$$

Theorem 2.18 *Every* L^1*-bounded sequence* $(\phi_n)_n$ *of measureable functions contains a subsequence that* K*-converges to some member* ϕ *of* L^1*.*

Remark 2.19 Let $(a_n)_n$ be a sequence in R and $a \in R$, then $(a_n)_n$ K-converges to a is equivalent to $a_n \to a$.

Finally, the weak-star (resp. the Mackey) topology of X^* is denoted by w^* (resp. τ). Recall that the Mackey topology on X^* is the topology of uniform convergence on symmetric convex weakly compact subsets of X.

3. Main results.

3.1. The Pettis and Aumann-Pettis integral for multifunctions.

Let $(J = [a,b], \mathcal{A}, \mu)$ be a Lebesgue measure space, X be a separable Banach space, X^* be its dual, a be a non negative real number, α be a positive real number, $L^1_X(J,\mathcal{A},\mu)$ be the set of all Bochner integrable functions from J to X, $L^{Pe}_X(J,\mathcal{A},\mu)$ be the set of all Pettis integrable functions from J to X, F be a multifunction from J to $\mathcal{P_C}(\mathcal{X})$, and S^1_F, S^{Pe}_F be the following sets

$$\begin{aligned} S^1_F &= \left\{ f \in L^1_X(J,\mathcal{A},\mu) : f(t) \in F(t) a.e. \text{ on } J \right\} \\ S^{Pe}_F &= \left\{ f \in L^{Pe}_X(J,\mathcal{A},\mu) : f(t) \in F(t) a.e. \text{ on } J \right\}. \end{aligned}$$

Definition 3.1 A measurable and scalary integrable multifunction $F : J \to \mathcal{P_{CC}}(\mathcal{X})$ is said to be *Pettis integrable of order* α over $[a,t]; t \in [a,b]$ if there exists a member $I^{P,\alpha}_a F(t) \in \mathcal{P_{CC}}(\mathcal{X})$ such that

$$\delta^*(y, I^{P,\alpha}_a F(t)) = \int_a^t \delta^*(y, \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F(s)) ds,$$

for each $y \in X^*$, where Γ is the gamma function. The set $I^{P,\alpha}_a F(t)$ is called the *Pettis or weak integral of order* α of F over $[a,t]$.

If $\mathcal{C}$ is a subspace of $\mathcal{P_{CC}}(\mathcal{X})$, we say that the multifunction $F : S \to \mathcal{C}$ is Pettis integrable of order α in $\mathcal{C}$ if $I^{P,\alpha}_a F(t)$ is a member of $\mathcal{C}$ for each interval $[a,t]; t \in [a,b]$.

Remark 3.2 It is clear that, if F is Pettis integrable, then it is Pettis integrable of order α. See Proposition 2.16.

Example 3.3 Let r be a positive real number, f be a function defined on J by

$$f(t) = t^\gamma, \gamma > -1,$$

and F be a multifunction defined on J by

$$F(t) = \bar{B}(f(t), r) = \text{the closed ball of radius } r \text{ centered at } f(t).$$

Clearly, F is measurable and scalary integrable multifunction, since

$$\delta^*(y, F(t)) = \langle y, f(t) \rangle + r\|y\|,$$

and f is scalary integrable function.

We can easily shown that for all $y \in R$

$$\begin{array}{l}\delta^*(y, \bar{B}(\int_a^t \phi_\alpha(t-s)f(s)\mu(ds), \int_a^t \phi_\alpha(t-s)r\mu(ds)) \\ = \int_a^t \delta^*(y, \phi_\alpha(t-s)F(s))\mu(ds), t \in J,\end{array}$$

then,

$$I_a^{P,\alpha}F(t) = \bar{B}(\int_a^t \phi_\alpha(t-s)f(s)\mu(ds), \int_a^t \phi_\alpha(t-s)r\mu(ds)), t \in J.$$

But, and it is known that (see [23] Pg. 47)

$$\begin{array}{ll}\int_a^t \phi_\alpha(t-s)s^\gamma\mu(ds) & = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}(t-a)^{\gamma+\alpha}, \\ \int_a^t \phi_\alpha(t-s)r\mu(ds) & = \frac{r(t-a)}{\Gamma(\alpha+1)}.\end{array}$$

Hence,

$$I_a^{P,\alpha}F(t) = \bar{B}(\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}(t-a)^{\gamma+\alpha}, \frac{r(t-a)}{\Gamma(\alpha+1)}), t \in J.$$

When $\alpha = 1$, we obtain

$$I_a^{P,1}F(t) = \bar{B}(\frac{(t-a)^{\gamma+1}}{\gamma+1}, r(t-a)), t \in J.$$

DEFINITION 3.4 Let $F : J \to \mathcal{P}_{\mathcal{C}}(\mathcal{X})$ be Aumann Pettis integrable multifunction, $\alpha > 0$. We define Aumann Pettis integral of order α for F over the interval $[a, t]; t \in [a, b]$ as

$$I_a^{AP,\alpha}F(t) = \left\{ w - \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds : f \in S_F^{Pe} \right\}.$$

The following theorems shows that when F is closed valued and Aumann Pettis integrable multifunction of order α, then it is also Pettis integrable of order α. Now, our aim is to obtain the relation between the Aumann-Pettis integral of order α and the Pettis integral of order α for multifunction.

THEOREM 3.5 *Let $F : J \to \mathcal{P}_{\mathcal{C}}(\mathcal{X})$ be an Aumann-Pettis integrable multifunction. Then, for every $y \in X^*$ and $t \in [a, b]$ we have*

$$\delta^*(y, I_a^{AP,\alpha}F(t)) = \int_a^t \delta^*(y, \phi_\alpha(t-s)F(s))\mu(ds), \tag{1}$$

where $\phi_\alpha(t) = \begin{cases} \frac{t^{\alpha-1}}{\Gamma(\alpha)} & t > 0 \\ 0 & t \leq 0 \end{cases}$ *and the common value in* (1) *can be finite or equal to* $+\infty$.

Proof Let t be a fixed point in $[a,b]$ and $y \in X^*$. Put $B = [a,t]$, we have

$$\begin{aligned}\delta^*(y, I_a^{AP,\alpha}F(t)) &= \sup\left\{\langle y, \textstyle\int_a^t \phi_\alpha(t-s)f(s)\rangle\mu(ds) : f \in S_F^{Pe}\right\}\\ &= \sup\left\{\textstyle\int_a^t \langle y, \phi_\alpha(t-s)f(s)\rangle\mu(ds) : f \in S_F^{Pe}\right\}\\ &\leq \textstyle\int_a^t \delta^*(y, \phi_\alpha(t-s)F(s))\mu(ds).\end{aligned}$$

To prove the converse inequality, we distinguish two cases:

Case 1: $\int_a^t \delta^*(y, \phi_\alpha(t-s)F(s))\mu(ds) < \infty$.

Given $\varepsilon > 0$, we have to show the existence of $g \in S_F^{Pe}$ such that

$$\int_a^t \delta^*(y, \phi_\alpha(t-s)F(s))\mu(ds) \leq \langle y, \int_a^t \phi_\alpha(t-s)g(s)\rangle\mu(ds) + \varepsilon,$$

and this will be satisfied if we prove that

$$\int_a^t \delta^*(y, \phi_\alpha(t-s)F(s))\mu(ds) \leq \int_a^t \langle y, \phi_\alpha(t-s)g(s)\rangle\mu(ds) + \varepsilon.$$

Fix $y \in X^*$, and define a multifunction G by

$$G(s) = F(s) \cap \{z \in X : \delta^*(y, \phi_\alpha(t-s)F(s)) \leq \langle y, \phi_\alpha(t-s)z\rangle + \varepsilon\}.$$

The graph of G is measurable, then we can find a measurable selection h, i.e.

$$h(s) \in G(s) \quad \text{a.e., and}$$

$$\delta^*(y, \phi_\alpha(t-s)F(s)) \leq \langle y, \phi_\alpha(t-s)h(s)\rangle + \varepsilon.$$

Now, for any integer $k \geq 1$, consider the measurable subsets B_k defined by

$$B_k = \{s \in B : \|h(s)\| \leq k\}.$$

Clearly,

$$B_1 \subseteq B_2 \subseteq \cdots \subseteq B_k \subseteq B_{k+1} \subseteq \cdots \subseteq B.$$

Then

$$\lim_{k\to\infty} \mu([a,t] \setminus B_k) = 0.$$

For every $k \geq 1$, consider the function $\mathbf{1}_{B_k}f$ which is defined by

$$\mathbf{1}_{B_k}f(x) = \begin{cases} f(x) & \text{if}\, x \in B\\ 0 & \text{if}\, x \notin B\end{cases}$$

Clearly, $\mathbf{1}_{B_k}f; k \geq 1$ is measurable and bounded, then it is Bochner integrable.

Let f_0 be a fixed Pettis integrable selection of F. Since the two functions $s \longmapsto \langle y, \phi_\alpha(t-s)f_0\rangle$ and $s \longmapsto \delta^*(y, \phi_\alpha(t-s)F(s))$ are integrable, we can find an integer k_0 large enough such that for every $k \geq k_0$

$$\int_{B\setminus B_k} |\langle y, \phi_\alpha(t-s)f_0(s)\rangle|\mu(ds) < \varepsilon \tag{2}$$

and

$$\int_{B\setminus B_k} |\delta^*(y,\phi_\alpha(t-s)F(s))|\mu(ds) < \varepsilon. \tag{3}$$

Now, set

$$g = \mathbf{1}_{B_k} h + \mathbf{1}_{B\setminus B_k} f_0, \text{ for some}\, k \geq k_0.$$

Then

$$\begin{aligned}\textstyle\int_B \langle y,\phi_\alpha(t-s)g(s)\rangle\mu(ds) &= \textstyle\int_{B_k}\langle y,\phi_\alpha(t-s)h(s)\rangle\mu(ds)\\ &\quad +\textstyle\int_{B\setminus B_k}\langle y,\phi_\alpha(t-s)f_0(s)\rangle\mu(ds)\\ &\geq \textstyle\int_B \delta^*(y,\phi_\alpha(t-s)F(s))\mu(ds) - \varepsilon\mu(B_k) - \varepsilon\\ &\geq \textstyle\int_B \delta^*(y,\phi_\alpha(t-s)F(s))\mu(ds) - (2+\mu(B))\varepsilon\end{aligned}$$

This being true for every $\varepsilon > 0$, the desired inequality is proved.

Case 2: $\int_a^t \delta^*(y,\phi_\alpha(t-s)F(s))\mu(ds) = +\infty$.

In this case, we define G by

$$G(s) = F(s) \cap \{x \in X : \langle y,\phi_\alpha(t-s)x\rangle \geq m\},$$

where m is positive real. Like in the proof of case 1, we consider a Pettis integrable selection f_0 of F and a measurable selection h of G, and it is enough to choose k_0 large enough so it satisfies (2) with $\alpha = 1$ and such that it satisfies

$$\mu(B_k) \geq d, \text{ for some}\, d \leq b-a,\ \forall k \geq k_0.$$

This entails

$$\begin{aligned}\textstyle\int_B \langle y,\phi_\alpha(t-s)g(s)\rangle\mu(ds) &= \textstyle\int_{B_k}\langle y,\phi_\alpha(t-s)h(s)\rangle\mu(ds)\\ &\quad +\textstyle\int_{B\setminus B_k}\langle y,\phi_\alpha(t-s)f_0(s)\rangle\mu(ds) \geq m\mu(B_k) - 1\\ &\geq md - 1.\end{aligned}$$

Since the right-hand side can be made arbitrarily large by choosing m large enough. This completes the proof. ■

Corollary 3.6 *Let $F : J \to \mathcal{P}_{\mathcal{C}}(\mathcal{X})$ be an Aumann-Pettis integrable multifunction, then the multifunction cl co F is Pettis integrable and its Pettis integral is equal to the closed convex hull of Aumann Pettis integral i.e.*

$$I_a^{P,\alpha} cl\, co F(t) = cl\, co I_a^{AP,\alpha} F(t)\, for\ every t \in [a,b].$$

Proof Clear from Definition 3.1 and Theorem 3.5. ■

Proposition 3.7 *Let $F : J \to \mathcal{P}_{\mathcal{CWK}}(\mathcal{X})$ be a measurable and scalary integrable multifunction. If every meaurable selection of F is Pettis integrable, then for every $t \in [a,b], I_a^{AP,\alpha}F(t)$ is closed.*

PROOF Let t be a fixed point in $[a, b]$ and put $B = [a, b]$. Let $(x_n)_{n\geq 1}$ be a sequence in $I_a^{AP,\alpha}F(t)$, converges to x. For every $n \geq 1$, there exists $f_n \in S_F^{Pe}$ such that

$$x_n = w - \int_a^t \phi_\alpha(t-s)f_n(s)\mu(ds).$$

Let D be a countable w^*-dense of B^*. By Theorem 2.18 and the diagonal extraction procedure, we can find a subsequence $(f_{n'})$ of (f_n) such that for each $y \in D$, the sequence $(\langle y, f_{n'}\rangle)_{n'\geq 1}$ K-converges to some ψ_y in L^1. for every $m \geq 1$, we define the function g_m and Z_m by

$$\begin{aligned} g_m(s) &= \tfrac{1}{m}\sum_{n'=1}^{m} f_{n'}(s), \\ Z_m(s) &= \phi_\alpha(t-s)g_m(s)\ \forall s \in B. \end{aligned}$$

Since $f_{n'} \in S_F^{Pe}$ and since for every $s \in B, F(s)$ is convex, then

$$g_m(s) \in F(s)\,\text{for every}\,m \geq 1\,\text{and every}\,s \in B.$$

Furthermore, since $F(s); s \in J$ is weakly compact and convex then, Elberlein-Smulyan theorem shows that the sequence $(g_m(s))_{m\geq 1}$ admits a weak cluster point denoted by $g(s)$. Hence, one can find a null set N such that

$$\phi_y(s) = \langle y, g(s)\rangle = \lim_{m\to\infty}\langle y, g_m(s)\rangle\ \forall s \in S \setminus N, y \in D. \tag{4}$$

This shows the uniqueness of the weak cluster point $g(s)$, hence the weak convergence of $(g_m(s))$ to $g(s)$. We also have.

$$\lim_{m\to\infty}\langle t, Z_m(s)\rangle = \langle y, Z(s)\rangle\ \forall s \in S\backslash N, y \in D,$$

where

$$Z(s) = \phi_\alpha(t-s)g(s)\ \ \forall s \in B.$$

Since

$$\|x - g(s)\| = \sup\{|\langle y, x - g(s)\rangle| : y \in D\}$$

valid for every $x \in X$ and every $s \in B$, then g is measurable. Further, for every $s \in S, y \in X^*$ and $m \geq 1$, the following inequalities hold

$$-\delta^*(-y, \phi_\alpha(t-s)F(s)) \leq \langle y, \phi_\alpha(t-s)g_m(s)\rangle \leq \delta^*(y, \phi_\alpha(t-s)F(s)). \tag{5}$$

Thus, taking into account (4), (5) and the scalar integrability of F, it is possible to apply Lebesgue's Deminated Convergence theorem,

$$\lim_{m\to\infty}\int_B \langle y, Z_m\rangle\mu(ds) = \int_B \langle y, Z\rangle\mu(ds)$$

and by Pettis integrability hypothesis,

$$\lim_{m\to\infty}\langle y, \int_B Z_m\mu(ds)\rangle = \langle y, \int_B Z\mu(ds)\rangle.$$

This yields

$$x = \int_B Z d\mu$$

and shows the closedness of $I_a^{AP,\alpha} F(t)$. ∎

In the following theorem we show that when the values of F are convex and weakly compact, then the Aumann-Pettis integral of order α and the Pettis integral of order α of F are coincide. On the other hand, this theorem gives factional version for Theorem 3.2 of [35].

THEOREM 3.8 *Let $F : J \to P_{CWK}(X)$ be a measurable and scalary integrable multifunction. Then, the following two statements are equivalent:*

- *(a) F is Pettis integrable of order α in $\mathcal{P_{CWK}(X)}$,*
- *(b) for every $t \in [a, b]$, the Aumann-Pettis integral of order α over $[a, t]$ is a member of $\mathcal{P_{CWK}(X)}$ and, for every $y \in X^*$, one has*

$$\delta^*(y, I_a^{AP,\alpha} F(t)) = \int_a^t \delta^*(y, \phi_\alpha(t - s)F(s))ds. \tag{6}$$

PROOF (a) ⇒ (b). Suppose that F is Pettis integrable of order α. Then, every measurable selection of F is Pettis integrable, by Proposition 2.13. This means that $S_F^{Pe} \neq \phi$, consequently F is Aumann Pettis integrable over any interval $[a, t]; t \in [a, b]$. So, our aim is to show that for every $t \in [a, b], I_a^{AP,\alpha} F(t)$ is a member of P_{CWK}. Let $B = [a, t]$. Without loss of generality, we fix $t \in [a, b]$. From Proposition 3.7, $I_a^{AP,\alpha} F(t)$ is convex and closed, then weakly closed by convexity. To show that $I_a^{AP,\alpha} F(t)$ is weakly compact it suffices, by James Pryce Theorem [27], to prove that for every $y \in X^*$ there is $z_y \in I_a^{AP,\alpha} F(t)$ such that

$$\langle y, z_y \rangle = \delta^*(y, I_a^{AP,\alpha} F(t)).$$

For this purpose we fix $y \in X^*$ and define a multifunction $G : B \to \mathcal{P_{CWK}(X)}$ by

$$G(s) = F(s) \cap \{x \in X : \langle y, x \rangle = \delta^*(y, \phi_\alpha(t - s)F(s))\}.$$

Since $\phi_\alpha(t - s)F(s) \in \mathcal{P_{CWK}(X)}$, then there is $u \in \phi_\alpha(t - s)F(s)$ such that

$$\langle y, u \rangle = \delta^*(y, \phi_\alpha(t - s)(s)).$$

This means that $G(s)$ is nonempty. Also, since $F(s); s \in B$ is convex and weakly compact, so is $G(s); s \in B. G$ is measurable and then it has a measurable selection $f : B \to X$. Hence, it is Pettis integrable. Let z_y be the Pettis integral of order α of f over B. Then.

$$\langle y, z_y \rangle = \int_a^t \langle y, \phi_\alpha(t - s)f(s) \rangle \mu(ds).$$

But for every $s \in B$,

$$\langle y, \phi_\alpha(t - s)F(s) \rangle = \delta^*(y, \phi_\alpha(t - s)f(s)).$$

Then,

$$\begin{aligned}\langle y, z_y\rangle &= \int_a^t \delta^*(y, \phi_\alpha(t-s)f(s))\mu(ds)\\ &= \langle y, I_a^{AP,\alpha}F(t)\rangle.\end{aligned}$$

(b) ⇒ (a) is obtained by setting, for every $t \in [a,b], I_a^{P,\alpha}F(t) = I_a^{AP,\alpha}F(t)$. ■

EXAMPLE 3.9 Let X be a reflexive Banach space, f be a Bochner integrable function from J to X and r be a function that belongs to $L^1(J,[0,\infty])$. Let F be a multifunction defined on J by

$$F(t) = \bar{B}(f(t), r(t)).$$

Clearly, the values of F are in $\mathcal{P}_{\mathcal{CWK}}(\mathcal{X})$. Since f is measurable, then F is also. On the other hand, we can show that for every $t \in J$ and $y \in X^*$, we have

$$\delta^*(y, F(t)) = \langle y, f(t)\rangle + r(t)\|y\|,$$

Then, from the scalar integrability of f, we can deduce that F is scalary integrable. Thus, F satisfies the assumptions of Theorem 3.8. Now, for every $y \in X^*$, we have

$$\begin{aligned}&\delta^*(y, \bar{B}(w - \int_a^t \phi_\alpha(t-s)f(s)\mu(ds)), w - \int_a^t \phi_\alpha(t-s)r(s)\mu(ds))\\ &= \int_a^t \delta^*(y, \phi_\alpha(t-s)F(s))\mu(ds), t \in J.\end{aligned}$$

Thus, F is Pettis integrable of order α and

$$I_a^{AP,\alpha}F(t) = I_\alpha^{P,\alpha}F(t) = \bar{B}(w - \int_a^t \phi_\alpha(t-s)f(s)\mu(ds), w - \int_a^t \phi_\alpha(t-s)r(s)\mu(ds)), t \in J.$$

COROLLARY 3.10 *Let* $F : J \to \mathcal{P}_{\mathcal{CWK}}(\mathcal{X})$ *be a measurable and scalary integrable multifunction, then the multifunction* $t \to I_a^{AP,\alpha}F(t)$ *is a measurable and scalary integrable multifunction.*

An analogous result for a multifunction with strongly compact values can be formulated.

THEOREM 3.11 *Let* $F : S \to \mathcal{P}_{\mathcal{CK}}(\mathcal{X})$ *be a measurable and scalary integrable multifunction. Then, the following two statements are equivalent.*

(a) F *is Pettis integrable of order* α *in* $\mathcal{P}_{\mathcal{CK}}(\mathcal{X})$*,*

(b) for every $t \in [a,b]$*, the Aumann-Pettis integral of order* α *over* $[a,t]$ *is a member of* $\mathcal{P}_{\mathcal{CK}}(\mathcal{X})$ *and, for every* $y \in X^*$*, one has*

$$\delta^*(y, I_a^{AP,\alpha}F(t)) = \int_a^t \delta^*(y, \phi_\alpha(t-s)F(s))ds.$$

Proof In order to prove that (a) $\Rightarrow$ (b), we use the proposition, which states that for every $C \in \mathcal{P}_{\mathcal{C}}, \mathcal{C} \in \mathcal{P}_{\mathcal{CK}}$ if and only if the restrection of $\delta^*(.,C)$ to B^* is w^*-continuous. Moreover, due to the separability of X, the restriction of w^* to B^* is metrizable, so we can restrict our attention to sequences. Without loss of generality, we fix $t \in [a,b]$. Put $B = [a,t]$. So let $(y_n)_{n\geq 1}$ be a sequence in B^* converging to some $y \in B^*$. Since $F(s) \in \mathcal{P}_{\mathcal{CK}}(\mathcal{X})$ for every $s \in J$, we have

$$\delta^*(y, \phi_\alpha(t-s)F(s)) = \lim_{n\to\infty} \delta^*(y_n, \phi_\alpha(t-s)F(s)) \quad \forall s \in J.$$

Since F is Pettis integrable of order α in $\mathcal{P}_{\mathcal{CK}}(\mathcal{X})$, then using Proposition 3.1 in [3] or Theorem 5.5 in [7] we know that the sequence $(\delta^*(y_n, F(s))_{n\geq 1}$ is uniformly integrable. Therefore, an appeal to the Lebesgue-Vitali theorem yields

$$\int_a^t \delta^*(y, \phi_\alpha(t-s)F(s))d\mu = \lim_{n\to\infty} \int_a^t \delta^*(y_n, \phi_\alpha(t-s)F(s))d\mu.$$

Since, by Theorem 3.8, equality (6) holds, we obtain

$$\delta^*(y, I_a^{AP,\alpha}F(t)) = \lim_{n\to\infty} \delta^*(y_n, I_a^{AP,\alpha}F(t)).$$

i.e. the restrection of $\delta^*(., I_a^{AP,\alpha}(t))$ to B^* is w^*-continuous. Then, $I_a^{AP,\alpha}F(t)$ is strongly compact.

Like in the proof of Theorem 3.8, implication (b) $\Rightarrow$ (a) is obtained by setting, for every $t \in [a,b], I_a^{P,\alpha}F(t) = I_a^{AP,\alpha}F(t)$. ■

The following proposition gives the relation between Aumann integral of order α and Aumann Pettis Integral of order α, when F is closed valued multifunction and its values are not necessarily convex and weakly compact:

Proposition 3.12 *If $F : J \to \mathcal{P}_{\mathcal{C}}(\mathcal{X})$ is an Aumann integrable multifunction (i.e. S_F^1 is nonempty), then for every $t \in [a,b]$ we have*

$$w\text{-}clI_a^{A,\alpha}F(t) = w\text{-}clI_a^{AP,\alpha}F(t), \tag{7}$$

where 'w-cl' denotes the weak-closure operation. Consequently, we have for every $t \in [a,b]$

$$cl\,coI_a^{A,\alpha}F(t) = cl\,coI_a^{AP,\alpha}F(t).$$

Proof Without loss of generality, we fix $t \in [a,b]$. Put $B = [a,t]$. Since $S_F^1 \subseteq S_F^{Pe}$, we clearly have $I_a^{A,\alpha}F(t) \subseteq I_a^{AP,\alpha}F(t)$, so $w-$cl $I_a^{A,\alpha}F(t) \subseteq w-$cl $I_a^{AP,\alpha}F(t)$.

Conversly, if x is in $I_a^{AP,\alpha}F(t)$, there exists $f \in S_F^{Pe}$ satisfying

$$x = w - \int_a^t \phi_\alpha(t-s)f(s)ds.$$

Let g be an arbitrary member of S_F^1, which is nonempty by the hypothesis. For each integer $k \geq 1$, we define the subset $B_k \in \mathcal{A}$ by

$$B_k = \{s \in [a,b] : \|f(s)\| \leq k\}$$

and the function f_k by

$$f_k(x) = \mathbf{1}_{B_k} f(x) + \mathbf{1}_{B_k^c} g(x) := \begin{cases} f(x) & x \in B_k \\ g(x) & x \in B_k^c \end{cases},$$

where B_k^c denotes the complement of B_k. Clearly, $\mathbf{1}_{B_k} f$ is measurable and bounded by k, so it is a member of S_F^1. Then, f_k is a member of S_F^1. Now, consider the sequence

$$x_k = \int_a^t \phi_\alpha(t-s) f_k(s) ds.$$

For every $y \in X^*$ and $k \geq 1$, one has

$$\langle y, x - x_k \rangle = \langle y, \int_{B_k^c} \phi_\alpha(t-s)\, f(s) - g(s) ds \rangle$$

whence

$$|\langle y, x - x_k \rangle| \leq \int_{B_k^c} \phi_\alpha(t-s) \langle y, f(s) \rangle ds + \int_{B_k^c} \phi_\alpha(t-s) \langle y, g(s) \rangle ds.$$

Since f and g are scalary integrable, we obtain for each $y \in X^*$.

$$\lim_{k \to \infty} |\langle y, x - x_k \rangle| = 0.$$

This shows that x is in the weak-closure of $I_a^{A,\alpha} F(t)$. Now if x is the weak limit of a sequence x_n, where $x_n \in I_a^{AP,\alpha} F(t)$, then from the above discussion $x_n \in w$-cl $I_a^{A,\alpha} F(t)$ and consequently $x \in w$-cl $I_a^{A,\alpha} F(t)$. The second statement follows easily by taking the closed convex hull of each side in (7). ■

3.2. Continuation Property.

In this section we prove the continuation property i.e., when $\alpha \to 1$ in the Aumann-Pettis integral of F of order α we obtain the regular Aumann-Pettis integral of F.

THEOREM 3.13 (CONTINUATION PROPERTY) *Let $F : J \to \mathcal{P}_{\mathcal{CWK}}(\mathcal{X})$ be a measurable and integrably bounded multifunction, then for all $t \in J$, we have:*

$$\lim_{\alpha \to 1} h(I_a^{AP,\alpha} F(t), I_a^{AP} F(t)) = 0 \ \ \forall t \in J, \text{ where } h \text{ is the Hausdorff distance.}$$

PROOF Let t be a fixed point in J, we have

$$\begin{aligned} h(I_a^{AP,\alpha} F(t), I_a^{AP} F(t)) &= \sup_{\|y\| \leq 1} |\delta^*(y, I_a^{AP,\alpha} F(t)) - \delta^*(y, I_a^{AP} F(t))| \\ &= \sup_{\|y\| \leq 1} |\textstyle\int_a^t (\delta^*(y, \varphi_\alpha(t-s) F(s)) - \delta^*(y, F(s)) \mu(ds)| \\ &= \sup_{\|y\| \leq 1} |\textstyle\int_a^t (\varphi_\alpha(t-s) - 1) \delta^*(y, F(s)) \mu(ds)|. \end{aligned}$$

Since F is integrably bounded, then there exists an integrable function

$$g : [a,b] \to [0, \infty] \text{ such that } \sup_{z \in f(s)} \|z\| \leq g(s), \forall s \in J.$$

Thus,

$$|\delta^*(y, F(s))| \leq \|y\||g(s)|.$$

Then,

$$h(I_a^{AP,\alpha}F(t), I_a^{AP}F(t)) \leq \sup_{\|y\|\leq 1} \int_a^t \varphi_\alpha(t-s)\|y\||g(s)|\mu(ds).$$

From the definition of ϕ_α one have

$$\lim_{\alpha\to 1} \phi_\alpha(t-s) = 0 \quad \forall s \in [a,t].$$

Then,

$$\lim_{\alpha\to 1} h(I_a^{AP,\alpha}F(t), I_a^{AP}F(t)) = 0.$$

3.3. Convergence Theorem. In the literature there are many papers dealing with versions for multifunction Fatou's lemma and dominated convergence theorem for Aumann-Pettis integral (see for example [4] and [6]). In this section, we give some versions for Fatou's lemma and dominated convergence theorem in the fractional case.

Now, let (A_n) be a sequence of closed subsets of X. We say that A_n is convergent to a closed subset A of X in the Kurotowski Mosco sense ($A_n \overset{K.M.}{\longrightarrow} A$) as $n \to \infty$ if and only if

$$\overline{\lim_{n\to\infty}} A_n \subset A \subset \underline{\lim_{n\to\infty}} A_n,$$

where

$$\underline{\lim_{n\to\infty}} A_n = \left\{x \in X : x = \lim_{n\to\infty} x_n, x_n \in A_n, \forall n \geq 1\right\},$$

and

$$\overline{\lim_{n\to\infty}} A_n = \left\{x \in X : x = \lim_{n\to\infty} x_{n_k}, x_{n_k} \in A_{n_k}, k \geq 1\right\}.$$

Theorem 3.14 (Dominated convergence theorem) *Let $F_n, F : J \to \mathcal{P}_{\mathcal{CWK}}(\mathcal{X})$ be a measurable and uniformly integrably bounded multifunctions by a function $g \in L^1$. If for all $t \in J, F_n(t) \overset{K.M.}{\longrightarrow} F(t)$ as $n \to \infty$, then for all $t \in J, I_a^{AP,\alpha}F_n(t) \overset{K.M.}{\longrightarrow} I_a^{AP,\alpha}F(t)$ as $n \to \infty$.*

Proof Let t be a fixed point in J. For each $n \geq 1$ we define $f_n, f : X^* \to R$ by

$$\begin{aligned} f_n(y) &= \delta^*(y, F_n(t)), \\ f(y) &= \delta^*(y, F(t)). \end{aligned}$$

Since F_n and F are uniformly integrably bounded by a function $g \in L^1$, then the functions f_n and f are equi-lower semicontinuous. Hence by Theorem 7 in [21], we obtain,

$$\lim_{n\to\infty} f_n(y) = f(y) \quad \forall y \in X^*.$$

This means that

$$\lim_{n\to\infty} \delta^*(y, F_n(t)) = \delta^*(y, F(t)) \quad \forall y \in X^*.$$

But for every $n \geq 1$ and for $s \in [a,t]$,

$$\begin{aligned}|\delta^*(y, \phi_\alpha(t-s)F_n(s))| &\leq \|y\||\phi_\alpha(t-s)g(s)|,\\ |\delta^*(y, \phi_\alpha(t-s)F(s))| &\leq \|y\||\phi_\alpha(t-s)g(s)|.\end{aligned}$$

Then, since the function $s \longmapsto \phi_\alpha(t-s)g(s)$ is real integrable function, we can deduce from the Lebesgue's dominated convergence theorem that

$$\lim_{n\to\infty} \int_a^t \delta^*(y, \phi_\alpha(t-s)F_n(s))\mu(ds) = \int_a^t \delta^*(y, \phi_\alpha(t-s)F(s))\mu(ds),$$

$$\lim_{n\to\infty} \delta^*(y, I_a^{AP,\alpha}F_n(s)) = \delta^*(y, I_a^{AP,\alpha}F(s)).$$

Applying Theorem 3.1 in [24]

$$I_a^{AP,\alpha}F_n(s) \xrightarrow{K.M.} I_a^{AP,\alpha}F(s) \,\text{as}\, n \to \infty.$$

Now we give an infinite dimensional approximate version of the Fatou's Lemma:

THEOREM 3.15 *Let $F_n : J \to \mathcal{P}_{\mathcal{CWK}}(\mathcal{X})$ be a sequence of multifunctions. If for each $n \geq 1$, F_n is measurable and integrably bounded by a function $g \in L^1([a,b],[0,\infty])$, and if for all $t \in J$, $\underline{\lim}_{n\to\infty} F_n(t) \in \mathcal{P}_{\mathcal{CWK}}(\mathcal{X})$. Then,*

$$I_a^{AP,\alpha}(\underline{\lim}_{n\to\infty} F_n(t)) \subset \underline{\lim}_{n\to\infty} I_a^{AP,\alpha}F_n(t), \forall t \in J.$$

PROOF From the assumption, its obvious that the set-valued function $t \to \Gamma(t) = \underline{\lim}_{n\to\infty} F_n(t)$ is measurable and integrably bounded by g, then for all $t \in [a,b], \phi(t) = I_a^{AP,\alpha}\Gamma(t)$ exists. Put, for all $n \geq 1, \phi_n(t) = I_a^{AP,\alpha}F_n(t)$, we want to prove that

$$\phi(t) \subset \underline{\lim}_{n\to\infty} \phi_n(t), \quad \forall t \in J.$$

So, let $f \in S_\Gamma^{Pe}$ and consider for each $n \geq 1$ the set valued function

$$t \to \Gamma_n(t) = \{x \in \phi_n(t) : \|x - \psi(t)\| = d(\psi(t), \phi_n(t))\}, \,\text{where}\, \psi(t) = I_a^{AP,\alpha}f(t).$$

Clearly Γ_n is measurable and hence, there exists a measurable selection

$$\psi_n(t) : [a,b] \to X \,\text{such that}\, \psi_n(t) \in \Gamma_n(t), \forall t \in J.$$

This means

$$\|\psi_n(t) - \psi(t)\| = d(\psi(t), \phi_n(t)) \,\text{and}\, \psi_n(t) \in \phi_n(t)\, \forall t \in J.$$

Thus for all $t \in J$ and $n \geq 1$, we have

$$\begin{aligned}
&\|\psi_n(t) - \psi(t)\| = d(\psi(t), \phi_n(t)) = \inf\left\{\|\psi(t) - I_a^{AP,\alpha} f_n(t)\|, f_n \in S_{F_n}^{Pe}\right\} \\
&= \inf\left\{\|(w) - \textstyle\int_a^t (t-s)^{\alpha-1}(f(s) - f_n(s))\mu(ds)/\Gamma(\alpha)\|, f_n \in S_{F_n}^{Pe}\right\} \\
&\leq (w) \textstyle\int_a^t (t-s)^{\alpha-1} \inf\left\{\|f(s) - f_n(s)\|, f_n \in S_{F_n}^{Pe}\right\} \mu(ds)/\Gamma(\alpha).
\end{aligned}$$

But $f(s) \in \Gamma(s) = \underline{\lim}_{n\to\infty} F_n(t)$. Then, $\inf\left\{\|f(s) - f_n(s)\|, f_n \in S_{F_n}^{Pe}\right\} = 0, \forall s \in J$. Thus

$$\psi(t) \in \underline{\lim}_{n\to\infty} \phi_n(t), \ \forall t \in J.$$

Theorem 3.16 *Let for each $n \geq 1, F_n$ be a measurable and integrably bounded multifunction from J to $\mathcal{P}_{\mathcal{CWK}}(\mathcal{X})$ such that for all $t \in J$, $\overline{\lim}_{n\to\infty} F_n(t)$ is nonempty. If $\cup S_{F_n}^{Pe}$ is weakly compact in L^∞, then for all $t \in J$, $\overline{\lim}_{n\to\infty} I_a^{AP,\alpha} F_n(t)$ is contained in $I_a^{AP,\alpha}(w - \overline{\lim}_{n\to\infty} F_n(t))$, where w denotes the weak limit.*

Proof For all $t \in J$, let us put: $\phi_n = I_a^{AP,\alpha} F_n(t), \Gamma(t) = w - \overline{\lim}_{n\to\infty} F_n(t)$. We want to show that $\overline{\lim}_{n\to\infty} \phi_n(t)$ is contained in $I_a^{AP,\alpha}\Gamma(t), \forall t \in J$. So, let $\psi(t) \in \overline{\lim}_{n\to\infty} \phi_n(t)$, for all $t \in J$. Then $\underline{\lim}_{n\to\infty} d(\psi(t), \phi_n(t)) = 0$ for all $t \in J$. As in Theorem 3.15, the multifunction

$$\Gamma_n(t) = \{x \in \phi_n(t) : \|x - \psi(t)\| = d(\psi(t), \phi_n(t))\}, t \in J$$

has a measurable selection $\psi_n : J \to X$ such that $\psi_n(t) \in \Gamma_n(t)$, for all $t \in J$. This means that

$$\psi_n(t) \in \phi_n(t) \text{ and } \underline{\lim}_{n\to\infty} \|\psi_n(t) - \psi(t)\| = \underline{\lim}_{n\to\infty} d(\psi(t), \phi_n(t)) = 0, \ \forall t \in J.$$

Now, for all $t \in J$,

$$\psi_n(t) = I_a^\alpha f_n(t) := \int_a^t \phi_\alpha(t-s) f_n(s)\mu(ds) \text{ for some } f_n \in S_F^{Pe}.$$

Since S_F^{Pe} is weakly compact in L^∞, then we can find a subsequence, called again f_n, such that

$$w - \lim_{n\to\infty} f_n = f \text{ in } L^\infty.$$

From [20], we have

$$\lim_{n\to\infty} \psi_n(t) = \lim_{n\to\infty} I_a^\alpha f_n(t) = I_a^\alpha f(t), \text{ weakly in } X \text{ and for all } t \in J.$$

Hence,

$$0 = \underline{\lim}_{n\to\infty} \|\psi_n(t) - \psi(t)\| \geq \|w - \lim_{n\to\infty} \psi_n(t) - \psi(t)\|.$$

So, for all $t \in J$, we have

$$\|I_a^\alpha f(t) - \psi(t)\| = 0,$$

which means that

$$\psi(t) \in w - \overline{\lim_{n\to\infty}} F_n(t), \quad \forall t \in J,$$

and this completes the proof. ■

Corollary 3.17 *If dim$(X) < \infty$, then, under the assumption of Theorem 3.16 we get*

$$\lim_{n\to\infty} I_a^{AP,\alpha} F_n(t) \subseteq I_a^{AP,\alpha} \lim_{n\to\infty} F_n(t), \quad \forall t \in J.$$

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