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Weak nearly uniform soothness and worth property of ψ-direct sums of Banach spaces

Abstract.

We shall characterize the weak nearly uniform smoothness of the ψ-direct sum $X \oplus_\psi Y$ of Banach spaces X and Y. The Schur and WORTH properties will be also characterized. As a consequence we shall see in the ℓ_∞-sums of Banach spaces there are many examples of Banach spaces with the fixed point property which are not uniformly non-square.

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1. Introduction. The ψ-direct sum $X \oplus_\psi Y$ of Banach spaces X and Y is the direct sum $X \oplus Y$ equipped with the norm $\|(x, y)\|_\psi = \|(\|x\|, \|y\|)\|_\psi$, where the $\|(\cdot, \cdot)\|_\psi$ term in the right hand side is the absolute normalized norm on $\mathbb{C}^2$ corresponding to a convex (continuous) function ψ with some conditions on the unit interval ([31]). This extends the notion of the ℓ_p-sum $X \oplus_p Y$. Recently various geometric properties of ψ-direct sums have been investigated by several authors ([31, 26, 16, 7, 17, 19, 5, 6, 8, 18, etc.]). In particular it is shown in [17] that $X \oplus_\psi Y$ is uniformly non-square if and only if X and Y are uniformly non-square and neither $\psi = \psi_1$ nor $\psi = \psi_\infty$, where $\psi_1(t) = 1$ and $\psi_\infty(t) = \max\{1 - t, t\}$ are the corresponding convex functions to the ℓ_1- and ℓ_∞-norms respectively.

The aim of this paper is to characterize the weak nearly uniform smoothness of $X \oplus_\psi Y$, which particularly implies the fixed point property (for nonexpansive

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mappings) ([10]). According to Garcia-Falset [9] *a Banach space X is weakly nearly uniformly smooth if and only if X is reflexive and $R(X) < 2$*, where $R(X)$ is the Garcia-Falset coefficient. Thus we shall treat the property $R(X) < 2$ for $X \oplus_\psi Y$ (the reflexivity is immediate to see). The WORTH property (Sims [29]) and the WORTHness (Sims [30]) will be discussed as well.

Section 2 is devoted to preliminary definitions and results. In Section 3 we shall show that under the condition $\psi \neq \psi_1$, $R(X \oplus_\psi Y) < 2$ if and only if $R(X) < 2$ and $R(Y) < 2$. As is readily seen, the reflexivity of $X \oplus_\psi Y$ is equivalent to that of X and Y. Combining these results we obtain that, when $\psi \neq \psi_1$, $X \oplus_\psi Y$ is weakly nearly uniformly smooth if and only if X and Y are weakly nearly uniformly smooth. (The case ψ is strictly convex is found in [6]). This result looks interesting as the function ψ_∞ is allowed in contrast with the above-mentioned result concerning the uniform non-squareness of $X \oplus_\psi Y$ where both of ψ_1 and ψ_∞ are excluded. As a consequence, if X and Y are weakly nearly uniformly smooth and $\psi \neq \psi_1$, then $X \oplus_\psi Y$ has the fixed point property. In the recent paper [11] Garcia-Falset, Llorens-Fuster and Mazcuñan-Navarro proved that all uniformly non-square spaces have the fixed point property. Our preceding result especially implies that $X \oplus_\infty Y$ with the above X and Y has the fixed point property, while it is not uniformly non-square. For the case $\psi = \psi_1$, $R(X \oplus_1 Y) < 2$ if and only if X and Y have the Schur property; and hence $X \oplus_1 Y$ is weakly nearly uniformly smooth if and only if X and Y are of finite dimension.

In the final section it will be shown that $X \oplus_\psi Y$ has WORTH if and only if X and Y have WORTH. According to Garcia-Falset [9], in the class of Banach spaces having WORTH, uniform non-squareness implies weak nearly uniform smoothness. Our foregoing results imply that if X and Y are uniformly non-square and have WORTH, then $X \oplus_\infty Y$ has WORTH and is weakly nearly uniformly smooth, but not uniformly non-square. Thus the converse of the above fact by Sims is not valid with many counter examples in ℓ_∞-sums $X \oplus_\infty Y$. Finally we shall discuss the WORTHness and the weak nearly uniform smoothness of $X \oplus_\infty Y$ for X and Y which may fail to have WORTH.

2. Preleminaries. Let Ψ be the family of all convex functions ψ on $[0,1]$ satisfying

$$\psi(0) = \psi(1) = 1 \quad \text{and} \quad \max\{1-t, t\} \leq \psi(t) \leq 1 \quad (0 \leq t \leq 1). \tag{1}$$

Let $\|\cdot\|$ be any absolute normalized norm on $\mathbb{C}^2$, that is, $\|(z,w)\| = \|(|z|,|w|)\|$ and $\|(1,0)\| = \|(0,1)\| = 1$ and let

$$\psi(t) = \|(1-t, t)\| \quad (0 \leq t \leq 1). \tag{2}$$

Then $\psi \in \Psi$. Conversely for any $\psi \in \Psi$ define

$$\|(z,w)\|_\psi = \begin{cases} (|z|+|w|)\psi\left(\frac{|w|}{|z|+|w|}\right) & \text{if } (z,w) \neq (0,0), \\ 0 & \text{if } (z,w) = (0,0). \end{cases} \tag{3}$$

Then $\|\cdot\|_\psi$ is an absolute normalized norm on $\mathbb{C}^2$ and satisfies (2) (Bonsall and Duncan [2], see also [27, 28]). The ℓ_p-norms $\|\cdot\|_p$ are such examples and for all absolute normalized norms $\|\cdot\|$ on $\mathbb{C}^2$ we have ([2])

$$\|\cdot\|_\infty \le \|\cdot\| \le \|\cdot\|_1. \tag{4}$$

By (2) the convex function corresponding to the ℓ_p-norm is given by

$$\psi_p(t) = \begin{cases} \{(1-t)^p + t^p\}^{1/p} & \text{if } 1 \le p < \infty, \\ \max\{1-t, t\} & \text{if } p = \infty. \end{cases} \tag{5}$$

The following monotonicity properties of absolute norms are essential in our discussions.

Lemma 2.1 (Bonsall and Duncan [2]) *Let $\|\cdot\|$ be an absolute normalized norm on $\mathbb{C}^2$. Then*

(i) *If $|p| \le |r|$ and $|q| \le |s|$, then $\|(p,q)\| \le \|(r,s)\|$,*

(ii) *If $|p| < |r|$ and $|q| < |s|$, then $\|(p,q)\| < \|(r,s)\|$.*

Lemma 2.2 (Kato, Saito and Tamura [17]) *Let $\psi \in \Psi$ and let $(p,q),\ (r,s) \in \mathbb{C}^2$.*

(i) *Let $|p| < |r|$ and $|q| = |s|$. Then $\|(p,q)\|_\psi = \|(r,s)\|_\psi$ if and only if $\|(p,q)\|_\psi = |q|$.*

(ii) *Let $|p| = |r|$ and $|q| < |s|$. Then $\|(p,q)\|_\psi = \|(r,s)\|_\psi$ if and only if $\|(p,q)\|_\psi = |p|$.*

Let X and Y be Banach spaces and let $\psi \in \Psi$. The ψ-direct sum $X \oplus_\psi Y$ of X and Y is the direct sum $X \oplus Y$ equipped with the norm

$$\|(x,\, y)\|_\psi = \|(\|x\|,\, \|y\|)\|_\psi, \tag{6}$$

where the $\|(\cdot,\,\cdot)\|_\psi$ term in the right hand side is the absolute normalized norm on $\mathbb{C}^2$ corresponding to the convex function ψ ([31, 16]; see [26] for several examples). It is obvious that $(x_n, y_n) \to (x, y)$ in $X \oplus_\psi Y$ if and only if $x_n \to x$ in X and $y_n \to y$ in Y as $n \to \infty$.

Now let us recall some properties of Banach spaces. As usual S_X and B_X stand for the unit sphere and the closed unit ball of a Banach space X. X is called *uniformly convex* provided for any $\epsilon > 0$ there exists $\delta > 0$ such that whenver $\|x-y\| \ge \epsilon,\ x, y \in S_X$, one has $\|(x+y)/2\| \le 1-\delta$, or equivalently, provided for any ε $(0 < \varepsilon < 2)$ one has $\delta_X(\varepsilon) > 0$, where $\delta_X(\varepsilon)$ is the *modulus of convexity of X*: $\delta_X(\epsilon) := \inf\{1 - \|(x+y)/2\|;\ \|x-y\| \ge \epsilon,\ x, y \in S_X\}$. X is called *uniformly non-square* ([14]; cf. [1, 23]) provided there exists a δ $(0 < \delta < 1)$ such that, whenever $\|(x-y)/2\| > 1-\delta,\ x, y \in S_X$, one has $\|(x+y)/2\| \le 1-\delta$, or equivalently, provided $\delta_X(\varepsilon) > 0$ for some $0 < \varepsilon < 2$. X is said to have the *Schur property* if every weakly convergent sequence in X is strongly convergent.

A sequence $\{x_n\}$ in X is called a *basic sequence* if it is a Schauder basis for its closed linear span, that is, every x in the span of $\{x_n\}$ has a unique representation of the form $x = \sum_{n=1}^{\infty} \alpha_n x_n$. According to Kutzarova, Prus and Sims [20] X is said to be *weakly nearly uniformly smooth* provided there exist $0 < \epsilon < 1$ and $\mu > 0$ such that for any $0 < t < \mu$ and any basic sequence $\{x_n\}$ in B_X there is $k > 0$ so that $\|x_1 + tx_k\| \leq 1 + t\epsilon$. Garcia-Falset [9] introduced the coefficient $R(X)$ of X, which is referred to as *Garcia-Falset coefficient* in [3, 4], is defined by

$$R(X) = \sup \{\liminf_{n\to\infty} \|x_n - x\|\}, \tag{7}$$

where the supremum is taken over all weakly null sequences $\{x_n\}$ in B_X and all x in B_X. As is readily seen, $1 \leq R(X) \leq 2$ and it is known that $R(c_0) = R(\ell_1) = 1$, $R(\ell_p) = 2^{1/p}$ $(1 < p < \infty)$ and $R(c) = R(\ell_\infty) = 2$ (cf. [22, p. 165]). It is known that uniformly convex, resp., uniformly smooth spaces are weakly nearly uniformly smooth (cf. [22, p. 165], [25, p. 508]), and X is weakly nearly uniformly smooth if and only if X is reflexive and $R(X) < 2$ (Garcia-Falset [9, Corollary 4.4]).

A Banach space X is said to have the *fixed point property* (*resp., weak fixed point property*) *for nonexpansive mappings* if for any nonempty closed bounded (resp., weakly compact) convex subset C of X, every nonexpansive mapping $T : C \to C$ ($\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$), has a fixed point. The former is called FPP in short. If $R(X) < 2$, X has the weak fixed point property (Garcia-Falset [10]); hence weakly nearly uniformly smooth spaces have FPP.

3. Weak nearly uniform smoothness of $X \oplus_\psi Y$. We need some lemmas for our main theorem. First we shall discuss the dual space of $X \oplus_\psi Y$ and generalized Hölder's inequality, which are also treated in [5] (see also [24]); however for convenience of the reader we shall present our proofs though they are similar to those in [5]. In the following X^* stands for the dual space of X.

LEMMA 3.1 (**Generalized Hölder's inequality**; CF. [5, 24]) *Let X and Y be Banach spaces and $\psi \in \Psi$. Let ψ^* be the function on $[0,1]$ defined by*

$$\psi^*(s) = \sup_{0\leq t\leq 1} \frac{(1-s)(1-t) + st}{\psi(t)}. \tag{8}$$

Then $\psi^* \in \Psi$ and

$$|x^*(x) + y^*(y)| \leq \|(x^*, y^*)\|_{\psi^*} \|(x, y)\|_\psi \tag{9}$$

for all $(x, y) \in X \oplus_\psi Y$ and $(x^*, y^*) \in X^* \oplus_{\psi^*} Y^*$.

PROOF It is easily seen that $\psi^* \in \Psi$. For any nonzero $(x, y) \in X \oplus_\psi Y$ and $(x^*, y^*) \in$

$X^* \oplus_{\psi^*} Y^*$ we have

$$\begin{aligned}
|x^*(x)+y^*(y)| &\le \|x^*\|\|x\|+\|y^*\|\|y\| \\
&= \|(x,y)\|_\psi(\|x^*\|+\|y^*\|)\frac{\frac{\|x^*\|}{\|x^*\|+\|y^*\|}\frac{\|x\|}{\|x\|+\|y\|}+\frac{\|y^*\|}{\|x^*\|+\|y^*\|}\frac{\|y\|}{\|x\|+\|y\|}}{\psi(\frac{\|y\|}{\|x\|+\|y\|})} \\
&\le \|(x,y)\|_\psi(\|x^*\|+\|y^*\|)\sup_{0\le t\le 1}\frac{\frac{\|x^*\|}{\|x^*\|+\|y^*\|}(1-t)+\frac{\|y^*\|}{\|x^*\|+\|y^*\|}t}{\psi(t)} \\
&= \|(x,y)\|_\psi(\|x^*\|+\|y^*\|)\psi^*\left(\frac{\|y^*\|}{\|x^*\|+\|y^*\|}\right) \\
&= \|(x,y)\|_\psi\|(x^*,y^*)\|_{\psi^*}. \qquad \blacksquare
\end{aligned}$$

Now we shall see $(X\oplus_\psi Y)^* = X^*\oplus_{\psi^*} Y^*$:

Proposition 3.2 (cf. [5, 24]) *Let X and Y be Banach spaces and $\psi \in \Psi$. Then $f \in (X\oplus_\psi Y)^*$ if and only if there exist unique $x^* \in X^*$ and $y^* \in Y^*$ such that*

$$f(x,y) = x^*(x)+y^*(y) \quad \textit{for all } (x,y)\in X\oplus_\psi Y. \tag{10}$$

Moreover, $\|f\| = \|(x^*,y^*)\|_{\psi^*}$.

Proof Let $x^* \in X^*$ and $y^* \in Y^*$. Define $f(x,y)$ by (10). Then by Lemma 3.1

$$|f(x,y)| = |x^*(x)+y^*(y)| \le \|(x^*,y^*)\|_{\psi^*}\|(x,y)\|_\psi,$$

from which it follows that $f \in (X\oplus_\psi Y)^*$ and $\|f\| \le \|(x^*,y^*)\|_{\psi^*}$. Conversely take an arbitrary $f \in (X\oplus_\psi Y)^*$ and let $x^* = f(\cdot,0)$, $y^* = f(0,\cdot)$. Then $x^* \in X^*$ and $y^* \in Y^*$ as

$$\|x^*\| = \sup_{\|x\|\le 1}|x^*(x)| = \sup_{\|x\|\le 1}|f(x,0)| \le \|f\| \sup_{\|x\|\le 1}\|(x,0)\|_\psi = \|f\|.$$

Obviously we have $f(x,y) = x^*(x)+y^*(y)$ for $(x,y)\in X\oplus_\psi Y$. We see $\|(x^*,y^*)\|_{\psi^*} \le \|f\|$. We may assume $(x^*,y^*) \ne (0,0)$. For any $\varepsilon > 0$ there exist $u \in S_X$ and $v \in S_Y$ for which $\|x^*\| \le x^*(u)+\varepsilon$ and $\|y^*\| \le y^*(v)+\varepsilon$. Then

$$\begin{aligned}
&\|(x^*,y^*)\|_{\psi^*} = (\|x^*\|+\|y^*\|)\psi^*(\frac{\|y^*\|}{\|x^*\|+\|y^*\|}) \\
&= (\|x^*\|+\|y^*\|)\sup_{0\le t\le 1}\frac{1}{\psi(t)}\left[\left(1-\frac{\|y^*\|}{\|x^*\|+\|y^*\|}\right)(1-t)+\frac{\|y^*\|}{\|x^*\|+\|y^*\|}t\right] \\
&= \sup_{0\le t\le 1}\frac{\|x^*\|(1-t)+\|y^*\|t}{\psi(t)} \\
&\le \sup_{0\le t\le 1}\frac{(x^*(u)+\varepsilon)(1-t)+(y^*(v)+\varepsilon)t}{\psi(t)} \\
&\le \sup_{0\le t\le 1}\left[x^*\left(\frac{1-t}{\psi(t)}u\right)+y^*\left(\frac{t}{\psi(t)}v\right)\right]+2\varepsilon.
\end{aligned}$$

Since

$$\left\|\left(\frac{1-t}{\psi(t)}u,\ \frac{t}{\psi(t)}v\right)\right\|_\psi \le \frac{1}{\psi(t)}\|(1-t,\ t)\|_\psi = 1$$

and ε is arbitrary, we have $\|(x^*,y^*)\|_{\psi^*} \le \|f\|$ (recall (10)). This completes the proof. ∎

Lemma 3.3 *Let X and Y be Banach spaces and $\psi \in \Psi$. Then $\{(x_n,y_n)\}$ tends weakly to 0 in $X \oplus_\psi Y$ if and only if $\{x_n\}$ and $\{y_n\}$ tend weakly to 0 in X and Y, respectively.*

Proof Let $\{x_n\}$ and $\{y_n\}$ be weakly null sequences. By Proposition 3.2 for any $f \in (X \oplus_\psi Y)^*$ there exist unique $x^* \in X^*$ and $y^* \in Y^*$ such that $f(x,y) = \langle (x^*,y^*),\,(x,y)\rangle = x^*(x)+y^*(y)$ for all $(x,y) \in X \oplus_\psi Y$. Therefore

$$\lim_{n\to\infty} f(x_n,y_n) = \lim_{n\to\infty}[x^*(x_n)+y^*(y_n)] = 0.$$

Conversely let $\{(x_n,y_n)\}$ tend weakly to 0 in $X \oplus_\psi Y$. Take arbitrary $x^* \in X^*$ and $y^* \in Y^*$. Since $(x^*,0) \in (X \oplus_\psi Y)^*$, we have

$$\lim_{n\to\infty} x^*(x_n) = \lim_{n\to\infty}\langle (x^*,0),(x_n,y_n)\rangle = 0.$$

In the same way we have $\lim_{n\to\infty} y^*(y_n) = 0$. ∎

The following lemma is essential in our later discussion.

Lemma 3.4 *Let $\{x_n^{(k)}\}$ and $\{y_n^{(k)}\}$ be double sequences with nonzero terms in a Banach space X such that $\lim\limits_{k\to\infty}\lim\limits_{n\to\infty}\|x_n^{(k)}\| > 0$ and $\lim\limits_{k\to\infty}\lim\limits_{n\to\infty}\|y_n^{(k)}\| > 0$. Then the following are equivalent.*

(i) $\lim\limits_{k\to\infty}\liminf\limits_{n\to\infty}\|x_n^{(k)}+y_n^{(k)}\| = \lim\limits_{k\to\infty}\lim\limits_{n\to\infty}(\|x_n^{(k)}\|+\|y_n^{(k)}\|)$.

(ii) $\lim\limits_{k\to\infty}\liminf\limits_{n\to\infty}\left\|\dfrac{x_n^{(k)}}{\|x_n^{(k)}\|}+\dfrac{y_n^{(k)}}{\|y_n^{(k)}\|}\right\| = 2.$

Proof Let $\lim_{k\to\infty}\lim_{n\to\infty}\|x_n^{(k)}\| = a,\ \lim_{k\to\infty}\lim_{n\to\infty}\|y_n^{(k)}\| = b$. We may assume that $0 < a \le b$. Suppose (i) to be true. Since

$$\begin{aligned}
2 &\ge \left\|\frac{x_n^{(k)}}{\|x_n^{(k)}\|}+\frac{y_n^{(k)}}{\|y_n^{(k)}\|}\right\| = \frac{1}{\|x_n^{(k)}\|\|y_n^{(k)}\|}\left\|\|y_n^{(k)}\|x_n^{(k)}+\|x_n^{(k)}\|y_n^{(k)}\right\| \\
&= \frac{1}{\|x_n^{(k)}\|\|y_n^{(k)}\|}\left\|\|y_n^{(k)}\|(x_n^{(k)}+y_n^{(k)})-(\|y_n^{(k)}\|-\|x_n^{(k)}\|)y_n^{(k)}\right\| \\
&\ge \frac{1}{\|x_n^{(k)}\|\|y_n^{(k)}\|}\left|\|y_n^{(k)}\|\|x_n^{(k)}+y_n^{(k)}\|-\left|\|y_n^{(k)}\|-\|x_n^{(k)}\|\right|\|y_n^{(k)}\|\right| \\
&= \frac{1}{\|x_n^{(k)}\|}\left[\|x_n^{(k)}+y_n^{(k)}\|-\left|\|y_n^{(k)}\|-\|x_n^{(k)}\|\right|\right],
\end{aligned}$$

we have

$$\begin{aligned}
2 &\geq \liminf_{n\to\infty} \left\| \frac{x_n^{(k)}}{\|x_n^{(k)}\|} + \frac{y_n^{(k)}}{\|y_n^{(k)}\|} \right\| \\
&\geq \liminf_{n\to\infty} \left[\frac{1}{\|x_n^{(k)}\|} \left| \|x_n^{(k)} + y_n^{(k)}\| - \left| \|y_n^{(k)}\| - \|x_n^{(k)}\| \right| \right| \right] \\
&= \lim_{n\to\infty} \frac{1}{\|x_n^{(k)}\|} \liminf_{n\to\infty} \left[\|x_n^{(k)} + y_n^{(k)}\| - \left| \|y_n^{(k)}\| - \|x_n^{(k)}\| \right| \right] \\
&\geq \lim_{n\to\infty} \frac{1}{\|x_n^{(k)}\|} \left[\liminf_{n\to\infty} \left[\|x_n^{(k)} + y_n^{(k)}\| \right] - \lim_{n\to\infty} \left| \|y_n^{(k)}\| - \|x_n^{(k)}\| \right| \right],
\end{aligned}$$

where the last term tends to $\frac{1}{a}[(a+b) - |b-a|] = 2$ as $k \to \infty$. Therefore we have (ii). Conversely assume that (ii) is true. Then

$$\begin{aligned}
\|x_n^{(k)}\| + \|y_n^{(k)}\| &\geq \|x_n^{(k)} + y_n^{(k)}\| \\
&\geq \|y_n^{(k)}\| \left\| \frac{x_n^{(k)}}{\|y_n^{(k)}\|} + \frac{y_n^{(k)}}{\|y_n^{(k)}\|} \right\| \\
&= \|y_n^{(k)}\| \left\| \left(\frac{x_n^{(k)}}{\|x_n^{(k)}\|} + \frac{y_n^{(k)}}{\|y_n^{(k)}\|} \right) - \left(\frac{x_n^{(k)}}{\|x_n^{(k)}\|} - \frac{x_n^{(k)}}{\|y_n^{(k)}\|} \right) \right\| \\
&\geq \|y_n^{(k)}\| \left(\left\| \frac{x_n^{(k)}}{\|x_n^{(k)}\|} + \frac{y_n^{(k)}}{\|y_n^{(k)}\|} \right\| - \left| \frac{1}{\|x_n^{(k)}\|} - \frac{1}{\|y_n^{(k)}\|} \right| \|x_n^{(k)}\| \right).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\lim_{n\to\infty} \left[\|x_n^{(k)}\| + \|y_n^{(k)}\| \right] \\
\geq\; &\liminf_{n\to\infty} \left[\|y_n^{(k)}\| \left(\left\| \frac{x_n^{(k)}}{\|x_n^{(k)}\|} + \frac{y_n^{(k)}}{\|y_n^{(k)}\|} \right\| - \left| \frac{1}{\|x_n^{(k)}\|} - \frac{1}{\|y_n^{(k)}\|} \right| \|x_n^{(k)}\| \right) \right] \\
\geq\; &\lim_{n\to\infty} \|y_n^{(k)}\| \left[\liminf_{n\to\infty} \left\| \frac{x_n^{(k)}}{\|x_n^{(k)}\|} + \frac{y_n^{(k)}}{\|y_n^{(k)}\|} \right\| - \lim_{n\to\infty} \left| \frac{1}{\|x_n^{(k)}\|} - \frac{1}{\|y_n^{(k)}\|} \right| \lim_{n\to\infty} \|x_n^{(k)}\| \right].
\end{aligned}$$

Since the first and the last terms tend to $a+b$ as $k \to \infty$, we have the conclusion.∎

The next lemma is useful to treat the coefficient $R(X)$.

LEMMA 3.5 *Let X be a Banach space. Then $R(X) = \sup\{\lim_{n\to\infty} \|x_n - x\|\}$, where the supremum is taken over all weakly null sequences $\{x_n\}$ in B_X and $x \in B_X$ such that $\{\|x_n - x\|\}$ and $\{\|x_n\|\}$ have limits.*

Proof For any $\varepsilon > 0$ there exist a weakly null sequence $\{x_n\}$ in B_X and $x \in B_X$ such that $\liminf_{n\to\infty} \|x_n - x\| \geq R(X) - \varepsilon$. As $\{\|x_n - x\|\}$ and $\{\|x_n\|\}$ are bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{\|x_{n_i} - x\|\}$ and $\{\|x_{n_i}\|\}$ have limits. Then we have $R(X) \geq \lim_{i\to\infty} \|x_{n_i} - x\| \geq R(X) - \varepsilon$. Since ε is arbitrary, we have the conclusion. ∎

Now we are in a position to present our main theorem.

Theorem 3.6 *Let X and Y be Banach spaces. Let $\psi \in \Psi$ and $\psi \neq \psi_1$. Then $R(X \oplus_\psi Y) < 2$ if and only if $R(X) < 2$ and $R(Y) < 2$.*

Proof In view of Lemma 3.3 it is obvious that $R(X \oplus_\psi Y) < 2$ implies $R(X) < 2$ and $R(Y) < 2$. For the converse let $R(X) < 2$ and $R(Y) < 2$. Suppose that $R(X \oplus_\psi Y) = 2$. Then by Lemma 3.5 we have weakly null sequences $\{(x_n^{(k)}, y_n^{(k)})\}_n$ $(k = 1, 2, \cdots)$ and a sequence $\{(x^{(k)}, y^{(k)})\}$ in the unit ball of $X \oplus_\psi Y$ such that

$$2 = R(X \oplus_\psi Y) = \lim_{k\to\infty} \lim_{n\to\infty} \|(\|x_n^{(k)} + x^{(k)}\|, \|y_n^{(k)} + y^{(k)}\|)\|_\psi.$$

Then for each k, $\{x_n^{(k)}\}_n$ and $\{y_n^{(k)}\}_n$ are weakly null sequences in X and Y respectively. We may assume the existence of all the following limits without loss of generality:

$$\lim_{k\to\infty} \lim_{n\to\infty} \|x_n^{(k)} + x^{(k)}\|, \quad \lim_{k\to\infty} \lim_{n\to\infty} \|y_n^{(k)} + y^{(k)}\|,$$
$$\lim_{k\to\infty} \lim_{n\to\infty} \|x_n^{(k)}\|, \quad \lim_{k\to\infty} \lim_{n\to\infty} \|y_n^{(k)}\|,$$
$$\lim_{k\to\infty} \|x^{(k)}\|, \quad \lim_{k\to\infty} \|y^{(k)}\|$$

(Indeed these sequences are bounded; so take subsequences necessary times.) Then we have

$$\begin{aligned} 2 &= \lim_{k\to\infty} \lim_{n\to\infty} \|(\|x_n^{(k)} + x^{(k)}\|, \|y_n^{(k)} + y^{(k)}\|)\|_\psi \\ &\leq \lim_{k\to\infty} \lim_{n\to\infty} \|(\|x_n^{(k)}\| + \|x^{(k)}\|, \|y_n^{(k)}\| + \|y^{(k)}\|)\|_\psi \\ &\leq \lim_{k\to\infty} \lim_{n\to\infty} \left\{\|(\|x_n^{(k)}\|, \|y_n^{(k)}\|)\|_\psi + \|(\|x^{(k)}\|, \|y^{(k)}\|)\|_\psi\right\} \leq 2, \end{aligned}$$

and hence

$$\begin{aligned} (11) \qquad & \lim_{k\to\infty} \lim_{n\to\infty} \|(\|x_n^{(k)} + x^{(k)}\|, \|y_n^{(k)} + y^{(k)}\|)\|_\psi \\ &= \lim_{k\to\infty} \lim_{n\to\infty} \|(\|x_n^{(k)}\| + \|x^{(k)}\|, \|y_n^{(k)}\| + \|y^{(k)}\|)\|_\psi \\ &= \lim_{k\to\infty} \lim_{n\to\infty} \left\{\|(\|x_n^{(k)}\|, \|y_n^{(k)}\|)\|_\psi + \|(\|x^{(k)}\|, \|y^{(k)}\|)\|_\psi\right\} = 2. \end{aligned}$$

Thus

$$\begin{aligned} (12) \qquad & \|(\lim_{k\to\infty} \lim_{n\to\infty} \|x_n^{(k)} + x^{(k)}\|, \lim_{k\to\infty} \lim_{n\to\infty} \|y_n^{(k)} + y^{(k)}\|)\|_\psi \\ &= \|(\lim_{k\to\infty} \lim_{n\to\infty} \{\|x_n^{(k)}\| + \|x^{(k)}\|\}, \lim_{k\to\infty} \lim_{n\to\infty} \{\|y_n^{(k)}\| + \|y^{(k)}\|\})\|_\psi = 2. \end{aligned}$$

Suppose here that $\lim_{k\to\infty}\lim_{n\to\infty}\|x_n^{(k)}+x^{(k)}\| < \lim_{k\to\infty}\lim_{n\to\infty}\{\|x_n^{(k)}\|+\|x^{(k)}\|\}$. Then by Lemma 2.1 and (12) we have $\lim_{k\to\infty}\lim_{n\to\infty}\|y_n^{(k)}+y^{(k)}\| = \lim_{k\to\infty}\lim_{n\to\infty}\{\|y_n^{(k)}\|+\|y^{(k)}\|\}$, and hence

$$\lim_{k\to\infty}\lim_{n\to\infty}\|y_n^{(k)}+y^{(k)}\| = \lim_{k\to\infty}\lim_{n\to\infty}\{\|y_n^{(k)}\|+\|y^{(k)}\|\} = 2$$

by Lemma 2.2, from which it follows that $R(Y) = 2$, a contradiction. Therefore we have

$$\lim_{k\to\infty}\lim_{n\to\infty}\|x_n^{(k)}+x^{(k)}\| = \lim_{k\to\infty}\lim_{n\to\infty}\{\|x_n^{(k)}\|+\|x^{(k)}\|\}. \tag{13}$$

In the same way

$$\lim_{k\to\infty}\lim_{n\to\infty}\|y_n^{(k)}+y^{(k)}\| = \lim_{k\to\infty}\lim_{n\to\infty}\{\|y_n^{(k)}\|+\|y^{(k)}\|\}. \tag{14}$$

Now we shall show that

$$\begin{aligned} &\max\{\lim_{k\to\infty}\lim_{n\to\infty}\|x_n^{(k)}\|, \lim_{k\to\infty}\|x^{(k)}\|\} \\ = \ &\max\{\lim_{k\to\infty}\lim_{n\to\infty}\|y_n^{(k)}\|, \lim_{k\to\infty}\|y^{(k)}\|\} = 1. \end{aligned} \tag{15}$$

We first see

$$\begin{aligned} &\min\{\lim_{k\to\infty}\lim_{n\to\infty}\|x_n^{(k)}\|, \lim_{k\to\infty}\|x^{(k)}\|\} \\ = \ &\min\{\lim_{k\to\infty}\lim_{n\to\infty}\|y_n^{(k)}\|, \lim_{k\to\infty}\|y^{(k)}\|\} = 0. \end{aligned} \tag{16}$$

Suppose that $\min\{\lim_{k\to\infty}\lim_{n\to\infty}\|x_n^{(k)}\|, \lim_{k\to\infty}\|x^{(k)}\|\} > 0$. By Lemma 3.4 and (13) we have

$$\lim_{k\to\infty}\liminf_{n\to\infty}\left\|\frac{x_n^{(k)}}{\|x_n^{(k)}\|}+\frac{x^{(k)}}{\|x^{(k)}\|}\right\| = 2,$$

which implies $R(X) = 2$, a contradiction. In the same way we have the latter identity of (16). To see (15) assume that

$$\lim_{k\to\infty}\|x^{(k)}\| = \min\{\lim_{k\to\infty}\lim_{n\to\infty}\|x_n^{(k)}\|, \lim_{k\to\infty}\|x^{(k)}\|\} = 0. \tag{17}$$

Since $\lim_{k\to\infty}\lim_{n\to\infty}\{\|(\|x_n^{(k)}\|,\|y_n^{(k)}\|)\|_\psi + \|(\|x^{(k)}\|,\|y^{(k)}\|)\|_\psi\} = 2$ by (11), we have

$$\lim_{k\to\infty}\lim_{n\to\infty}\|(\|x_n^{(k)}\|,\|y_n^{(k)}\|)\|_\psi = \lim_{k\to\infty}\|(\|x^{(k)}\|,\|y^{(k)}\|)\|_\psi = 1. \tag{18}$$

From (17) and (18) follows $\lim_{k\to\infty}\|y^{(k)}\| = 1$ and hence $\lim_{k\to\infty}\lim_{n\to\infty}\|y_n^{(k)}\| = 0$ by (16). Therefore we obtain $\lim_{k\to\infty}\lim_{n\to\infty}\|x_n^{(k)}\| = 1$ by (18), or the first half assertion of (15). In the case where

$$\lim_{k\to\infty}\lim_{n\to\infty}\|x_n^{(k)}\| = \min\{\lim_{k\to\infty}\lim_{n\to\infty}\|x_n^{(k)}\|, \lim_{k\to\infty}\|x^{(k)}\|\} = 0,$$

a parallel argument works and we have $\lim_{k\to\infty}\|x^{(k)}\| = 1$. Thus we obtain (15). Consequently, according to (16) and (15) we have

$$\begin{aligned} 2 &= \lim_{k\to\infty}\lim_{n\to\infty}\|(\|x_n^{(k)}+x^{(k)}\|, \|y_n^{(k)}+y^{(k)}\|)\|_\psi \\ &= \lim_{k\to\infty}\lim_{n\to\infty}\|(\|x_n^{(k)}\|+\|x^{(k)}\|, \|y_n^{(k)}\|+\|y^{(k)}\|)\|_\psi \\ &= \Big\|(\max\{\lim_{k\to\infty}\lim_{n\to\infty}\|x_n^{(k)}\|, \lim_{k\to\infty}\|x^{(k)}\|\}, \max\{\lim_{k\to\infty}\lim_{n\to\infty}\|y_n^{(k)}\|, \lim_{k\to\infty}\|y^{(k)}\|\})\Big\|_\psi \\ &= \|(1,1)\|_\psi, \end{aligned}$$

or $\psi(1/2) = 1$, which implies that $\psi = \psi_1$, a contradiction. This completes the proof. ∎

From the fundamental fact (4) for absolute normalized norms it immediately follows that $X \oplus_\psi Y$ is reflexive if and only if X and Y are reflexive. Combining Theorem 3.6 and this result, we obtain our next main theorem.

THEOREM 3.7 *Let X and Y be Banach spaces and $\psi \in \Psi$, $\psi \neq \psi_1$. Then the following are equivalent.*
(i) *$X \oplus_\psi Y$ is weakly nearly uniformly smooth.*
(ii) *X and Y are weakly nearly uniformly smooth.*

We note that the case ψ is strictly convex is found in [6]. According to Garcia-Falset [10] *weakly nearly uniformly smooth spaces have the fixed point property for nonexpansive mappings.* Thus we obtain the following.

COROLLARY 3.8 *Let X and Y be weakly nearly uniformly smooth Banach spaces and let $\psi \in \Psi$, $\psi \neq \psi_1$. Then $X \oplus_\psi Y$ has FPP.*

We recall here a recent result of Kato-Saito-Tamura [17]:

THEOREM 3.9 (KATO, SAITO AND TAMURA [17]; SEE ALSO [18]) *Let X and Y be Banach spaces and $\psi \in \Psi$. Then the following are equivalent.*
(i) *$X \oplus_\psi Y$ is uniformly non-square.*
(ii) *X and Y are uniformly non-square and $\psi \neq \psi_1, \psi_\infty$.*

REMARK 3.10 Recently Garcia-Falset, Llorens-Fuster and Mazcuñan-Navarro [11] showed that *all uniformly non-square Banach spaces have FPP.* Our Theorem 3.7, or Corollary 3.8 especially asserts that *for weakly nearly uniformly smooth Banach spaces X and Y, $X \oplus_\infty Y$ has FPP, whereas $X \oplus_\infty Y$ is not uniformly non-square* by Theorem 3.9.

Now we shall apply our result to the $\ell_{p,q}$-sum of Banach spaces X and Y ([26]). Let $1 \le q \le p \le \infty$. Let $\|\cdot\|_{p,q}$ be the (Lorentz) $\ell_{p,q}$-norm: $\|(z_1, z_2)\|_{p,q} = \left\{z_1^{*q} + 2^{(q/p)-1} z_2^{*q}\right\}^{1/q}$, where $\{z_1^*, z_2^*\}$ is the non-increasing rearrangement of

$\{|z_1|, |z_2|\}$. Then $\|\cdot\|_{p,q}$ is an absolute normalized norm on $\mathbb{C}^2$ with the corresponding convex function

$$\psi_{p,q}(t) = \begin{cases} \{(1-t)^q + 2^{q/p-1}t^q\}^{1/q} & \text{if } 0 \le t \le 1/2, \\ \{t^q + 2^{q/p-1}(1-t)^q\}^{1/q} & \text{if } 1/2 \le t \le 1. \end{cases}$$

The $\ell_{p,q}$ sum $X \oplus_{p,q} Y$ is the direct sum of X and Y with the norm $\|(x,y)\|_{p,q} := \|(\|x\|, \|y\|)\|_{p,q}$.

Corollary 3.11 *Let X and Y be Banach spaces and let $1 \le q \le p \le \infty$, not $p = q = 1$. Then $X \oplus_{p,q} Y$ is weakly nearly uniformly smooth if and only if X and Y are weakly nearly uniformly smooth. In particular the same is true for $X \oplus_p Y$, $1 < p \le \infty$.*

Our next concern is the case $\psi = \psi_1$, namely the ℓ_1-sum $X \oplus_1 Y$.

Proposition 3.12 *Let X and Y be Banach spaces and $\psi \in \Psi$. Then $X \oplus_\psi Y$ has the Schur property if and only if X and Y have the Schur property.*

Proof As the Schur property is inherited by subspaces, the necessity is trivial. Conversely let X and Y have the Schur property. Let $\{(x_n, y_n)\}$ be a weakly null sequence in $X \oplus_\psi Y$. Then $\{x_n\}$ and $\{y_n\}$ tend weakly to 0 in X and Y, respectively, and hence they tend strongly to 0, from which it follows that $\{(x_n, y_n)\}$ tends (strongly) to $(0,0)$ in $X \oplus_\psi Y$ as $\|(x_n, y_n)\|_\psi \le \|x_n\| + \|y_n\|$. ∎

Theorem 3.13 *Let X and Y be Banach spaces. Then $R(X \oplus_1 Y) < 2$ if and only if X and Y have the Schur property.*

Proof Assume that X and Y have the Schur property. Let $\{(x_n, y_n)\}$ be any weakly null sequence in $B_{X\oplus_1 Y}$ and $(x,y) \in B_{X\oplus_1 Y}$. Then $\{(x_n, y_n)\}$ converges strongly to $(0, 0)$ in $X \oplus_1 Y$. Therefore

$$\liminf_{n\to\infty} \|(x_n, y_n) - (x,y)\|_1 = \|(x,y)\|_1 \le 1,$$

which implies $R(X \oplus_1 Y) \le 1$ and hence $R(X \oplus_1 Y) = 1$. For the converse we assume that Y does not have the Schur property without loss of generality. Then there exists a sequence $\{y_n\}$ in Y which converges weakly to 0, but does not converge strongly. Then there exists $\varepsilon_0 > 0$ such that $\|y_n\| \ge \varepsilon_0$ for infinitely many n. We may assume $y_n \ne 0$ for all n. Let $v_n = y_n/\|y_n\| \in S(Y)$. Then $\{v_n\}$ converges weakly to 0, whence $\{(0, v_n)\}$ converges weakly to 0 in $B_{X\oplus_1 Y}$ and $\|(x,0)\|_1 = 1$. Take an arbitrary $x \in S(X)$. Then since

$$R(X \oplus_1 Y) \ge \liminf_{n\to\infty} \|(0, v_n) + (x,0)\|_1 = \|(1,1)\|_1 = 2,$$

we have $R(X \oplus_1 Y) = 2$. This completes the proof. ∎

By Theorem 3.13 we obtain the following

THEOREM 3.14 *Let X and Y be Banach spaces. Then the following are equivalent.*
(i) *$X \oplus_1 Y$ is weakly nearly uniformly smooth.*
(ii) *X and Y are reflexive and have the Schur property.*
(iii) *X and Y are of finite dimension.*

By Theorems 3.13 and 3.14 we have the following.

COROLLARY 3.15 *Let X and Y be weakly nearly uniformly smooth and let $\psi \neq \psi_1$. Then $X \oplus_\psi Y$ is weakly nearly uniformly smooth. The converse holds true if X or Y is of infinite dimension.*

4. WORTH property of $X \oplus_\psi Y$. A Banach space X is said to have the property *WORTH* (Sims [29]) if

$$\lim_{n\to\infty}\Big| \|x_n + x\| - \|x_n - x\| \Big| = 0 \tag{19}$$

for all weakly null sequences $\{x_n\}$ in X and for all $x \in X$. It is obvious that all Banach spaces with the Schur property (e.g. finite dimensional spaces and ℓ_1) have WORTH. Also Hilbert spaces, ℓ_p $(1 < p < \infty)$ and c_0 have WORTH, while $L_p[0,1]$ $(1 \leq p \leq \infty, p \neq 2)$ (Sims [30, p. 528]) and c do not have WORTH (cf. Example 1 below). We first recall the next result which is a direct consequence of Garcia-Falset [9].

THEOREM 4.1 (CF. GARCIA-FALSET [9] PROPOSITION 3.6) *If a Banach space X is uniformly non-square and has WORTH, X is weakly nearly uniformly smooth.*

THEOREM 4.2 *Let X and Y be Banach spaces and let $\psi \in \Psi$. Then $X \oplus_\psi Y$ has WORTH if and only if X and Y have WORTH.*

PROOF Assume that X and Y have WORTH. Let $\{(x_n, y_n)\}$ be a weakly null sequence in $X \oplus_\psi Y$ and $(x, y) \in X \oplus_\psi Y$. Then $\{x_n\}$ and $\{y_n\}$ tend weakly to 0 in X and Y respectively and

$$\begin{aligned}
& \Big| \|(x_n, y_n) + (x, y)\|_\psi - \|(x_n, y_n) - (x, y)\|_\psi \Big| \\
= & \Big| \|(\|x_n + x\|,\ \|y_n + y\|)\|_\psi - \|(\|x_n - x\|,\ \|y_n - y\|)\|_\psi \Big| \\
\leq & \ \|(\|x_n + x\| - \|x_n - x\|,\ \|y_n + y\| - \|y_n - y\|)\|_\psi \\
\leq & \Big| \|x_n + x\| - \|x_n - x\| \Big| + \Big| \|y_n + y\| - \|y_n - y\| \Big| \to 0
\end{aligned}$$

as $n \to \infty$. Thus $X \oplus_\psi Y$ has WORTH. The converse assertion is obvious as the property WORTH is inherited by subspaces. ∎

By Theorems 4.1 and 3.13 we obtain the following.

Corollary 4.3 *Let X and Y be uniformly non-square Banach spaces with the property WORTH and $\psi \in \Psi$, $\psi \neq \psi_1$. Then $X \oplus_\psi Y$ is weakly nearly uniformly smooth.*

Remark 4.4 According to Theorem 4.1, *under the condition WORTH uniform non-squareness implies weak nearly uniform smoothness.* Theorem 4.2 and Corollary 4.3 especially asserts that if X and Y are uniformly non-square Banach spaces with WORTH, then $X \oplus_\infty Y$ has WORTH and is weakly nearly uniformly smooth, whereas it is not uniformly non-square. Thus the converse of the above statement is not true. Similarly, *under the condition of uniform non-squareness, WORTH implies weak nearly uniform smoothness.* The converse assertion is not valid as well. Indeed, $L_p[0,1]$ $(1 < p < \infty,\ p \neq 2)$ does not have WORTH though it is uniformly non-square and weakly nearly uniformly smooth.

In the rest of the paper we shall discuss the case where X and Y fail to have WORTH. Sims [30] introduced the degree of WORTHness of a Banach space X by

$$w(X) = \sup\Big\{ r > 0 : r \liminf_{n\to\infty} \|x_n + x\| \le \liminf_{n\to\infty} \|x_n - x\| \text{ whenever } x_n \rightharpoonup 0,\ x \in X \Big\} \tag{20}$$

(cf. [10]). It is known that $1/3 \le w(X) \le 1$ ([21, p. 62]) and X has WORTH if and only if $w(X) = 1$ (Sims [30, p. 527]). We shall see these facts for convenience of the reader below.

Let $\{x_n\}$ be an arbitrary weakly null sequence in X and $x \in X$. Then

$$\liminf_{n\to\infty} \|x_n + x\| \le \liminf_{n\to\infty} \|x_n - x\| + 2\|x\| \le 3 \liminf_{n\to\infty} \|x_n - x\|$$

(Llorens [21, 22]), whence $w(X) \ge 1/3$. Next assume that $\liminf_{n\to\infty} \|x_n + x\| \ge \liminf_{n\to\infty} \|x_n - x\|$ and $r \liminf_{n\to\infty} \|x_n + x\| \le \liminf_{n\to\infty} \|x_n - x\|$. Then we have $r \le 1$ and hence $w(X) \le 1$.

To see the latter statement the next proposition is helpful.

Proposition 4.5 *A Banach spaces X has the WORTH property if and only if*

$$\lim_{n\to\infty} \|x_n + x\| = \lim_{n\to\infty} \|x_n - x\| \tag{21}$$

for all weakly null sequences $\{x_n\}$ in X and all $x \in X$ such that both of the limits exist.

Proof The necessity is clear. Assume that (21) holds true for all weakly null sequences $\{x_n\}$ and all x in X for which the both limits in (21) exist. Now let $\{x_n\}$

be an arbitrary weakly null sequence in X and $x \in X$. Take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{\|x_{n_j}+x\|\}$ and $\{\|x_{n_j}-x\|\}$ have limits and

$$\limsup_{n\to\infty}\Big|\|x_n+x\|-\|x_n-x\|\Big| = \lim_{j\to\infty}\Big|\|x_{n_j}+x\|-\|x_{n_j}-x\|\Big|.$$

Then by the assumption we obtain $\limsup_{n\to\infty}\big|\|x_n+x\|-\|x_n-x\|\big| = 0$, which implies (19). Now assume that $w(X)=1$. Then by replacing x with $-x$ in (20) we

have $\liminf_{n\to\infty}\|x_n+x\| = \liminf_{n\to\infty}\|x_n-x\|$ for every weakly null sequences $\{x_n\}$ and element x in X. Therefore by Proposition 4.5 X has the WORTH property. Conversely assume that X has WORTH. Then for any weakly null sequence $\{x_n\}$ in X and $x\in X$, $\liminf_{n\to\infty}\|x_n+x\| = \liminf_{n\to\infty}\{\|x_n-x\|+(\|x_n+x\|-\|x_n-x\|)\} = \liminf_{n\to\infty}\|x_n-x\|$, whence we have $w(X)=1$. ■

Example 4.6 Let c be the space of real convergent sequences (with the sup norm). Then $w(c)=1/3$.

Indeed let $x_n = (0,\cdots,0,\overset{n}{\check{1}},0,\cdots)$ and $x=(1/2,1/2,\ldots)$. Then $\{x_n\}$ converges weakly to 0. Since $\lim_{n\to\infty}\|x_n+x\| = 3/2$ and $\lim_{n\to\infty}\|x_n-x\|=1/2$, we have $w(c)\le 1/3$. Therefore $w(c)=1/3$.

Theorem 4.2 is sharpened as follows.

Theorem 4.7 *For any Banach spaces X and Y*, $w(X\oplus_\psi Y)=\min\{w(X),w(Y)\}$.

Proof Since X and Y are identified with subspaces of $X\oplus_\psi Y$, $w(X\oplus_\psi Y)\le \min\{w(X),w(Y)\}$. Let $\{(x_n,y_n)\}$ be an arbitrary weakly null sequence and (x,y) in the unit ball of $X\oplus_\psi Y$. Put $w=\min\{w(X),w(Y)\}$. We take a subsequence $\{(x_{n_j},y_{n_j})\}$ of $\{(x_n,y_n)\}$ such that $\{\|x_{n_j}\pm x\|\}$ and $\{\|y_{n_j}\pm y\|\}$ have limits and

$$\liminf_{n\to\infty}\|(\|x_n-x\|,\|y_n-y\|)\|_\psi = \lim_{j\to\infty}\|(\|x_{n_j}-x\|,\|y_{n_j}-y\|)\|_\psi.$$

Then

$$\begin{aligned}
\frac{1}{w}\liminf_{n\to\infty}\|(\|x_n-x\|,\|y_n-y\|)\|_\psi &= \frac{1}{w}\lim_{j\to\infty}\|(\|x_{n_j}-x\|,\|y_{n_j}-y\|)\|_\psi\\
&= \|(\frac{1}{w}\lim_{k\to\infty}\|x_{n_j}-x\|,\frac{1}{w}\lim_{j\to\infty}\|y_{n_j}-y\|)\|_\psi\\
&\ge \|(\lim_{j\to\infty}\|x_{n_j}+x\|,\lim_{j\to\infty}\|y_{n_j}+y\|)\|_\psi\\
&= \lim_{j\to\infty}\|(\|x_{n_j}+x\|,\|y_{n_j}+y\|)\|_\psi\\
&\ge \liminf_{n\to\infty}\|(\|x_n+x\|,\|y_n+y\|)\|_\psi,
\end{aligned}$$

which implies $w(X\oplus_\psi Y)\ge\min\{w(X),w(Y)\}$. Thus $w(X\oplus_\psi Y)=\min\{w(X),w(Y)\}$. ■

Now Garcia-Falset [10, Theorem 4] proved that *if* $\varepsilon_0(X) < 2w(X)$*, then* $R(X) < 2$, where $\varepsilon_0(X)$ is the characteristic of convexity of X, that is, $\varepsilon_0(X) = \sup\{\varepsilon \geq 0 : \delta_X(\epsilon) > 0\}$. From this result he derived the following.

THEOREM 4.8 (GARCIA-FALSET [10]COROLLARY 8) *Let* X *be a Banach space with* $\varepsilon_0(X) < 2w(X)$*. Then* X *is weakly nearly uniformly smooth.*

This generalizes Theorem 4.1. Indeed X is uniformly non-square and has WORTH if and only if $\varepsilon_0(X) < 2$ and $w(X) = 1$. Combining Theorems 4.8 and 3.13, for Banach spaces X and Y which may fail to have WORTH we obtain the following.

THEOREM 4.9 *Let* $\varepsilon_0(X) < 2w(X)$ *and* $\varepsilon_0(Y) < 2w(Y)$*. Let* $\psi \in \Psi,\ \psi \neq \psi_1$*. Then* $X \oplus_\psi Y$ *is weakly nearly uniformly smooth.*

REMARK 4.10 Let $\varepsilon_0(X) < 2w(X)$ and $\varepsilon_0(Y) < 2w(Y)$. Then $X \oplus_\infty Y$ is a counterexample to the converse assertion of Theorem B. In fact, $X \oplus_\infty Y$ is weakly nearly uniformly smooth; however as $X \oplus_\infty Y$ is not uniformly non-square, or $\varepsilon_0(X \oplus_\psi Y) = 2$, we have $2w(X \oplus_\infty Y) = 2\min\{w(X), w(Y)\} \leq 2 = \varepsilon_0(X \oplus_\infty Y)$ by Theorem 4.7.

REMARK 4.11 In several preceding results on ψ-direct sums such as strict and uniform convexity ([31, 26, 16]), smoothness ([24]), uniform non-squareness ([17]; see Theorem 5 in this paper) etc., the convex functions ψ_1 and ψ_∞ are excluded. In this point of view Theorems 3.9, 3.13 and Corollary 4.3 are interesting as the function ψ_∞ is allowed there. Owing to these results the ℓ_∞-sum $X \oplus_\infty Y$ works as an effective counterexample in Remarks 3.10-4.10.

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