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Explicit Difference Schemes for Nonlinear Differential Functional Parabolic Equations with Mixed Derivatives - Convergence Analysis

Abstract. We study the initial-value problem for parabolic equations with mixed partial derivatives and constant coefficients, and with nonlinear and nonlocal right-hand sides. Nonlocal terms appear in the unknown function and its gradient. We analyze convergence of explicit finite difference schemes by means of discrete fundamental solutions.

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1. Introduction. The paper presents a convergence analysis for explicit finite difference methods (FDM's) consistent with parabolic equations whose leading terms include mixed derivatives. The right-hand side contains nontrivial nonlocal operators (delays, integrals), acting on the unknown function and its derivatives. We show that discrete solutions and their spatial difference quotients converge uniformly to the exact solution and its gradient. Unlike [1], [3], [6], [7] the maximum principle is not applicable in that case, because the gradient essentially depends on functional arguments (delays, integrals). Applying a mixed combinatorial approach, together with recurrence inequalities, we generalize some results of [5], where an analogous convergence theorem was proved for nonlocal heat equations i.e., the leading term is just the Laplacean.

The paper is organized as follows: 1) formulation of the differential-functional problem and standard assumptions on coercivity and boundedness of the leading term, 2) formulation of the difference scheme and auxiliary lemmas on the positivity of its coefficients and properties of discrete fundamental solution, 3) the key role plays Lemma 1.6 on estimates of finite differences between two values of the fundamental

solution, 4) formulation of assumptions on the right-hand side function, natural in the consistency and stability theory of difference schemes, 5) concluding convergence theorem from consistency and stability lemmas, 6) the results are illustrated by numerical experiments performed in \mathbb{R}^2 .

1.1. Formulation of the Differential and Difference Problem. Suppose that we have $b > 0$, $\tau_0, \tau_1, \dots, \tau_n \in \mathbb{R}_+$ and $[-\tau, \tau] = [-\tau_1, \tau_1] \times \dots \times [-\tau_n, \tau_n]$. Let $E = [0, b] \times \mathbb{R}^n$, $E_0 = [-\tau_0, 0] \times \mathbb{R}^n$. Let $C(B, \mathbb{R})$ denote the set of all continuous functions from B to \mathbb{R} , where $B = [-\tau_0, 0] \times [-\tau, \tau]$. If $u : E_0 \cup E \rightarrow \mathbb{R}$ and $(t, x) \in E$, then we define the Hale-type functional $u_{(t,x)} : B \rightarrow \mathbb{R}$ by $u_{(t,x)}(s, y) = u(t + s, x + y)$ for $(s, y) \in B$. If $U = (u_1, \dots, u_n) : E_0 \cup E \rightarrow \mathbb{R}^n$, then $U_{(t,x)} = ((u_1)_{(t,x)}, \dots, (u_n)_{(t,x)})$. Suppose that $a_{kl} \in \mathbb{R}$, $\phi : E_0 \rightarrow \mathbb{R}$ is continuous, $\Omega := E \times C(B, \mathbb{R}) \times C(B, \mathbb{R}^n)$, $f : \Omega \rightarrow \mathbb{R}$.

Consider the Cauchy problem

$$(1) \quad \partial_t u(t, x) = \sum_{k,l=1}^n a_{kl} \partial_{x_k x_l} u(t, x) + f(t, x, u_{(t,x)}, (\partial_x u)_{(t,x)}) \quad \text{on } E,$$

$$(2) \quad u(t, x) = \phi(t, x) \quad \text{on } E_0.$$

We will use the following assumptions:

ASSUMPTION 1 Suppose that $a_{kl} = a_{lk}$ for $k, l = 1, \dots, n$ and the matrix $[a_{kl}]_{k,l=1,\dots,n}$ is positive definite.

ASSUMPTION 2 There are positive real numbers C_1, \dots, C_n such that

$$1 - \sum_{k=1}^n 2C_k^2 a_{kk} + \sum_{k,l=1, k \neq l}^n C_k C_l |a_{kl}| \geq 0 \quad \text{on } E$$

for $k = 1, \dots, n$.

ASSUMPTION 3 Matrix $[m_{kl}]_{k,l=1,\dots,n} = \text{diag}[1/C_1, \dots, 1/C_n][a_{kl}]\text{diag}[C_1, \dots, C_n]$ is diagonally dominant, i.e. $m_{kk} > \sum_{l=1, l \neq k}^n |m_{kl}|$ for all $1 \leq k \leq n$, where $\text{diag}[C_1, \dots, C_n]$ is the diagonal $n \times n$ matrix whose diagonal consists of C_1, \dots, C_n .

Fix $(\bar{h}_0, \bar{h}') \in (0, b) \times (0, \infty)^n$ and $C_1, \dots, C_n \in (0, \infty)$. Define the set of admissible steps:

$$(3) \quad I_C = \left\{ h = (h_0, h') : \begin{array}{l} h_0 \in (0, b), \quad h' = (h_1, \dots, h_n) \in \mathbb{R}_+^n, \\ h_i^2 C_i^2 = h_0, \quad (i = 1, \dots, n) \end{array} \right\}.$$

Here, for the sake of simplicity, we fix proportions of h_0 to h_i^2 (in the literature it is assumed that h_0 is sufficiently small compared with h_i^2).

We introduce a regular mesh. Denote $t^\alpha = \alpha h_0$ and $x^\beta = (\beta_1 h_1, \dots, \beta_n h_n)$ for $\alpha \in \mathbb{Z}$ and $\beta \in \mathbb{Z}^n$. Let $Z_h = \{(t^\alpha, x^\beta) : (\alpha, \beta) \in \mathbb{Z}^{1+n}\}$. Define

$$E_h^0 = E_0 \cap Z_h, \quad E_h = E \cap Z_h, \quad \tilde{E}_h^0 = E_h^0 \cup E_h,$$

$$E_h^+ = \{(t^\alpha, x^\beta) \in E_h : (t^{\alpha+1}, x^\beta) \in E_h\}, \quad B_h = B \cup Z_h.$$

We will use the difference operators δ_0^+ , δ_k and δ_{kl} ($k, l = 1, \dots, n$):

$$\delta_0^+ u^{(\alpha, \beta)} = \frac{u^{(\alpha+1, \beta)} - u^{(\alpha, \beta)}}{h_0},$$

$$\delta_k u^{(\alpha, \beta)} = \frac{u^{(\alpha, \beta+e_k)} - u^{(\alpha, \beta-e_k)}}{2h_k},$$

$$\delta_{kk} u^{(\alpha, \beta)} = \frac{u^{(\alpha, \beta+e_k)} - 2u^{(\alpha, \beta)} + u^{(\alpha, \beta-e_k)}}{h_k^2},$$

$$\delta_{kl} u^{(\alpha, \beta)} = \begin{cases} \frac{1}{2} [\delta_k^+ \delta_l^+ u^{(\alpha, \beta)} + \delta_k^- \delta_l^- u^{(\alpha, \beta)}], & \text{when } a_{kl} \geq 0, \\ \frac{1}{2} [\delta_k^- \delta_l^+ u^{(\alpha, \beta)} + \delta_k^+ \delta_l^- u^{(\alpha, \beta)}], & \text{when } a_{kl} < 0, \end{cases}$$

where

$$\delta_k^+ u^{(\alpha, \beta)} = \frac{1}{h_k} [u^{(\alpha, \beta+e_k)} - u^{(\alpha, \beta)}], \quad \delta_k^- u^{(\alpha, \beta)} = \frac{1}{h_k} [u^{(\alpha, \beta)} - u^{(\alpha, \beta-e_k)}],$$

and $e_k = (\delta_{1,k}, \dots, \delta_{n,k})$, $\delta_{j,k}$ is the Kronecker symbol for $j, k = 1, \dots, n$. The difference operators δ_0^+ , δ_k , δ_{kl} approximate the respective partial derivatives ∂_t , ∂_{x_k} , $\partial_{x_k x_l}$. Define

$$c_0 = 1 - \sum_{k=1}^n \frac{2h_0}{h_k^2} a_{kk} + \sum_{k,l=1, k \neq l}^n |a_{kl}| \frac{h_0}{h_k h_l},$$

$$c_{\pm e_k} = \frac{h_0}{h_k^2} a_{kk} - \sum_{l=1, l \neq k}^n |a_{kl}| \frac{h_0}{h_k h_l},$$

$$c_{\pm e_k, \pm e_l} = \theta_{kl} a_{kl} \frac{h_0}{2h_k h_l}, \quad c_{\pm e_k, \mp e_l} = -(1 - \theta_{kl}) a_{kl} \frac{h_0}{2h_k h_l}, \quad (k \neq l),$$

where $\theta_{kl} \in \{0, 1\}$ and $\theta_{kl} = 1$ for $a_{kl} \geq 0$, $\theta_{kl} = 0$ for $a_{kl} < 0$. Put $c_s = 0$ for the remaining multiindices $s \in \mathbb{Z}^n$.

LEMMA 1.1 *Suppose that Assumptions 1-3 are satisfied and $h \in I_C$, given by (3), then*

$$\sum_{s \in \mathbb{Z}^n} c_s = 1, \quad c_s \geq 0, \quad \text{for } (t^\alpha, x^\beta) \in \mathbb{R}_+ \times \mathbb{R}^n \text{ on } E_h,$$

and $c_{\pm e_k} \geq \epsilon > 0$ for $k = 1, \dots, n$.

PROOF From the definition of coefficients we have that $c_{\pm e_k, \pm e_l}, c_{\pm e_k, \mp e_l} \geq 0$. Assumption 2 gives that $c_0 \geq 0$, while the Assumption 3 gives that there exists $\epsilon > 0$ such that $c_{\pm e_k} \geq \epsilon > 0$ for $k = 1, \dots, n$. The result $\sum_{s \in \mathbb{Z}^n} c_s = 1$ follows from direct summation of all coefficients c_s . ■

The finite difference approximation of problem (1)-(2) takes the form

$$(4) \quad \delta_0^+ u^{(\alpha, \beta)} = \sum_{k, l=1}^n a_{kl} \delta_{kl} u^{(\alpha, \beta)} + f[u]^{(\alpha, \beta)} \quad \text{on } E_h^+,$$

$$(5) \quad u^{(\alpha, \beta)} = \bar{\phi}^{(\alpha, \beta)} \quad \text{on } E_h^0,$$

where $\bar{\phi}$ is a discrete perturbed counterpart of the function ϕ ,

$$(6) \quad f[u]^{(\alpha, \beta)} = f_h(t^\alpha, x^\beta, u_{[\alpha, \beta]}, (\delta u)_{[\alpha, \beta]}), \quad \delta u = (\delta_1 u, \dots, \delta_n u),$$

and

$$u_{[\alpha, \beta]}(t^{\tilde{\alpha}}, x^{\tilde{\beta}}) = u(t^{\tilde{\alpha} + \alpha}, x^{\tilde{\beta} + \beta}) \quad \text{for } (t^{\tilde{\alpha}}, x^{\tilde{\beta}}) \in B_h,$$

$$(\delta u)_{[\alpha, \beta]} = ((\delta_1 u)_{[\alpha, \beta]}, \dots, (\delta_n u)_{[\alpha, \beta]}).$$

Equation (4) can be rewritten in the explicit form

$$(7) \quad u^{(\alpha+1, \beta)} = \sum_{s \in \mathbb{Z}^n} c_s u^{(\alpha, \beta+s)} + h_0 f[u]^{(\alpha, \beta)} \quad \text{on } E_h^+.$$

Formula (7) is crucial in further theoretical considerations. First we investigate basic properties of solutions of such equations.

LEMMA 1.2 *Suppose that Assumptions 1-3 are satisfied. If $h \in I_C$, $g : E_h^+ \rightarrow \mathbb{R}$, and $u : \tilde{E}_h \rightarrow \mathbb{R}$ satisfies the equation*

$$(8) \quad u^{(\alpha+1, \beta)} = \sum_{s \in \mathbb{Z}^n} c_s u^{(\alpha, \beta+s)} + h_0 g^{(\alpha, \beta)} \quad \text{on } E_h^+,$$

then

$$(9) \quad \|u^{(\alpha)}\|_\infty \leq \|u^{(0)}\|_\infty + h_0 \sum_{\mu=0}^{\alpha-1} \|g^{(\mu)}\|_\infty \leq \|u^{(0)}\|_\infty + t^\alpha \|g^{(\alpha-1)}\|_\infty,$$

where $\|v^{(\alpha)}\|_\infty = \sup_{\tilde{\alpha} \leq \alpha, \beta \in \mathbb{Z}^n} |v^{(\tilde{\alpha}, \beta)}|$ for any discrete function $(\alpha, \beta) \mapsto v^{(\alpha, \beta)}$, and there is the unique representation of the solution

$$(10) \quad u^{(\alpha, \beta)} = \sum_{\eta \in \mathbb{Z}^n} \Gamma^{(\alpha, \beta, 0, \eta)} u^{(0, \eta)} + h_0 \sum_{\zeta=1}^{\alpha} \sum_{\eta \in \mathbb{Z}^n} \Gamma^{(\alpha, \beta, \zeta, \eta)} g^{(\zeta-1, \eta)} \quad \text{on } E_h,$$

where $\Gamma^{(\alpha, \beta, \zeta, \eta)}$ is the discrete fundamental solution, determined by the relations

$$(11) \quad \Gamma^{(\alpha, \beta, \alpha, \eta)} = \delta_{0, |\beta - \eta|},$$

$$(12) \quad \Gamma^{(\alpha+1, \beta, \zeta, \eta)} = \sum_{s \in \mathbb{Z}^n} c_s \Gamma^{(\alpha, \beta+s, \zeta, \eta)} \quad 0 \leq \zeta \leq \alpha,$$

where $\delta_{0, |\beta - \eta|}$ is the Kronecker symbol.

PROOF Formula (9) follows from Lemma 1.1. Formulas (10) and (11)-(12) are proved by induction on α . ■

REMARK 1.3 It follows from Lemma 1.2 that

$$\Gamma^{(\alpha, \beta, \zeta, \eta)} = \sum_{s_1 \in \mathbb{Z}^n} \cdots \sum_{s_{\alpha-\zeta} \in \mathbb{Z}^n} \prod_{i=1}^{\alpha-\zeta} c_{s_i}^{(\alpha-i)} \delta_{0, |\eta - \beta - \sum_{i=1}^{\alpha-\zeta} s_i|}.$$

We give further properties of the discrete fundamental solution. The following lemma is a simple consequence of Lemma 1.1 and the recurrence relations (11) and (12).

LEMMA 1.4 Under the assumptions of Lemma 1.2 we have

$$(13) \quad \Gamma^{(\alpha, \beta, \zeta, \eta)} \geq 0, \quad \sum_{\eta \in \mathbb{Z}^n} \Gamma^{(\alpha, \beta, \zeta, \eta)} = 1$$

for $\alpha, \zeta = 0, 1, \dots, \beta \in \mathbb{Z}^n, \alpha \geq \zeta$.

To obtain apriori estimates of the difference operators for the fundamental solution we will use the following auxiliary lemma. The symbol $[r]$ stands for the integer part of $r \in \mathbb{R}$, i.e., the integer k such that $k \leq r < k + 1$.

LEMMA 1.5 If $0 < b \leq 1/2$, then

$$\sum_{k=0}^i \binom{i}{k} (1-2b)^{i-k} b^k \binom{k}{[k/2]} \leq \frac{1}{\sqrt{2b(i+1)}}.$$

PROOF The proof is based on the estimate

$$\binom{i}{[i/2]} \leq \frac{2^i}{\sqrt{i+1}}.$$

Then, after simple calculations, using Schwarz's inequality and formula

$$\binom{i}{k} \frac{1}{k+1} = \binom{i+1}{k+1} \frac{1}{i+1},$$

we get the assertion of the Lemma 1.5. ■

LEMMA 1.6 Suppose that Assumptions 1 - 3 are satisfied and $h \in I_C$. Then

$$\sum_{\zeta=1}^{\alpha} \sum_{\eta \in \mathbb{Z}^n} |\Gamma^{(\alpha, \beta + e_j, \zeta, \eta)} - \Gamma^{(\alpha, \beta - e_j, \zeta, \eta)}| \leq \frac{2\sqrt{2\alpha}}{\sqrt{d}} + \frac{4(n-1)\sqrt{2\alpha}}{\sqrt{\epsilon}},$$

where

$$d = c_{+e_j} + \sum_{i=1, i \neq j}^n (c_{+e_j \pm e_i} + c_{\pm e_i + e_j}), \quad 0 < \epsilon = \min_{j=1, \dots, n} c_{+e_j}.$$

PROOF Without loss of generality we assume that $j = 1$. Define

$$\Delta\Gamma^{(\alpha,\zeta)} = \sum_{\eta \in \mathbb{Z}^n} |\Gamma^{(\alpha,\beta+e_1,\zeta,\eta)} - \Gamma^{(\alpha,\beta-e_1,\zeta,\eta)}|.$$

Thus from Remark 1.3 we have

$$\begin{aligned} \Delta\Gamma^{(\alpha,\zeta)} &= \sum_{\eta \in \mathbb{Z}^n} \left| \sum_{s_1 \in \mathbb{Z}^n} \cdots \sum_{s_{\alpha-\zeta} \in \mathbb{Z}^n} \prod_{i=1}^{\alpha-\zeta} c_{s_i} \delta_{0,|\eta-\beta-e_1-\sum_{i=1}^{\alpha-\zeta} s_i|} \right. \\ &\quad \left. - \sum_{s_1 \in \mathbb{Z}^n} \cdots \sum_{s_{\alpha-\zeta} \in \mathbb{Z}^n} \prod_{i=1}^{\alpha-\zeta} c_{s_i} \delta_{0,|\eta-\beta+e_1-\sum_{i=1}^{\alpha-\zeta} s_i|} \right|. \end{aligned}$$

Because each multiindex in \mathbb{Z}^n can be decomposed to $me_1 + \eta$, where $\eta_1 = 0$ and $m \in \mathbb{Z}$, we rearrange the above formula as follows

$$(14) \quad \Delta\Gamma^{(\alpha,\zeta)} = \sum_{m \in \mathbb{Z}} \sum_{\eta \in \mathbb{Z}^n, \eta_1=0} \left| \sum_{\sum_i s_i = (m+1)e_1 + \eta} \prod_{i=1}^{\alpha-\zeta} c_{s_i} - \sum_{\sum_i s_i = (m-1)e_1 + \eta} \prod_{i=1}^{\alpha-\zeta} c_{s_i} \right|.$$

Since $c_{+e_1} = c_{-e_1}$, a suitable replacement $+e_1$ by $-e_1$ (or $-e_1$ by $+e_1$) in the set of multiindices $(s_1, \dots, s_{\alpha-\zeta})$ allows us to rewrite (14) in the form

$$\begin{aligned} \Delta\Gamma^{(\alpha,\zeta)} &= \sum_{\eta \in \mathbb{Z}^n, \eta_1=0} \left| \sum_{\sum_i s_i = +e_1 + \eta} \prod_{i=1}^{\alpha-\zeta} c_{s_i} - \sum_{\sum_i s_i = -e_1 + \eta} \prod_{i=1}^{\alpha-\zeta} c_{s_i} \right| \\ &+ 2 \sum_{m \in \mathbb{N}} \sum_{\eta \in \mathbb{Z}^n, \eta_1=0} \left| \sum_{\sum_i s_i = (m+1)e_1 + \eta} \prod_{i=1}^{\alpha-\zeta} c_{s_i} - \sum_{\sum_i s_i = (m-1)e_1 + \eta} \prod_{i=1}^{\alpha-\zeta} c_{s_i} \right|. \end{aligned}$$

Hence

$$(15) \quad \Delta\Gamma^{(\alpha,\zeta)} = 2 \sum_{\eta \in \mathbb{Z}^n, \eta_1=0} \left(\sum_{\sum_i s_i = 0 + \eta} \prod_{i=1}^{\alpha-\zeta} c_{s_i} + \sum_{\sum_i s_i = +e_1 + \eta} \prod_{i=1}^{\alpha-\zeta} c_{s_i} \right) + 2 \sum_{m \in \mathbb{N}} A_m,$$

where A_m consists of products of coefficients c_{s_i} whose sum of indices is equal to $me_1 + \eta$ and which are not cancelled by any product of coefficients c_{s_i} whose sum of indices is equal to $(m-2)e_1 + \eta$.

Now we consider the first term in (15). It follows from (15) that we have to consider only sequences $(s_1, \dots, s_{\alpha-\zeta})$ such that

$$(16) \quad \left(\sum_i s_i = 0 + \eta, \quad \eta_1 = 0 \right) \quad \text{or} \quad \left(\sum_i s_i = +e_1 + \eta, \quad \eta_1 = 0 \right).$$

These sequences will be classified according to the appearance of three categories of coefficients:

- the first one for which $s_i = +e_1$ or $s_i = +e_1 \pm e_k$ or $s_i = \pm e_k + e_1$, $k = 2, \dots, n$,
- the second one for which $s_i = -e_1$ or $s_i = -e_1 \pm e_k$ or $s_i = \pm e_k - e_1$, $k = 2, \dots, n$,
- the third one - all remaining indices.

Let $J = \{1, \dots, \alpha - \zeta\}$. Denote by A and B all disjoint subsets of J such that their cardinal numbers $\#A$ and $\#B$ either are equal to each other or satisfy the relation $\#B = \#A - 1$. The set A is related to the indices $s = +e_1$, $s = +e_1 \pm e_k$ and $s = \pm e_k + e_1$, $k = 2, \dots, n$, while the set B to the indices $s = -e_1$, $s = -e_1 \pm e_k$ and $s = \pm e_k - e_1$, $k = 2, \dots, n$, so condition (16) is met. Since $A \cap B = \emptyset$ and $A \cup B \subset J$, it is obvious that $2\#A \leq \#J + 1$. We have $c_{+e_1} = c_{-e_1}$ and $A \cap B = \emptyset$, so it will cause no confusion if we use the notation $c_{\pm e_1}$ instead of c_{+e_1} and c_{-e_1} to simplify some of derived formulas. Fixed in (15) any $k \in J$ and any $(s_i)_{i \neq k}$, the sum over $s_k \in \mathbb{Z}^n$, $s_k \neq \pm e_1$, $s_k \neq \pm e_1 \pm e_l$ and $s_k \neq \pm e_1 \mp e_l$ yields $(1 - 2d) \prod_{i \in J, i \neq k} c_{s_i}$. Thus we can represent the first term in formula (15) as follows

$$(17) \quad 2 \sum_{\substack{A, B \subset J \\ A \cap B = \emptyset \\ \#A - \#B \in \{0, 1\}}} \prod_{k \in A} d \prod_{k \in B} d \prod_{k \in J \setminus (A \cup B)} (1 - 2d),$$

where $d = c_{+e_1} + \sum_{i=2}^n (c_{+e_1 \pm e_i} + c_{\pm e_i + e_1})$. Let $i = \#(A \cup B)$. Observe that $0 \leq i \leq \alpha - \zeta$. Thus, by basic combinatorics, we have the estimate of the first term in (15) by

$$2 \sum_{i=0}^{\alpha - \zeta} \binom{\alpha - \zeta}{i} \binom{i}{[i/2]} (1 - 2d)^{\alpha - \zeta - i} d^i.$$

Now we estimate the term $2 \sum_{m \in \mathbb{N}} A_m$ in (15). Without loss of generality we assume that $c_{+e_1 - e_k} = 0$ for $k = 2, \dots, n$. Consider all sequences $(s_1, \dots, s_{\alpha - \zeta})$ such that $\eta_1 > 0$, where $\eta = (\eta_1, \dots, \eta_n) = \sum_i s_i$. These sequences can be divided into n categories:

- (a) the first one for which $\#\{i : s_i = +e_1\} > \#\{i : s_i = -e_1\}$,
- (b) the categories $j = 2, \dots, n$ for which $\#\{i : s_i = +e_1 + e_j\} > \#\{i : s_i = -e_1 - e_j\}$, not belonging to the category (a). ■

The sequences from the category (a) vanish, because there is a replacement of indices $s = +e_1$ by $s = -e_1$ like in the case without mixed derivatives in (1). The sequences from the category (b) for $j = 2$ vanish if there is a replacement of $(+e_1 + e_2, -e_2)$ by $(-e_1 - e_2, +e_2)$, so it depends on indices $s = \pm e_2$, as we have the inequality $\#\{i : s_i = +e_1 + e_2\} > \#\{i : s_i = -e_1 - e_2\}$. Hence the remaining products of coefficients, whose sequences of indices belong to the category (b) for $j = 2$ can be estimated similarly as in the case without mixed derivatives in (1). In that case the indices can be divided into three categories like it the estimation of the first term in (15). The first category is related to the indices $s = -e_2$ (the set A), the second category to the indices $s = +e_2$ (the set B) and the third category to the remaining

indices. So we get the estimate of the products of the coefficients whose sequences of indices belong to the category **(b)** for $j = 2$

$$4 \sum_{i=0}^{\alpha-\zeta} \binom{\alpha-\zeta}{i} \binom{i}{[i/2]} (1 - 2c_{\pm e_2})^{\alpha-\zeta-i} c_{\pm e_2}^i.$$

In the same way we treat sequences from that category for $j = 3, \dots, n$. Hence we get the estimate of the products of the coefficients whose sequences of indices belong to the category **(b)** for any $j = 2, \dots, n$

$$4 \sum_{i=0}^{\alpha-\zeta} \binom{\alpha-\zeta}{i} \binom{i}{[i/2]} (1 - 2c_{\pm e_j})^{\alpha-\zeta-i} c_{\pm e_j}^i.$$

Applying Lemma 1.5 we get

$$(18) \quad \Delta\Gamma(\alpha, \zeta) \leq \frac{2}{\sqrt{2d(\alpha-\zeta+1)}} + \sum_{k=2}^n \frac{4}{\sqrt{2c_{\pm e_k}(\alpha-\zeta+1)}}$$

Observe that, if $c_{+e_1 \pm e_k} = 0$ for some $2 \leq k \leq n$, then k -th member of the above sum is equal to 0. Summing over ζ in (18) and using Assumption 3 yields

$$\sum_{\zeta=1}^{\alpha} \sum_{\eta \in \mathbb{Z}^n} |\Gamma(\alpha, \beta + e_1, \zeta, \eta) - \Gamma(\alpha, \beta - e_1, \zeta, \eta)| \leq \frac{2\sqrt{2\alpha}}{\sqrt{d}} + \frac{4(n-1)\sqrt{2\alpha}}{\sqrt{\epsilon}}.$$

REMARK 1.7 Suppose that Assumption 3 is not satisfied, and $c_{\pm e_k} = 0$ for all k . Then we have

$$\sum_{\eta \in \mathbb{Z}^n} |\Gamma(\alpha, \beta + e_j, \zeta, \eta) - \Gamma(\alpha, \beta - e_j, \zeta, \eta)| = 2.$$

This means that the solution of scheme (4) will not converge to the solution of (1).

REMARK 1.8 Changing definition of category **(b)** for all $j = 2, \dots, n$, such that there are disjoint, it allows to obtain the estimate

$$\sum_{\zeta=1}^{\alpha} \sum_{\eta \in \mathbb{Z}^n} |\Gamma(\alpha, \beta + e_1, \zeta, \eta) - \Gamma(\alpha, \beta - e_1, \zeta, \eta)| \leq \frac{2\sqrt{2\alpha}}{\sqrt{d}} + \frac{2\sqrt{2\alpha}}{\sqrt{\epsilon}}.$$

However, the proof is more complicated in that case.

COROLLARY 1.9 *It follows from Lemmas 1.2 and 1.6 that*

$$|\delta_j u^{(\alpha, \beta)}| \leq \|\delta_j u^{(0)}\|_{\infty} + C_j \left(\frac{\sqrt{2t^{\alpha+1}}}{\sqrt{d}} + \frac{2(n-1)\sqrt{2t^{\alpha+1}}}{\sqrt{\epsilon}} \right) \|g^{(\alpha)}\|_{\infty}.$$

2. Stability and convergence. The defect of scheme (4)-(5) will be defined by

$$\Theta[u, h]^{(\alpha, \beta)} = \delta_0^+ u^{(\alpha, \beta)} - \sum_{k, l=1}^n a_{kl} \delta_{kl} u^{(\alpha, \beta)} - f[u]^{(\alpha, \beta)}.$$

If $\Theta[u, h]^{(\alpha, \beta)} \equiv 0$, then $\{u^{(\alpha, \beta)}\}$ is an exact solution of (4)-(5). We formulate sufficient consistency and stability conditions. All supremum norms will be denoted by $\|\cdot\|_\infty$.

ASSUMPTION 4 Suppose that there are constants $L_1, L_2 \in \mathbb{R}_+$ such that

$$|f_h(t, x, p, q) - f_h(t, x, \tilde{p}, \tilde{q})| \leq L_1 \|p - \tilde{p}\|_\infty + L_2 \|q - \tilde{q}\|_\infty$$

for all $(t, x, p, q), (t, x, \tilde{p}, \tilde{q}) \in E_h^+ \times {}^{B_h}\mathbb{R} \times ({}^{B_h}\mathbb{R})^n$.

ASSUMPTION 5 Let the discrete function $f_h = f[u]^{(\alpha, \beta)}$ (see (6)) satisfies

$$|f_h(t^\alpha, x^\beta, u_{[\alpha, \beta]}, (\delta u)_{[\alpha, \beta]}) - f(t^\alpha, x^\beta, u_{(t^\alpha, x^\beta)}, (\partial_x u)_{(t^\alpha, x^\beta)})| \leq C \|h\|_\infty,$$

with a constant C dependent on $u, \partial_{x_1} u, \dots, \partial_{x_n} u$ and Lipschitz constant of $\partial_{x_i} u$ with respect to x .

It can be shown (see [2]) that there exists an interpolation operator T_h such that $T_h w \in C(B, \mathbb{R})$ for arbitrary $w : B_h \rightarrow \mathbb{R}$, and there exist constants $C, \bar{C} > 0$ such that

$$|T_h(v|_{B_h})(x) - v(x)| \leq C \|h\|_\infty \quad \text{for } v \in C^1(B)$$

and

$$\|T_h w - T_h \bar{w}\|_\infty \leq \bar{C} \|w - \bar{w}\|_\infty \quad \text{for } w, \bar{w} : B_h \rightarrow \mathbb{R}.$$

If we define

$$f_h(t^\alpha, x^\beta, w, \delta w) = f(t^\alpha, x^\beta, T_h w, T_h(\delta w))$$

for $w : B_h \rightarrow \mathbb{R}$, then Assumption 5 is satisfied, provided that f fulfills the Lipschitz condition with respect to p, q and $u \in C^2(E, \mathbb{R})$.

ASSUMPTION 6 Suppose that there exists the unique solution $u \in C(E_0 \cup E, \mathbb{R}) \cap C^{1,3}(E, \mathbb{R})$ of the Cauchy problem (1), (2).

LEMMA 2.1 (*consistency*) Suppose that Assumptions 5-6 are satisfied. Then scheme (4), (5) is consistent with the Cauchy problem (1), (2), on its solution $u \in C(E_0 \cup E, \mathbb{R}) \cap C^{0,2}(E, \mathbb{R})$.

PROOF The consistency is obtained by using Taylor expansions at nodal points. ■

LEMMA 2.2 (*stability*) If $u, v : \tilde{E}_h \rightarrow \mathbb{R}$ and Assumption 4 is satisfied, and

$$\begin{aligned} |u^{(\alpha,\beta)} - v^{(\alpha,\beta)}| &\leq \tilde{C}\|h\|_\infty \quad \text{on } E_h^0, \\ |\delta_j(u-v)^{(\alpha,\beta)}| &\leq \tilde{C}\|h\|_\infty \quad \text{on } E_h^0, j = 1, \dots, n, \\ \Theta[u, h]^{(\alpha,\beta)} = 0, \quad |\Theta[v, h]^{(\alpha,\beta)}| &\leq \bar{C}\|h\|_\infty \quad \text{on } E_h^+, \\ L_1 b + \hat{C} \left(\sqrt{\frac{2b}{d}} + \frac{2(n-1)\sqrt{2b}}{\sqrt{\epsilon}} \right) L_2 &< 1, \end{aligned}$$

where

$$\begin{aligned} \hat{C} &= \max_{j \in \{1, \dots, n\}} C_j, \quad 0 < \epsilon = \min_{j=1, \dots, n} c_{+e_j}, \\ d &= \max_{j=1, \dots, n} \left(c_{+e_j} + \sum_{i=1, i \neq j}^n (c_{+e_j \pm e_i} + c_{\pm e_i + e_j}) \right), \end{aligned}$$

and there is $\tilde{\alpha} > 0$ such that for all $\alpha \leq \tilde{\alpha}$, $t^\alpha \leq b$, then

$$\sup_{\alpha \leq \tilde{\alpha}, \beta \in \mathbb{Z}^n} \|(u^{(\alpha,\beta)} - v^{(\alpha,\beta)}, \delta(u-v)^{(\alpha,\beta)})\|_\infty \rightarrow 0 \quad \text{as } \|h\|_\infty \rightarrow 0.$$

PROOF Denote

$$\begin{aligned} \omega^{(\alpha,\beta)} &= u^{(\alpha,\beta)} - v^{(\alpha,\beta)}, \\ \gamma^{(\alpha,\beta)} &= f[u]^{(\alpha,\beta)} - f[v]^{(\alpha,\beta)} - \Theta[v, h]^{(\alpha,\beta)}. \end{aligned}$$

Form Lemmas 1.2 and 1.6 it follows that

$$\begin{aligned} \|\omega^{(\alpha)}\|_\infty &\leq \|\omega^{(0)}\|_\infty + t^{\alpha+1} \|\gamma^{(\alpha)}\|_\infty \\ &\leq \|\omega^{(0)}\|_\infty + t^{\alpha+1} \{L_1 \|\omega^{(\alpha)}\|_\infty + L_2 \|\delta_h \omega^{(\alpha)}\|_\infty\} + t^{\alpha+1} \bar{C} \|h\|_\infty \end{aligned}$$

and

$$\begin{aligned} \|\delta \omega^{(\alpha)}\|_\infty &\leq \|\delta \omega^{(0)}\|_\infty + \hat{C} \left(\sqrt{\frac{2t^{\alpha+1}}{d}} + \frac{2(n-1)\sqrt{2t^{\alpha+1}}}{\sqrt{\epsilon}} \right) \|\gamma^{(\alpha)}\|_\infty \\ &\leq \|\delta \omega^{(0)}\|_\infty + \hat{C} \left(\sqrt{\frac{2t^{\alpha+1}}{d}} + \frac{2(n-1)\sqrt{2t^{\alpha+1}}}{\sqrt{\epsilon}} \right) \{L_1 \|\omega^{(\alpha)}\|_\infty + L_2 \|\delta_h \omega^{(\alpha)}\|_\infty\} \\ &\quad + \bar{C} \hat{C} \left(\sqrt{\frac{2t^{\alpha+1}}{d}} + \frac{2(n-1)\sqrt{2t^{\alpha+1}}}{\sqrt{\epsilon}} \right) \|h\|_\infty. \end{aligned}$$

If we denote

$$\zeta^{(\alpha)} = L_1 \|\omega^{(\alpha)}\|_\infty + L_2 \|\delta_h \omega^{(\alpha)}\|_\infty,$$

then, since for small $t^{\alpha+1}$ i.e. $t^{\alpha+1} \leq b$

$$L_1 t^{\alpha+1} + \hat{C} \left(\sqrt{\frac{2t^{\alpha+1}}{d}} + \frac{2(n-1)\sqrt{2t^{\alpha+1}}}{\sqrt{\epsilon}} \right) L_2 < 1,$$

we get

$$\begin{aligned} \zeta^{(\alpha)} &\leq \zeta^{(0)} + \left(L_1 t^{\alpha+1} + \hat{C} \left(\sqrt{\frac{2t^{\alpha+1}}{d}} + \frac{2(n-1)\sqrt{2t^{\alpha+1}}}{\sqrt{\epsilon}} \right) L_2 \right) \zeta^{(\alpha)} + \\ &\quad \left(L_1 t^{\alpha+1} + \hat{C} \left(\sqrt{\frac{2t^{\alpha+1}}{d}} + \frac{2(n-1)\sqrt{2t^{\alpha+1}}}{\sqrt{\epsilon}} \right) L_2 \right) \bar{C} \|h\|_\infty. \end{aligned}$$

Thus $\|\zeta\|_\infty$ (which means that also $\|\omega\|_\infty$ and $\|\delta\omega\|_\infty$) tend to 0 as $\|h\|_\infty \rightarrow 0$, assuming $t^{\alpha+1} \leq b$. ■

THEOREM 2.3 (convergence) *Suppose that Assumptions 4-6 are satisfied and*

$$\|(\phi^{(\alpha,\beta)} - \bar{\phi}^{(\alpha,\beta)}, \delta(\phi - \bar{\phi})^{(\alpha,\beta)})\|_\infty \rightarrow 0 \quad \text{as } \|h\|_\infty \rightarrow 0 \text{ on } E_h^0.$$

Then the solution of scheme (4),(5) converges to the solution of the differential-functional problem (1),(2).

PROOF The assertion of Theorem 2.3 follows from Lemmas 2.1 and 2.2. ■

3. Numerical experiments. Fix $n = 2$ and consider the equations

$$(19) \quad \partial_t u(t, x_1, x_2) - \sum_{k,l=1}^2 a_{kl} \partial_{x_k x_l} u(t, x_1, x_2) = f_1(t, x_1, x_2),$$

$$(20) \quad \begin{aligned} &\partial_t u(t, x_1, x_2) - \sum_{k,l=1}^2 a_{kl} \partial_{x_k x_l} u(t, x_1, x_2) \\ &= \sin(u(t, x_1 + \sin(x_1), x_2)) + f_2(t, x_1, x_2), \end{aligned}$$

$$(21) \quad \begin{aligned} &\partial_t u(t, x_1, x_2) - \sum_{k,l=1}^2 a_{kl} \partial_{x_k x_l} u(t, x_1, x_2) \\ &= \sin \left(\int_{x_1-1}^{x_1+1} u(t, s, x_2) ds \right) + f_3(t, x_1, x_2), \end{aligned}$$

$$(22) \quad \begin{aligned} &\partial_t u(t, x_1, x_2) - \sum_{k,l=1}^2 a_{kl} \partial_{x_k x_l} u(t, x_1, x_2) \\ &= \sin(\partial_{x_1} u(t, x_1, x_2)) + f_4(t, x_1, x_2), \\ &\partial_t u(t, x_1, x_2) - \sum_{k,l=1}^2 a_{kl} \partial_{x_k x_l} u(t, x_1, x_2) \end{aligned}$$

$$(23) \quad = \sin(\partial_{x_1} u(t, x_1 + \sin(x_1), x_2)) + f_5(t, x_1, x_2),$$

$$\partial_t u(t, x_1, x_2) - \sum_{k,l=1}^2 a_{kl} \partial_{x_k x_l} u(t, x_1, x_2)$$

$$(24) \quad = \sin(\partial_{x_1} u(t - \tau_0, x_1, x_2)) + f_6(t, x_1, x_2),$$

$$\partial_t u(t, x_1, x_2) - \sum_{k,l=1}^2 a_{kl} \partial_{x_k x_l} u(t, x_1, x_2)$$

$$(25) \quad = \sin(u(t - \tau_0, x_1, x_2)) + f_7(t, x_1, x_2),$$

$$\partial_t u(t, x_1, x_2) - \sum_{k,l=1}^2 a_{kl} \partial_{x_k x_l} u(t, x_1, x_2)$$

$$(26) \quad = \sin\left(\int_{t-\tau_0}^t \int_{x_1-1}^{x_1+1} u(s, z, x_2) dz ds\right) + f_8(t, x_1, x_2).$$

Table 1: The maximum error for $(t, x, y) \in [0, 1] \times [-1, 1]^2$ with $m=10$ (number of added rectangles from each side), $h_1 = h_2 = 1/n$ and $h_0 = 1/4n^2$

step time	1/5		1/10		1/20	
	$t \in [0, 0.5]$	$t \in [0, 1]$	$t \in [0, 0.5]$	$t \in [0, 1]$	$t \in [0, 0.5]$	$t \in [0, 1]$
Eq. (19)	1.74e-03	5.02e-03	4.34e-04	1.26e-03	1.08e-04	3.14e-04
Eq. (20)	1.98e-03	6.08e-03	4.94e-04	1.53e-03	1.23e-04	3.81e-04
Eq. (21)	1.54e-03	4.52e-03	3.82e-04	1.13e-03	9.53e-05	2.83e-04
Eq. (22)	2.09e-03	6.01e-03	5.19e-04	1.50e-03	1.30e-04	3.75e-04
Eq. (23)	1.90e-03	5.54e-03	4.73e-04	1.41e-03	1.19e-04	3.55e-04
Eq. (24)						
$(\tau_0 = 0.5)$	1.74e-03	5.09e-03	4.34e-04	1.27e-03	1.08e-04	3.17e-04
Eq. (24)						
$(\tau_0 = 0.25)$	1.66e-03	4.72e-03	4.15e-04	1.18e-03	1.04e-04	2.95e-04
Eq. (25)						
$(\tau_0 = 0.5)$	1.74e-03	4.98e-03	4.34e-04	1.25e-03	1.08e-04	3.12e-04
Eq. (25)						
$(\tau_0 = 0.25)$	1.72e-03	4.92e-03	4.29e-04	1.23e-03	1.07e-04	3.07e-04
Eq. (26)						
$(\tau_0 = 0.5)$	2.54e-03	5.20e-03	6.35e-04	1.31e-03	1.59e-04	3.27e-04
Eq. (26)						
$(\tau_0 = 0.25)$	2.54e-03	5.21e-03	6.36e-04	1.31e-03	1.59e-04	3.27e-04

We perform numerical experiments for equations (19)-(26) with $a_{11} = a_{22} = 1$, $a_{12} = a_{21} = -0.25$. The results were compared with the prescribed solution

$$u(t, x_1, x_2) = \cos(t \sin(t + x_1 + x_2)).$$

The right-hand sides $f_1(t, x, y), \dots, f_8(t, x, y)$ are designed to obtain the prescribed solution. Since the computations require a large amount of time and computer memory (especially for small h_0), the domain considered in computations is cut-off to $[0, 1] \times [-11, 11]^2$ with boundary values equal to the initial values at $t = 0$. This may cause large errors near the cut boundary, so we present the maximal errors on a smaller domain, namely $[0, 1] \times [-1, 1]^2$ instead of $[0, 1] \times [-11, 11]^2$. The maximal errors on $[0, 1] \times [-1, 1]^2$ are presented in Tables 1 and 2. All numerical experiments confirm convergence of a discrete function to the exact solution.

Table 2: The maximum error for $u_x(t, x, y)$ with the same parameters as in Table 1

step time	1/5		1/10		1/20	
	$t \in [0, 0.5]$	$t \in [0, 1]$	$t \in [0, 0.5]$	$t \in [0, 1]$	$t \in [0, 0.5]$	$t \in [0, 1]$
Eq. (22)	3.42e-03	1.01e-02	8.55e-04	2.52e-03	2.14e-04	6.29e-04
Eq. (23)	4.31e-03	1.31e-02	1.12e-03	3.37e-03	2.82e-04	8.54e-04
Eq. (24)						
($\tau_0 = 0.5$)	3.23e-03	8.92e-03	8.25e-04	2.28e-03	2.06e-04	5.69e-04
Eq. (24)						
($\tau_0 = 0.25$)	3.12e-03	9.31e-03	7.90e-04	2.33e-03	1.97e-04	5.84e-04

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