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On homeomorphisms of the density type topologies

Abstract. This paper is dealing of the homeomorphisms of the density type topologies introduced in [3].

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Through the paper we shall use the standard notation: $\mathbb{R}$ will be the set of real numbers, $\mathcal{L}$ the family of Lebesgue measurable subsets of $\mathbb{R}$ and $l(A)$ the Lebesgue measure of a measurable set A. By $\mathbb{N}$ we shall denote the set of all positive integers ans by $\mathcal{S}$ the family of all unbounded and nondecreasing sequences of positive reals. If $\{s_n\}_{n\in\mathbb{N}} \in \mathcal{S}$, we shall denote $\langle s\rangle = \{s_n\}_{n\in\mathbb{N}}$.

Definition 1 (cf.[3]) We shall say that $x \in \mathbb{R}$ is a density point of a set $A \in \mathcal{L}$ with respect to a sequence $\langle s\rangle \in \mathcal{S}$ (in abbr. $\langle s\rangle$-density point) if

$$\lim_{n\to\infty} \frac{l(A \cap [x - \frac{1}{s_n}; x + \frac{1}{s_n}])}{\frac{2}{s_n}} = 1.$$

Considering the expression

$$\lim_{n\to\infty} \frac{l(A \cap [x; x + \frac{1}{s_n}])}{\frac{1}{s_n}} = 1$$

we say that x is a *right side $\langle s\rangle$-density point* of a set A In the same way we define the *left side $\langle s\rangle$-density point* of the set A.

Remindig the concept of ordinary density point it is worth observing (see [1], [9]) that a point $x \in \mathbb{R}$ is a density point of a set $A \in \mathcal{L}$ if

$$\lim_{n\to\infty} \frac{l(A \cap [x - \frac{1}{n}; x + \frac{1}{n}])}{\frac{2}{n}} = 1.$$

Definition 2 We shall say that $x \in \mathbb{R}$ is a dispersion point of a set $A \in \mathcal{L}$ with respect to a sequence $\langle s \rangle \in \mathcal{S}$ (in abbr. $\langle s \rangle$-dispersion point) if x is $\langle s \rangle$-density point of the set $\mathbb{R} \setminus A$. Precisely, we have that $x \in \mathbb{R}$ is $\langle s \rangle$-dispersion point of the set A if

$$\lim_{n\to\infty} \frac{l(A \cap [x - \frac{1}{s_n}; x + \frac{1}{s_n}])}{\frac{2}{s_n}} = 0.$$

It is clear that x is $\langle s \rangle$-density point of a set $A \in \mathcal{L}$ if and only if 0 is $\langle s \rangle$-density point of the set $A - x$, where $A - x = \{y : y = a - x, a \in A\}$. The same is true for $\langle s \rangle$-dispersion point. Let $\langle s \rangle \in \mathcal{S}$, $A \in \mathcal{L}$. Putting

$$\Phi_{\langle s \rangle}(A) = \{\ x \in \mathbb{R} : x \text{ is } \langle s \rangle - \text{density point of } A\ \},$$

we are getting that operator $\Phi_{\langle s \rangle} : \mathcal{L} \to \mathcal{L}$ is the lower density operator (see [3]). Let $\mathcal{T}_{\langle s \rangle} = \{A \in \mathcal{L} : A \subset \Phi_{\langle s \rangle}(A)\}$. Then by the general theory of liftings (cf. [6]) family $\mathcal{T}_{\langle s \rangle}$ forms topology on the real line. If $\langle s \rangle = \{n\}_{n\in\mathbb{N}}$, then $\mathcal{T}_{\langle s \rangle}$ is simply the classical density topology, which is denoted by $\mathcal{T}_d$. It is clear that $\mathcal{T}_d \subset \mathcal{T}_{\langle s \rangle}$ for every $\langle s \rangle \in \mathcal{S}$. The following result is presented in [3]:

Theorem 3 *Let* $\langle s \rangle \in \mathcal{S}$. *Then* $\mathcal{T}_{\langle s \rangle} = \mathcal{T}_d$ *if and only if* $\liminf_{n\to\infty} \frac{s_n}{s_{n+1}} > 0$.

Let $\mathcal{S}_0 = \{\ \langle s \rangle \in \mathcal{S} : \liminf_{n\to\infty} \frac{s_n}{s_{n+1}} = 0\}$. Several properties of such topologies, especially where $\langle s \rangle \in \mathcal{S}_0$, have been established in [3], [2], [4] and [5]. In the following theorem the natural properties of $\mathcal{T}_{<s>}$-topologies are listed.

Theorem 4 (cf [3], Theorem 4) *With the above notations we have*

(i) $\forall_{<s>\in\mathcal{S}} \forall_{x\in\mathbb{R}} (A \in \mathcal{T}_{<s>} \Rightarrow A + x \in \mathcal{T}_{<s>})$

(ii) $\forall_{<s>\in\mathcal{S}} (A \in \mathcal{T}_{<s>} \Rightarrow -A \in \mathcal{T}_{<s>})$

(iii) $\forall_{<s>\in\mathcal{S}\setminus\mathcal{S}_0} \forall_{m\in\mathbb{R}\setminus\{0\}} (A \in \mathcal{T}_{<s>} \Rightarrow mA \in \mathcal{T}_{<s>})$

(iv) $\forall_{<s>\in\mathcal{S}} \forall_{|m|\geqslant 1} (A \in \mathcal{T}_{<s>} \Rightarrow mA \in \mathcal{T}_{<s>})$

(v) $\forall_{<s>\in\mathcal{S}_0} \forall_{|m|<1} \exists_{A\in\mathcal{L}} (A \in \mathcal{T}_{<s>} \wedge mA \notin \mathcal{T}_{<s>})$

Our main goal is the investigation that in some cases the spaces $(\mathbb{R}, \mathcal{T}_{\langle s \rangle})$ and $(\mathbb{R}, \mathcal{T}_{\langle t \rangle})$ for $\langle s \rangle, \langle t \rangle, \in S$ are not homeomorphic. Some essential properties of the family of continuous functions with respect to $\mathcal{T}_{<s>}$-topology will be helpfull to get the main result. Let us recall the concept of $\langle s \rangle$-approximate continuity.

Definition 5 (cf [4]) Let $\langle s\rangle \in \mathcal{S}$. We shall say that $f:\mathbb{R}\to\mathbb{R}$ is $\langle s\rangle$-approximately continuous at $x_0\in\mathbb{R}$ if there exists a set $A\in\mathcal{L}$ such that $x_0\in A\cap\Phi_{\langle s\rangle}(A)$ and $f_{|A}$ is continuous at x_0.

Theorem 6 (cf [4]) *Let $\langle s\rangle\in\mathcal{S}$. Then a function $f:(\mathbb{R},\mathcal{T}_{\langle s\rangle})\to(\mathbb{R},\mathcal{T}_0)$, where $\mathcal{T}_0$ denotes the natural topology on $\mathbb{R}$, is continuous (we say that f is $\langle s\rangle$-continuous function) if and only if the function f is $\langle s\rangle$-approximately continuous at x_0 for every $x_0\in\mathbb{R}$.*

Let $\langle s\rangle\in\mathcal{S}$ and $F:\mathbb{R}\to\mathbb{R}$. By $F'^{(+)}_{\langle s\rangle}(x)$ we shall denote the right-side derivate number of the function F at a point x with respect to the sequence $\langle s\rangle$; it means that

$$F'^{(+)}_{\langle s\rangle}(x)=\lim_{n\to\infty}\frac{F(x+\frac{1}{s_n})-F(x)}{\frac{1}{s_n}}.$$

In the similar way

$$F'^{(-)}_{\langle s\rangle}(x)=\lim_{n\to\infty}\frac{F(x)-F(x-\frac{1}{s_n})}{\frac{1}{s_n}}$$

If $F'^{(+)}_{\langle s\rangle}(x)=F'^{(-)}_{\langle s\rangle}(x)$ then this common value we shall denote by $F'_{\langle s\rangle}(x)$ and it will be called the derivate number at the point x with respect to the sequence $\langle s\rangle$ (cf [7]).

Theorem 7 *Let $\langle s\rangle\in\mathcal{S}$ and $f:\mathbb{R}\to\mathbb{R}$ be a locally bounded $\langle s\rangle$-continuous function. Then there exists a continuous function $F:\mathbb{R}\to\mathbb{R}$ such that $F'_{\langle s\rangle}(x)=f(x)$ for every $x\in\mathbb{R}$.*

Proof Let f be an $\langle s\rangle$-continuous function which is locally bounded. Since f is measurable, we can define for a fixed point $a\in\mathbb{R}$, a function

$$F(x)=\int_a^x f(t)dt,$$

which is continuous on $\mathbb{R}$ .We will prove that $F'^{(+)}_{\langle s\rangle}(x)=f(x)$ for every $x\in\mathbb{R}$.

Let us fix $x_0\in\mathbb{R}$. There exists an interval (α,β) containing x_0 and a real number $M>0$ such that $|f(x)|<M$ for $x\in(\alpha,\beta)$. Let $\varepsilon>0$. The function f is $\langle s\rangle$-approximately continuous at x_0 hence there exists a measurable set A such that $x_0\in\Phi_{\langle s\rangle}(A)$ and $f_{|A}$ is continuous at x_0. Assume that $A\subset(\alpha,\beta)$. Hence

(1) there exists a number $\delta>0$ such that conditions $x\in A$ and $|x-x_0|<\delta$ implies that $|f(x)-f(x_0)|<\frac{\varepsilon}{2}$;

(2) there exists $n_0\in\mathbb{N}$ such that, for any $n\geqslant n_0$, $\frac{1}{s_n}<\delta$, $x_0+\frac{1}{s_n}<\beta$ and

$$\frac{l(A'\cap[x_0,x_0+\frac{1}{s_n}])}{\frac{1}{s_n}}<\frac{\varepsilon}{4M}.$$

Therefore for $n \geqslant n_0$ we get

$$\left|\frac{F(x_0+\frac{1}{s_n})-F(x_0)}{\frac{1}{s_n}}-f(x_0)\right| = \left|s_n\int\limits_{x_0}^{x_0+\frac{1}{s_n}} f(t)dt - f(x_0)\right| =$$

$$= \left|s_n\int\limits_{x_0}^{x_0+\frac{1}{s_n}} f(t)dt - s_n\int\limits_{x_0}^{x_0+\frac{1}{s_n}} f(x_0)dt\right| = s_n\left|\int\limits_{x_0}^{x_0+\frac{1}{s_n}} (f(t)-f(x_0))dt\right| \leqslant$$

$$\leqslant s_n\int\limits_{x_0}^{x_0+\frac{1}{s_n}} |f(t)-f(x_0)|\,dt = s_n\int\limits_{A\cap[x_0,x_0+\frac{1}{s_n}]} |f(t)-f(x_0)|\,dt +$$

$$+ s_n\int\limits_{A'\cap[x_0,x_0+\frac{1}{s_n}]} |f(t)-f(x_0)|\,dt < s_n\cdot\frac{\varepsilon}{2}\cdot l([x_0,x_0+\frac{1}{s_n}]\cap A) +$$

$$+2M\cdot l(A'\cap[x_0,x_0+\frac{1}{s_n}]) < s_n\cdot\frac{\varepsilon}{2}\cdot\frac{1}{s_n}+2M\cdot\frac{\varepsilon}{4M}=\varepsilon.$$

Similarly, $F'^{(-)}_{\langle s\rangle}(x)=f(x)$ for every $x\in\mathbb{R}$. ■

Theorem 8 (cf [7], Theorem 6) *Let $\langle s\rangle\in\mathcal{S}$. If $g:\mathbb{R}\to\mathbb{R}$ is a continuous function and there exists $g'_{\langle s\rangle}(x)$ for every $x\in\mathbb{R}$ then the function $f:\mathbb{R}\to\mathbb{R}$ of the form $f(x)=g'_{\langle s\rangle}(x)$ has the Darboux property.*

Corollary 9 *Every $\langle s\rangle$-approximately continuous function has the Darboux property.*

Basing on this corollary and following the proof of the form of connected sets in the density topology T_d (see [9], Theorem 3.7) we are getting

Theorem 10 *Let $\langle s\rangle\in\mathcal{S}$. The family of $\mathcal{T}_{\langle s\rangle}$-connected sets is identical with the family of connected sets with respect to natural topology.*

There are two sequences $\langle s\rangle,\langle t\rangle\in\mathcal{S}_0$ pointed in [3] such that $\mathcal{T}_{\langle s\rangle}\setminus\mathcal{T}_{\langle t\rangle}\neq\emptyset$ and $\mathcal{T}_{\langle t\rangle}\setminus\mathcal{T}_{\langle s\rangle}\neq\emptyset$. We will prove that the spaces $(\mathbb{R},\mathcal{T}_{\langle s\rangle})$ and $(\mathbb{R},\mathcal{T}_{\langle t\rangle})$ having the indicated properties are not homeomorphic. Let us start with the following lemma:

Lemma 11 *Let $\langle s\rangle,\langle t\rangle\in\mathcal{S}$. If $\mathcal{T}_{\langle s\rangle}\setminus\mathcal{T}_{\langle t\rangle}\neq\emptyset$ then there exist: a right side interval set $B=\bigcup_{k=1}^{\infty}[a_k,b_k]$, $\lim_{k\to\infty}b_k=0$, $b_{k+1}<a_k<b_k$ for $k\in\mathbb{N}$ and a subsequence $\{t_{n_k}\}_{k\in\mathbb{N}}$ of the sequence $\{t_n\}_{n\in\mathbb{N}}$, and a real number $\alpha>0$ such that*

(1) *0 is a right side $\langle s\rangle$-dispersion point of the set B*

(2) $$\frac{l([a_k;b_k]\cap[0;\frac{1}{t_{n_k}}])}{\frac{1}{t_{n_k}}}>\alpha \quad \textit{for every } k\in\mathbb{N}.$$

PROOF By the assumption there exists a measurable set $A\in\mathcal{T}_{\langle s\rangle}$ such that $A\notin\mathcal{T}_{\langle t\rangle}$. It implies that there exists a point $x_0\in A$ such that $x_0\notin\Phi_{\langle t\rangle}(A)$. Let $C=A-x_0=\{a-x_0: a\in A\}$. Since the topologies $\mathcal{T}_{\langle s\rangle}$ and $\mathcal{T}_{\langle t\rangle}$ are invariant with respect to translation, we have that $C\in\mathcal{T}_{\langle s\rangle}$ and $0\notin\Phi_{\langle t\rangle}(C)$. Therefore there exists a real number $\alpha>0$ and a subsequence $\{t_{n_i}\}_{i\in\mathbb{N}}$ of the sequence$\{t_n\}_{n\in\mathbb{N}}$ such that

$$\frac{l(C'\cap[0;\frac{1}{t_{n_i}}])}{\frac{1}{t_{n_i}}}>2\alpha,$$

where C' denotes the complement of C, for every $i\in\mathbb{N}$. There is no loss of generality in assuming that

(3) $$\frac{l(C'\cap[0;\frac{1}{t_n}])}{\frac{1}{t_n}}>2\alpha$$

for every $n\in\mathbb{N}$. Let $n_1=1$. Because $\{t_n\}_{n\in\mathbb{N}}\nearrow\infty$ then there exists $n_2\in\mathbb{N}$ such that

(4) $$\frac{l(C'\cap[\frac{1}{t_{n_2}};\frac{1}{t_{n_1}}])}{\frac{1}{t_{n_1}}}>\alpha.$$

Let $a_1=\frac{1}{t_{n_1}}-l(C'\cap[\frac{1}{t_{n_2}};\frac{1}{t_{n_1}}])$ and $b_1=\frac{1}{t_{n_1}}$. It is clear that $a_1\in(0;b_1)$ and by (4) we have

$$t_{n_1}\cdot l([a_1;b_1]\cap[0;\frac{1}{t_{n_1}}])=t_{n_1}\cdot(b_1-a_1)>\alpha.$$

Let us assume that we have already defined the segments $[a_i;b_i]$ for $i=1,...,k-1$, and the positive integers n_i for $i=1,...,k$. By (3), we know that there exists $n_{k+1}\in\mathbb{N}$ such that $t_{n_k}\cdot l(C'\cap[\frac{1}{t_{n_{k+1}}};\frac{1}{t_{n_k}}])>\alpha$. Let $a_k=\frac{1}{t_{n_k}}-l(C'\cap[\frac{1}{t_{n_{k+1}}};\frac{1}{t_{n_k}}])$ and $b_k=\frac{1}{t_{n_k}}$. Then

$$t_{n_k}\cdot l([a_k;b_k]\cap[0;\frac{1}{t_{n_k}}])=t_{n_k}\cdot(b_k-a_k)>\alpha.$$

Putting

$$B=\bigcup_{k=1}^{\infty}[a_k;b_k]$$

we have obtained that the set B and the sequence $\{t_{n_k}\}_{k\in\mathbb{N}}$ satisfy condition (2).

Showing that 0 is the right side $\langle s\rangle$-dispersion point of the set B we shall finish the proof. Firstly we prove that

(5) $$l(B\cap[0;h])\leqslant l(C'\cap[0;h])$$

for any $h \in (0; b_1]$. For every $k \in \mathbb{N}$

$$l(B \cap [0; b_k]) = \sum_{i=k}^{\infty} l(B \cap (b_{i+1}; b_i]) = \sum_{i=k}^{\infty} l([a_i; b_i]) =$$

$$= \sum_{i=k}^{\infty} l(C' \cap [b_{i+1}; b_i]) = l(C' \cap [0; b_k]).$$

Let $h \in (0; b_1]$. There exists a positive integer k such that $h \in (b_{k+1}; b_k] = (b_{k+1}; a_k] \cup (a_k; b_k]$. If $h \in (b_{k+1}; a_k]$ then $l(B \cap [0; h]) = l(B \cap [0; b_{k+1}]) = l(C' \cap [0; b_{k+1}]) \leqslant l(C' \cap [0; h])$. If $h \in (a_k; b_k]$ then $l(B \cap [0; h]) = l(B \cap [0; b_k]) - l(B \cap [h; b_k]) = l(C' \cap [0; b_k]) - l([h; b_k]) \leqslant l(C' \cap [0; b_k]) - l(C' \cap [h; b_k]) = l(C' \cap [0; h])$. By (5) we have that 0 is the right side $\langle s \rangle$-dispersion point of the set B because 0 is the right side $\langle s \rangle$-dispersion point of the set C'. ∎

THEOREM 12 *Let $\langle s \rangle, \langle t \rangle \in S_0$. If $\mathcal{T}_{\langle s \rangle} \setminus \mathcal{T}_{\langle t \rangle} \neq \emptyset$ and $\mathcal{T}_{\langle t \rangle} \setminus \mathcal{T}_{\langle s \rangle} \neq \emptyset$ then the spaces $\left(\mathbb{R}, \mathcal{T}_{\langle s \rangle}\right)$ and $\left(\mathbb{R}, \mathcal{T}_{\langle t \rangle}\right)$ are not homeomorphic.*

PROOF Let us suppose that there exists a homeomorphism $h : \left(\mathbb{R}, \mathcal{T}_{\langle s \rangle}\right) \to \left(\mathbb{R}, \mathcal{T}_{\langle t \rangle}\right)$. Observe first that, for every open interval $(a; b) \subset \mathbb{R}$, the sets $h^{-1}((a; b))$ and $h((a; b))$ are open intervals. From Theorem 10 and continuity of the function h we know that that the image of an interval is an interval. The same is true for h^{-1}.

Let (a, b) be an open interval. As $(a; b) \in \mathcal{T}_{\langle t \rangle}$, we have that $J = h^{-1}((a; b)) \in \mathcal{T}_{\langle s \rangle}$.We also know that J is an interval (bounded or unbounded). Let us suppose that the left endpoint of J, denoted by x, belongs to J. Since x is not the point of the left side $\langle s \rangle$-density of J, we obtain that $J \notin \mathcal{T}_{\langle s \rangle}$. This contradiction proves that J is left side open. Similarly we get that J is right side open. By the same arguments we obtain that $h((a, b))$ is an open interval.

Hence h and h^{-1} are continuous transformations from $(\mathbb{R}, \mathcal{T}_0)$ onto $(\mathbb{R}, \mathcal{T}_0)$, where $\mathcal{T}_0$ is natural topology on $\mathbb{R}$. It implies that h is a homeomorphism, so h is strictly monotone. Let us suppose that h is increasing.

We shall prove that the functions h and h^{-1} satisfy (N)-Lusin condition. Let us suppose that h does not satisfy condition (N). Then there exists a set $P \subset \mathbb{R}$ such that P is the Lebesgue measure zero set while $h(P)$ is not.

The set P and all its subsets are closed with respect to the $\mathcal{T}_{\langle s \rangle}$ topology. Hence $h(P)$ is closed with respect to the $\mathcal{T}_{\langle t \rangle}$ topology. Thus the set $h(P)$ is measurable and $l(h(P)) > 0$. It is clear that every subset of the set $h(P)$ is an image of a fixed subset of P with respect to h. In that way an arbitrary subset of $h(P)$ is measurable. This contradiction proves that the function h satisfies (N)-Lusin condition. In the similar way we get that h^{-1} also satisfies (N)-Lusin condition.

The functions h and h^{-1}, as monotone functions, are differentiable almost everywhere.We next show that there exists a point $x_0 \in \mathbb{R}$ such that the function h is differentiable at x_0 and the function h^{-1} is differentiable at $h(x_0)$, and

$$h'(x_0) \in (0, 1) \cup (1, \infty).$$

Let A be the set of points of nondifferentiability of the function h and B - the set of points of nondifferentiability of h^{-1}. From what has already been proved, it follows that $l(A \cup h^{-1}(B)) = 0$. Simultaneously $h'(x) \geqslant 0$ for $x \in \mathbb{R} \setminus (A \cup h^{-1}(B))$ because the function h is increasing. Observe that it is not true that h' is equal to 1 almost everywhere on $\mathbb{R}$, because otherwise h would be identity everywhere on $\mathbb{R}$, which contradicts the fact that $\mathcal{T}_{\langle s \rangle} \neq \mathcal{T}_{\langle t \rangle}$. If we denote $C = R \setminus \{x : h'(x) = 1\}$, then C is a measurable set and $l(C) > 0$. Let $D = \{x : h'(x) = 0\}$ and suppose that $l(C \setminus D) = 0$. Then obviously $l(D) > 0$ and $l(h(D)) > 0$. But for each $y \in h(D)$ we have $(h^{-1})'(y) = +\infty$, which is impossible by virtue of Theorem 4.4, chapter IX, p. 270 in [8]. Hence $l(C \setminus D) > 0$ and there exists $x_0 \in \mathbb{R}$ such that $h'(x_0) \in (0,1) \cup (1,\infty)$ because h is differentiable almost everywhere.

We can assume that $h(0) = 0$ and $h'(0) \in (0,1) \cup (1,\infty)$. Indeed, since the topologies $\mathcal{T}_{\langle s \rangle}$ and $\mathcal{T}_{\langle t \rangle}$ are invariant with respect to translation, the function $\overline{h}$ defined by formula

$$\overline{h}(x) = h(x + x_0) - h(x_0)$$

for every $x \in \mathbb{R}$, is a homeomorphism from $\left(\mathbb{R}, \mathcal{T}_{\langle s \rangle}\right)$ onto $\left(\mathbb{R}, \mathcal{T}_{\langle t \rangle}\right)$. For the simplicity of denotation, assume that $h = \overline{h}$.

Firstly, let $a = h'(0) < 1$. Since $\mathcal{T}_{\langle s \rangle} \setminus \mathcal{T}_{\langle t \rangle} \neq \emptyset$ then, by Lemma 11, there exists a subsequence $\{t_{n_k}\}_{k \in \mathbb{N}}$ of the sequence $\{t_n\}_{n \in \mathbb{N}}$ and a number $\alpha > 0$, and a set

$$B = \bigcup_{k=1}^{\infty} [a_k; b_k] = \bigcup_{k=1}^{\infty} [a_k; \frac{1}{t_{n_k}}]$$

such that $\lim_{k \to \infty} b_k = 0$, $b_{k+1} < a_k < b_k$ for $k \in \mathbb{N}$ and zero is $\langle s \rangle$-dispersion point of the set B. Moreover, for any $k \in \mathbb{N}$

$$\frac{l([a_k; b_k] \cap [0; \frac{1}{t_{n_k}}])}{\frac{1}{t_{n_k}}} = \frac{b_k - a_k}{b_k} > \alpha \tag{6}$$

It is easy to check that $B' \in \mathcal{T}_{\langle s \rangle}$. Further, we shall prove that $(h(B))' \notin \mathcal{T}_{\langle t \rangle}$. By the properties of the function h we have that

$$h(B) = \bigcup_{k=1}^{\infty} [h(a_k); h(b_k)]$$

Since $\{a_n\}_{n \in \mathbb{N}}$ and $\{b_n\}_{n \in \mathbb{N}}$ tend to zero and $h'(0) = a$ we get that for any $\varepsilon > 0$ there exists a natural number k_0 such that

$$\left| \frac{h(a_k)}{a_k} - a \right| < \varepsilon \quad \text{and} \quad \left| \frac{h(b_k)}{b_k} - a \right| < \varepsilon$$

for any $k > k_0$. Let $\varepsilon = \frac{a \cdot \alpha}{4}$. There exists $k_0 \in \mathbb{N}$ such that for $k > k_0$ we have

$$(a - \varepsilon) \cdot b_k < h(b_k) < (a + \varepsilon) \cdot b_k \quad \text{and}$$
$$(a - \varepsilon) \cdot a_k < h(a_k) < (a + \varepsilon) \cdot a_k$$

By the inequality $h'(0) < 1$, there exists $K \in \mathbb{N}$ such that for $k > K$

$$h(b_k) < b_k \tag{7}$$

For every $k > \max(k_0, K)$

$$(8)\quad \begin{aligned} h(b_k) - h(a_k) &\geqslant (a-\varepsilon)\cdot b_k - (a+\varepsilon)\cdot a_k = a\cdot(b_k - a_k) - \varepsilon\cdot(b_k + a_k) \\ &\geqslant a\cdot(b_k - a_k) - 2b_k\varepsilon \end{aligned}$$

and by (7)

$$h(B)\cap[0;\frac{1}{t_{n_k}}] = h(B)\cap[0;b_k] \supset [h(a_k);h(b_k)]\ .$$

In that way, taking into account (8) and (6), we get

$$\frac{l(h(B)\cap[0;\frac{1}{t_{n_k}}])}{\frac{1}{t_{n_k}}} \geqslant \frac{h(b_k)-h(a_k)}{b_k} \geqslant a\frac{b_k - a_k}{b_k} - 2\varepsilon > a\alpha - \frac{a\alpha}{2} = \frac{a\alpha}{2}.$$

It means that zero is not $\langle t\rangle$-dispersion point of the set $(h(B))'$; hence $(h(B))' \notin \mathcal{T}_{\langle t\rangle}$. It is a contradiction with the fact that h is a homeomorphism.

Let $a = h'(0) > 1$. Because the function h^{-1} is differentiable at the point $h(0) = 0$ then $(h^{-1})'(0) = \frac{1}{a} < 1$ Repeating the construction of the set B and using the fact that $\mathcal{T}_{\langle t\rangle} \setminus \mathcal{T}_{\langle s\rangle} \neq \emptyset$ we are getting the contradiction by the same arguments like in the previous part. ■

Now we shall discuss that in Theorem 12 the assumptions are essential. It is not possible to assume only the one condition that $\mathcal{T}_{\langle s\rangle} \setminus \mathcal{T}_{\langle t\rangle} \neq \emptyset$ or $\mathcal{T}_{\langle t\rangle} \setminus \mathcal{T}_{\langle s\rangle} \neq \emptyset$. We have the following observation:

Remark 13 Suppose that $\langle s\rangle \in \mathcal{S}$ and $m > 0$. Let $\langle ms\rangle$ denote the sequence $\{ms_n\}_{n\in\mathbb{N}}$ and $\frac{1}{m}\mathcal{T}_{\langle s\rangle} = \{\frac{1}{m}A : A \in \mathcal{T}_{\langle s\rangle}\}$, where $\frac{1}{m}A = \{\frac{1}{m}a : a \in A\}$. Then $\mathcal{T}_{\langle ms\rangle} = \frac{1}{m}\mathcal{T}_{\langle s\rangle}$.

Remark 14 Let $\langle s\rangle \in \mathcal{S}_0$. Then

1^0 $\mathcal{T}_{\langle s\rangle} \subsetneq \mathcal{T}_{\langle ms\rangle}$ for any $m > 1$,

2^0 $\mathcal{T}_{\langle ms\rangle} \subsetneq \mathcal{T}_{\langle s\rangle}$ for any $m < 1$.

The fact that $\mathcal{T}_{\langle s\rangle} \neq \mathcal{T}_{\langle ms\rangle}$ is a simple consequence of Theorem 4 and Remark 13. It is obvious that in both cases 1^0 and 2^0 the spaces $(\mathbb{R}, \mathcal{T}_{\langle s\rangle})$ and $(\mathbb{R}, \mathcal{T}_{\langle ms\rangle})$ are homeomorphic. It is sufficient to take $h(x) = \frac{1}{m}x$ and $h(x) = mx$, respectively.

Corollary 15 *For every sequence $\langle s\rangle \in \mathcal{S}_0$ there exists a sequence $\langle t\rangle \in \mathcal{S}_0$ such that $\mathcal{T}_{\langle t\rangle} \subsetneq \mathcal{T}_{\langle s\rangle}$ and the spaces $(\mathbb{R}, \mathcal{T}_{\langle t\rangle})$ and $(\mathbb{R}, \mathcal{T}_{\langle s\rangle})$ are homeomorphic.*

Corollary 16 *For every sequence $\langle s\rangle \in \mathcal{S}_0$ there exists a sequence $\langle t\rangle \in \mathcal{S}_0$ such that $\mathcal{T}_{\langle s\rangle} \subsetneq \mathcal{T}_{\langle t\rangle}$ and the spaces $(\mathbb{R}, \mathcal{T}_{\langle t\rangle})$ and $(\mathbb{R}, \mathcal{T}_{\langle s\rangle})$ are homeomorphic.*

We are finishing our considerations with the following:

THEOREM 17 *For every $\langle s\rangle \in \mathcal{S}$ the spaces $(\mathbb{R}, \mathcal{T}_d)$ and $(\mathbb{R}, \mathcal{T}_{\langle s\rangle})$ are not homeomorphic.*

PROOF Let $\langle s\rangle \in \mathcal{S}$. Let us suppose that there exists a homeomorphism $h : (\mathbb{R}, \mathcal{T}_{\langle s\rangle}) \to (\mathbb{R}, \mathcal{T}_d)$. It is easy to check that for every $\alpha > 0$ the function $h_\alpha(x) = \alpha \cdot h(x)$ is also a homeomorphism. Following the proof of the main Theorem 12 we are able to get that h is a such homeomorphism that $h(0) = 0$ and $h'(0) \in (0, 1) \cup (1, \infty)$. If $h'(0) < 1$ that taking into account that $\mathcal{T}_{\langle s\rangle} \setminus \mathcal{T}_d \neq \emptyset$, by the same arguments as presented in Theorem 12 we are getting a contradiction. If $h'(0) > 1$ then let us define $h_\alpha(x) = \frac{1}{\alpha} \cdot h(x)$, where $\alpha > h'(0)$. In that way the homeomorphism h_α satisfies conditions: $h_\alpha(0) = 0$ and $h'_\alpha(0) < 1$. This observation ends the proof. ■

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