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A comparison of some conditional functional equations

Abstract. For real inner product spaces we consider and compare the classes of mappings satisfying some conditional and unconditional functional equations.

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Let X and Y be real inner product spaces and let $\dim X \geq 2$. We consider the classes of functions preserving the inner product or its absolute value, i.e., we consider the functional equations with the unknown function $f : X \to Y$:

$$\text{(O)} \qquad \langle f(x)|f(y)\rangle = \langle x|y\rangle \qquad \text{for } x, y \in X$$

(the *orthogonality equation*) and

$$\text{(W)} \qquad |\,\langle f(x)|f(y)\rangle\,| = |\,\langle x|y\rangle\,| \qquad \text{for } x, y \in X$$

(the *Wigner equation*).

We consider also the class of *isometries*:

$$\text{(I)} \qquad \|f(x) - f(y)\| = \|x - y\| \qquad \text{for } x, y \in X \quad \text{and} \quad f(0) = 0,$$

the class of *additive* functions, i.e., satisfying the *Cauchy equation*:

$$\text{(A)} \qquad f(x + y) = f(x) + f(y) \qquad \text{for } x, y \in X$$

and the class of solutions of the *Fischer-Muszély equation*:

$$\text{(B)} \qquad \|f(x + y)\| = \|f(x) + f(y)\| \qquad \text{for } x, y \in X.$$

The following connections between the above equations hold:

$$\text{(1)} \qquad \text{(O)} \Leftrightarrow \text{(I)}, \quad \text{(A)} \Leftrightarrow \text{(B)}, \quad \text{(I)} \Rightarrow \text{(A)} \ \text{ and } \ \text{(A)} \not\Rightarrow \text{(I)}.$$

Indeed, if f satisfies (O), then it has to be linear and norm-preserving. This implies (I). Conversely, f satisfying (I) is (see [2]) linear. Moreover, $\|f(x)\| = \|x\|$ whence $\|f(x) + f(y)\| = \|x + y\|$ and $\|f(x) - f(y)\| = \|x - y\|$ for all $x, y \in X$. Therefore, using the polarization formula, we get that f preserves the inner product.

It follows from [3] that for inner product spaces (actually, strict convexity suffices) the equation (A) is equivalent to (B).

As we have already noticed, each isometry is additive, but if we take, e.g., $X = Y$ and $f(x) = 2x$, we would have an additive function which is not an isometry.

A solution of the equation (W) is (cf. [6], [5]) *phase-equivalent* to a linear isometry (i.e., $f(x) = \sigma(x)I(x)$ for $x \in X$ where I denotes an isometry and $|\sigma(x)| = 1$ for $x \in X$).

C. Alsina and J.L. Garcia-Roig [1] introduced the Cauchy equation *on spheres*:

$$(\mathrm{A}_{\|\cdot\|}) \qquad f(x + y) = f(x) + f(y) \quad \text{for } x, y \in X \text{ such that } \|x\| = \|y\|.$$

This conditional equation has been treated also by Ger and Sikorska [4]. They considered a more general situation where the domain X of the unknown function f was a real linear space with $\dim X \geq 2$, the codomain Y was an Abelian group and the condition $\|x\| = \|y\|$ was replaced by $\varphi(x) = \varphi(y)$ for φ being a given function mapping X into an arbitrary nonempty set Z and satisfying the conditions:

(i) for any two linearly independent vectors $x, y \in X$ there exist linearly independent vectors $u, v \in \mathrm{Lin}\,\{x, y\}$ such that $\varphi(u + v) = \varphi(u - v)$;

(ii) if $x, y \in X$, $\varphi(x + y) = \varphi(x - y)$, then $\varphi(\alpha x + y) = \varphi(\alpha x - y)$ for all $\alpha \in \mathbb{R}$;

(iii) for all $x \in X$ and $\lambda \in (0, \infty)$ there exists a $y \in X$ such that $\varphi(x+y) = \varphi(x-y)$ and $\varphi((\lambda + 1)x) = \varphi((\lambda - 1)x - 2y)$;

(iv) φ is even.

Ger and Sikorska showed (Theorem 1 in [4]) that the conditional Cauchy equation:

$$(\mathrm{A}_{\varphi}) \qquad f(x + y) = f(x) + f(y) \quad \text{for } x, y \in X \text{ such that } \varphi(x) = \varphi(y)$$

is equivalent to the unconditional one (A). In particular, this equivalency holds with the function $\varphi := \|\cdot\|$ which, in the case where the norm comes from an inner product, satisfies the assumptions (i)-(iv) (i.e., $(\mathrm{A}_{\|\cdot\|}) \Leftrightarrow$ (A)).

From now on, we assume that φ is a given function defined on a real inner product space X with values in a nonempty set Z and satisfies the conditions (i)-(iv). Let us define the conditional versions of the remaining considered equations:

$$(\mathrm{O}_{\varphi}) \qquad \langle f(x)|f(y)\rangle = \langle x|y\rangle \qquad \text{for } x, y \in X \text{ such that } \varphi(x) = \varphi(y),$$

$$(\mathrm{W}_{\varphi}) \qquad |\,\langle f(x)|f(y)\rangle\,| = |\,\langle x|y\rangle\,| \qquad \text{for } x, y \in X \text{ such that } \varphi(x) = \varphi(y),$$

$$(\mathrm{I}_\varphi) \qquad \begin{cases} \|f(x)-f(y)\| = \|x-y\| & \text{for } x,y \in X \text{ such that } \varphi(x)=\varphi(y) \\ f(0)=0, \end{cases}$$

and

$$(\mathrm{B}_\varphi) \qquad \|f(x+y)\| = \|f(x)+f(y)\| \qquad \text{for } x,y \in X \text{ such that } \varphi(x)=\varphi(y).$$

In particular, in each case for $\varphi = \|\cdot\|$, we get the equation *on spheres*; for φ being a constant function we obtain the respective unconditional equation.

We are going to establish the connections between the conditional equations and their unconditional counterparts as well as with other considered equations. Obviously, we have

$$(2) \qquad (\mathrm{O}) \Rightarrow (\mathrm{O}_\varphi), \quad (\mathrm{W}) \Rightarrow (\mathrm{W}_\varphi), \quad (\mathrm{I}) \Rightarrow (\mathrm{I}_\varphi), \quad (\mathrm{B}) \Rightarrow (\mathrm{B}_\varphi).$$

None of the converse implications are true:

$$(3) \qquad (\mathrm{O}_\varphi) \not\Rightarrow (\mathrm{O}), \quad (\mathrm{W}_\varphi) \not\Rightarrow (\mathrm{W}), \quad (\mathrm{I}_\varphi) \not\Rightarrow (\mathrm{I}), \quad (\mathrm{B}_\varphi) \not\Rightarrow (\mathrm{B}).$$

More precisely, here and later on, we mean that there exist spaces X, Y and suitable mappings φ for which the given implications do not hold. In all the (counter)examples in the paper we take $Z = \mathbb{R}$ and $\varphi(x) := \|x\|$. Now, let us consider the Euclidean space $X = Y = \mathbb{R}^2$ and let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$(*) \qquad f(x_1,x_2) := \begin{cases} (-x_2, x_1) & \text{if } x_1^2+x_2^2 = 1, \\ (x_1,x_2) & \text{if } x_1^2+x_2^2 \neq 1. \end{cases}$$

Then f satisfies (O_φ) (whence also (W_φ)) but not (W) (whence neither (O) nor (I)).

For an arbitrary normed space $X = Y$ and for an arbitrary unit vector $e \in X$, defining $f(x) := x + \|x\|e$ we obtain a mapping satisfying (I_φ) but not (I) as we have $\|f(e)\| \neq \|e\|$.

Now, let us consider the function $f : X \to X$ given by $f(x) = D(\|x\|)x$ where $D : \mathbb{R} \to \{-1, 1\}$ takes the value 1 for rational and -1 for non-rational numbers. It is easy to see that (B_φ) holds. Let e be a unit vector and let $x = e$, $y = \sqrt{2}e$. Then $\|f(x)+f(y)\| = \|(1-\sqrt{2})e\| = \sqrt{2}-1$ whereas $\|f(x+y)\| = \|-(1+\sqrt{2})e\| = 1+\sqrt{2}$, i.e., (B) does not hold. This implies, in particular, that (A_φ) and (B_φ) are not equivalent (of course (A_φ) implies (B_φ)).

Next, we show:

$$(4) \qquad \begin{array}{lcl} (\mathrm{O}_\varphi) \Rightarrow (\mathrm{I}_\varphi) & \text{and} & (\mathrm{I}_\varphi) \not\Rightarrow (\mathrm{O}_\varphi), \\ (\mathrm{W}_\varphi) \not\Rightarrow (\mathrm{I}_\varphi) & \text{and} & (\mathrm{I}_\varphi) \not\Rightarrow (\mathrm{W}_\varphi). \end{array}$$

Suppose that f satisfies (O_φ). Taking $x = y$ we obtain $\|f(x)\| = \|x\|$ for any $x \in X$; in particular $f(0) = 0$. Now, for arbitrary $x, y \in X$ such that $\varphi(x) = \varphi(y)$ we have

$$\begin{aligned} \|f(x)-f(y)\|^2 &= \|f(x)\|^2 - 2\langle f(x)|f(y)\rangle + \|f(y)\|^2 \\ &= \|x\|^2 - 2\langle x|y\rangle + \|y\|^2 \\ &= \|x-y\|^2, \end{aligned}$$

i.e., f satisfies (I_φ). The converse implication as well as $(\mathrm{I}_\varphi) \Rightarrow (\mathrm{W}_\varphi)$ are not true. Once again, the function $f(x) := x + \|x\|e$ acts as a counterexample. Finally, function $f(x) = \sigma(x)x$, where $\sigma : X \to \{-1, 1\}$ is an arbitrary but not constant function, satisfies (W) but not necessarily (I_φ).

$$(\mathrm{O}_\varphi) \not\Rightarrow (\mathrm{A}_\varphi) \quad \text{and} \quad (\mathrm{I}_\varphi) \not\Rightarrow (\mathrm{A}_\varphi). \tag{5}$$

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x) := D(\|x\|)x$ for $x \in \mathbb{R}^2$. Then f satisfies (O_φ) but not (A_φ) — for let take $x = (1, 0)$ and $y = (0, 1)$. Thus $(\mathrm{O}_\varphi) \not\Rightarrow (\mathrm{A}_\varphi)$ and, because of (4), $(\mathrm{I}_\varphi) \not\Rightarrow (\mathrm{A}_\varphi)$.

A function f preserving the inner product (unconditionally) has to be additive. On the other hand, as we have seen, a function which satisfies the orthogonality equation on spheres need not be additive on spheres. However, we show that it satisfies some kind of conditional additivity:

$$(\mathrm{O}_\varphi) \Rightarrow (\mathrm{B}_\varphi) \quad \text{and} \quad (\mathrm{B}_\varphi) \not\Rightarrow (\mathrm{O}_\varphi). \tag{6}$$

Suppose that f is a solution of (O_φ). Then $\|f(x)\| = \|x\|$ for all $x \in X$. Let $\varphi(x) = \varphi(y)$; then we have

$$\begin{aligned} \|f(x) + f(y)\|^2 &= \|f(x)\|^2 + \|f(y)\|^2 + 2\langle f(x)|f(y)\rangle \\ &= \|x\|^2 + \|y\|^2 + 2\langle x|y\rangle = \|x + y\|^2 \\ &= \|f(x + y)\|^2. \end{aligned}$$

The implication $(\mathrm{B}_\varphi) \Rightarrow (\mathrm{O}_\varphi)$ does not hold as it is easily seen, e.g., from the example $f(x) = 2x$.

$$(\mathrm{I}_\varphi) \not\Rightarrow (\mathrm{B}_\varphi) \quad \text{and} \quad (\mathrm{B}_\varphi) \not\Rightarrow (\mathrm{I}_\varphi). \tag{7}$$

In order to prove $(\mathrm{I}_\varphi) \not\Rightarrow (\mathrm{B}_\varphi)$ it suffices to consider the function $f(x) = x + \|x\|e$ (where $\|e\| = 1$). The function $f(x) = 2x$ shows that $(\mathrm{B}_\varphi) \not\Rightarrow (\mathrm{I}_\varphi)$.

$$(\mathrm{W}) \not\Rightarrow (\mathrm{B}_\varphi) \quad \text{and} \quad (\mathrm{B}) \not\Rightarrow (\mathrm{W}_\varphi). \tag{8}$$

For an arbitrary X and $0 \neq x_0 \in X$, the function $f : X \to X$ defined by

$$f(x) := \begin{cases} x, & x \neq x_0, \\ -x_0, & x = x_0 \end{cases}$$

satisfies (W) but not (B_φ) as $0 = \|f(x_0 - x_0)\| \neq 2\|f(x_0)\|$. Once again, $f(x) = 2x$ is additive but it does not satisfy (W_φ).

To sum up, one can derive easily from (1) – (8) the following result.

THEOREM 1 *Let X and Y be real inner product spaces with* $\dim X \geq 2$. *Then the following implications hold* $(+)$ *or not* $(-)$*:*

$\Rightarrow$	O, I	A, B, A_φ	W	O_φ	I_φ	B_φ	W_φ
O, I	+	+	+	+	+	+	+
A, B, A_φ	−	+	−	−	−	+	−
W	−	−	+	−	−	−	+
O_φ	−	−	−	+	+	+	+
I_φ	−	−	−	−	+	−	−
B_φ	−	−	−	−	−	+	−
W_φ	−	−	−	−	−	−	+

Note that (0) and (I) as well as (A), (B) and (A_φ) are equivalent.

On establishing the possible connections between the considered single equations we may deal with combinations of them.

$$(A_\varphi) \wedge (W_\varphi) \Rightarrow (O). \tag{9}$$

Under our assumptions (A_φ) is equivalent to (A). We have also $\|f(x)\| = \|x\|$ for $x \in X$. Therefore f satisfies (I) (which is equivalent to (O)).

$$(A_\varphi) \wedge (I_\varphi) \Rightarrow (O). \tag{10}$$

Similarly as above f is additive whence odd. Thus from (I_φ), for arbitrary $x \in X$, we have (by evenness of φ)

$$\|f(x) - f(-x)\| = \|x - (-x)\|$$

which gives $\|f(x)\| = \|x\|$. This, together with (A) implies (I) and, equivalently, (O).

In (9) and (10) we cannot replace (A_φ) by weaker (B_φ):

$$(B_\varphi) \wedge (W_\varphi) \not\Rightarrow (W) \quad \text{and} \quad (B_\varphi) \wedge (I_\varphi) \not\Rightarrow (W). \tag{11}$$

As a counterexample one can take the function f defined by ($*$).

We can prove that

$$(B_\varphi) \wedge (W_\varphi) \Rightarrow (I_\varphi). \tag{12}$$

(B_φ) implies that f is odd. (W_φ) gives $\|f(x)\| = \|x\|$ for all $x \in X$. Therefore, for $x, y \in X$ such that $\varphi(x) = \varphi(y)$ we have

$$\|f(x) - f(y)\| = \|f(x) + f(-y)\| = \|f(x - y)\| = \|x - y\|,$$

i.e., (I_φ) holds.

Now, let us consider a condition similar to (O_φ) (we write it in two equivalent forms):

$$(O'_\varphi) \quad \begin{array}{ll} \langle f(x+y)|f(y)\rangle = \langle x+y|y\rangle & \text{for } x, y \in X \text{ such that } \varphi(x) = \varphi(y), \\ \langle f(x)|f(y)\rangle = \langle x|y\rangle & \text{for } x, y \in X \text{ such that } \varphi(x-y) = \varphi(y). \end{array}$$

The conjunction of (O_φ) and (O'_φ) means that f preserves all the inner products of x, y and $x+y$ for all vectors x, y satisfying the condition $\varphi(x) = \varphi(y)$. Equivalently, in the case where $\varphi(x)$ denotes the norm of x, it means geometrically that the inner product of vectors x and y is preserved provided that the triangle $x, y, x-y$ is isosceles. However, this assumption implies that f preserves the inner product unconditionally. Indeed, we have the following result.

THEOREM 2 *Let X and Y be real inner product spaces and let* $\dim X \geq 2$. *If $f : X \to Y$ satisfies* (O_φ) *and* (O'_φ), *then f preserves the inner product on X, i.e., f satisfies* (O).

PROOF From (O_φ) we have $\|f(x)\| = \|x\|$ for all $x \in X$. Let $\varphi(x) = \varphi(y)$. Then we have

$$\begin{aligned}
&\|f(x+y) - f(x) - f(y)\|^2 \\
&\qquad = \|f(x+y)\|^2 + \|f(x)\|^2 + \|f(y)\|^2 + 2\langle f(x)|f(y)\rangle \\
&\qquad\quad -2\langle f(x+y)|f(x)\rangle - 2\langle f(x+y)|f(y)\rangle \\
&\qquad = \|x\|^2 + \|y\|^2 + 2\langle x|y\rangle + \|x\|^2 + \|y\|^2 + 2\langle x|y\rangle \\
&\qquad\quad -2\langle x+y|x\rangle - 2\langle x+y|y\rangle \\
&\qquad = 0.
\end{aligned}$$

Thus f satisfies (A_φ) which is equivalent to the additivity of f. As an additive function preserving the norm, f preserves also the inner product on the whole X.■

The above theorem fails to hold in the case $\dim X = 1$; it suffices to consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = D(x)x$, $x \in \mathbb{R}$ and $\varphi(x) := |x|,\ x \in \mathbb{R}$.

As we have shown above (O_φ) does not imply (O). Therefore, (O_φ) does not imply (O'_φ). The question arises whether the condition (O'_φ) implies (O_φ) or, equivalently, whether (O'_φ) implies (O). If it were true, (O'_φ) would imply any condition from all considered in the paper.

As we have mentioned, a solution of (W) is phase-equivalent to a solution of (O). We can show an analogous property for conditional equations. However, we need some assumption on $\varphi : X \to Z$. For the final part of the paper, instead of (i)-(iv) we consider the condition:

(v) $\varphi(x) \neq \varphi(0), \qquad x \neq 0;$
$\forall\, r \in \varphi(X \setminus \{0\})\ \forall\, x \in X \setminus \{0\}\ \exists\, \lambda \in (0, \infty) :\ \varphi(\lambda x) = r.$

As an example consider the Minkowski functional

$$\varphi_A(x) := \inf\{\lambda > 0 :\ x \in \lambda A\}, \qquad x \in X$$

for a given nonempty, bounded, convex and radial set $A \subset X$. Although φ_A need not satisfy (i)-(iv) (unless we make additional assumptions on A), it satisfies (v). Indeed, for $x \neq 0$ we have $\varphi_A(x) \neq 0$ and, for any $r > 0$, we define $\lambda := \frac{r}{\varphi_A(x)}$ and we have

$$\varphi_A(\lambda x) = \lambda \varphi_A(x) = r.$$

In particular, (v) holds for φ being an arbitrary norm in X (not necessarily coming from an inner product).

THEOREM 3 *Let X and Y be real inner product spaces. Let Z be an arbitrary non-empty set and $\varphi : X \to Z$ be a given function satisfying the condition* (v). *If $f : X \to Y$ satisfies* (W_φ), *then there exists a function $\sigma : X \to \{-1, 1\}$ and a mapping $I : X \to Y$ satisfying the conditional equation* (O_φ) *such that*

$$f(x) = \sigma(x)I(x) \qquad \textit{for } x \in X.$$

In other words, a solution of (W_φ) is phase-equivalent to a solution of (O_φ).

PROOF Let us establish an arbitrary mapping

$$\varphi(X \setminus \{0\}) \times (X \setminus \{0\}) \ni (r, x) \mapsto \lambda(r, x) \in (0, \infty)$$

such that

$$\varphi(\lambda(r, x)x) = r \quad \text{for } r \in \varphi(X \setminus \{0\}),\ x \in X \setminus \{0\}$$

and

$$\text{if } \ \varphi(x) = r, \ \text{ then } \ \lambda(r, x) = 1.$$

For $r \in \varphi(X \setminus \{0\})$ let us define

$$g_r(x) := \begin{cases} \frac{1}{\lambda(r,x)} f\left(\lambda(r, x)x\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

For each fixed $r \in \varphi(X)$, we have $\varphi(\lambda(r, x)x) = r$ for $x \in X$ whence g_r satisfies (W). Therefore, for each $r \in \varphi(X \setminus \{0\})$, there exist $\sigma_r : X \to \{-1, 1\}$ and $I_r : X \to Y$ such that

$$g_r(x) = \sigma_r(x)I_r(x), \qquad x \in X$$

and

$$\langle I_r(x)|I_r(y)\rangle = \langle x|y\rangle, \qquad x, y \in X.$$

From the definition of the mapping λ we have that if $\varphi(x) = r$, then $f(x) = g_r(x)$. Now, we define

$$\sigma(x) := \sigma_{\varphi(x)}(x) \quad \text{and} \quad I(x) := I_{\varphi(x)}(x) \qquad \text{for } x \in X \setminus \{0\},$$

$$\sigma(0) := 1 \quad \text{and} \quad I(0) := 0.$$

Then we have $f(x) = g_{\varphi(x)}(x) = \sigma_{\varphi(x)}(x)I_{\varphi(x)}(x)$ for $x \neq 0$ and $f(0) = 0 = \sigma(0)I(0)$. Therefore

$$f(x) = \sigma(x)I(x) \qquad \text{for } x \in X,$$

i.e., f is phase-equivalent to I. Now, let us assume $x \neq 0$, $y \neq 0$, $\varphi(x) = \varphi(y)$. Then

$$\langle I(x)|I(y)\rangle = \left\langle I_{\varphi(x)}(x)|I_{\varphi(y)}(y)\right\rangle = \langle x|y\rangle$$

and the same is obviously true if $x = 0$ or $y = 0$. This completes the proof. ∎

Finally, let us remark that for φ satisfying (v) and f being positively homogeneous (O_φ) implies (O). Indeed, using the notation from the above proof, for an arbitrary $r \in \varphi(X \setminus \{0\})$, we have for $x, y \in X \setminus \{0\}$

$$\langle f(x)|f(y)\rangle = \frac{1}{\lambda(r,x)} \cdot \frac{1}{\lambda(r,y)} \langle f(\lambda(r,x)x)|f(\lambda(r,y)y)\rangle = \langle x|y\rangle$$

and obviously for $x = 0$ or $y = 0$.

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