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Satit Saejung*

Some Convexity properties in Musielak-Orlicz sequence spaces endowed with the Luxemburg Norm

Abstract. Criteria for k-strict convexity, uniform convexity in every direction, property K, property H, and property G in Musielak-Orlicz sequence spaces and their subspaces endowed with the Luxemburg norm are presented. In particular, we obtain a characterization of such properties of Nakano sequence spaces.

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1. Introduction. Convexity properties of Banach spaces play essential role in the theory of approximation and optimization. For example, the property of k-strict convexity, introduced by I. Singer, ensures that the dimension of the set $P_M(x)$, the Chebyshev map or the best approximation operator, is not greater than k and vice versa (see [16]).

Now we introduce the basic notions and definitions. A convex function $\varphi : \mathbb{R} \to \mathbb{R}_+ = [0, \infty)$ is called an *Orlicz function* if it vanishes at zero and is convex even on the whole line $\mathbb{R}$ and is not identically equal to zero. Denote by l the space of all real sequences $x = (x(i))$. For a given *Musielak-Orlicz function* $\Phi = (\Phi_i)$, i.e. a sequence $\Phi = (\Phi_i)$ of Orlicz functions, we define a *convex semimodular* $I_\Phi : l \to [0, \infty]$ by the formula

$$I_\Phi(x) = \sum_{i=1}^{\infty} \Phi_i(x(i)).$$

The *Musielak-Orlicz sequence space* l_Φ is the space

$$l_\Phi := \{x \in l : I_\Phi(cx) < \infty \text{ for some } c > 0\}.$$

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We consider l_Φ equipped with the *Luxemburg norm*

$$\|x\| = \inf\{k > 0 : I_\Phi(x/k) \leq 1\}.$$

To simplify notation, we put $l_\Phi := (l_\Phi, \|\cdot\|)$. It is known that l_Φ is a Banach space (see [13]).

The subspace h_Φ, called *the space of finite* (or *order continuous*) *elements*, is defined by

$$h_\Phi := \{x \in l_\Phi : I_\Phi(\lambda x) < \infty \text{ for all } \lambda > 0\}.$$

We say a Musielak-Orlicz function $\Phi = (\Phi_i)$ *satisfies the δ_2-condition* ($\Phi \in \delta_2$) if there exist constants $K \geq 2$, $u_0 > 0$ and a sequence (c_i) of positive numbers such that $\sum_{i=1}^{\infty} c_i < \infty$ and the inequality

$$\Phi_i(2u) \leq K\Phi_i(u) + c_i$$

holds for every $i \in \mathbb{N}$ and $u \in \mathbb{R}$ satisfying $\Phi_i(u) \leq u_0$.

It is well known that $h_\Phi = l_\Phi$ if and only if $\Phi \in \delta_2$ (see [8]).

We also say a Musielak-Orlicz function $\Phi = (\Phi_i)$ *satisfies the condition* $(*)$ if for any $\varepsilon \in (0, 1)$ there exists a $\delta > 0$ such that, for all $i \in \mathbb{N}$ and $u \in \mathbb{R}$, $\Phi_i((1+\delta)u) \leq 1$ whenever $\Phi_i(u) \leq 1 - \varepsilon$ (see [11]).

The paper is organized as follows: In Section 2, we give a criteria for a point in the unit sphere of Musielak-Orlicz sequence space l_Φ and of the subspace h_Φ to be a k-extreme point and then k-strict convexity of l_Φ and of h_Φ are characterized. Uniform convexity in every direction of l_Φ and h_ϕ are equivalent and presented in Section 3. Also, property K, property H, and property G of l_Φ and of h_Φ are characterized in Section 4. Finally, in Section 5, we conclude all corresponding results in the Nakano sequence spaces.

2. k-Strict Convexity. Let X be a Banach space. Denoted by $S(X)$ and $B(X)$ the unit sphere and the unit ball of X, respectively. A point $x \in S(X)$ is called an *extreme point* if for any two elements x_1 and x_2 in $B(X)$ satisfying $x = \frac{x_1+x_2}{2}$ we have that $x_1 = x_2$.

A point $x \in S(X)$ is called a *k-extreme point*, for $k \in \mathbb{N}$, if for any $k+1$ elements $x_1, x_2, \ldots, x_{k+1}$ in $B(X)$ satisfying $x = \frac{x_1+x_2+\cdots+x_{k+1}}{k+1}$ implies that the set $\{x_1, x_2, \ldots, x_{k+1}\}$ is linearly dependent. It is clear from the definition that if the dimension of a Banach space X is less than or equal to k, then every point in $S(X)$ is always a k-extreme point.

Remark 2.1 With the above notation we have

1. A point $x \in S(X)$ is a 1-extreme point if and only if it is an extreme point.
2. If a point x in $S(X)$ is a k-extreme point, then it is also a $(k+1)$-extreme point.

An interval $[a, b]$ is called a *structural affine interval* (SAI) of an Orlicz function φ if φ is affine on $[a, b]$, i.e.,

$$\varphi(\lambda a + (1-\lambda)b) = \lambda\varphi(a) + (1-\lambda)\varphi(b)$$

for all $\lambda \in [0,1]$, but not affine either on $[a-\varepsilon, b]$ or $[a, b+\varepsilon]$ for any $\varepsilon > 0$. Let $\{[a_n, b_n]\}_n$ be the set of all SAIs of φ. Put

$$SC_\varphi = \mathbb{R} \setminus \cup_n (a_n, b_n).$$

Let $a_\varphi = \sup\{u \in \mathbb{R} : \varphi(u) = 0\}$.

THEOREM 2.2 *A point $x = (x(i)) \in S(l_\Phi)$ is a k-extreme point if and only if the following conditions are satisfied:*

(i) $I_\Phi(x) = 1$,

(ii) $\#\{i \in \mathbb{N} : |x(i)| \in [0, a_{\Phi_i})\} \leq k-1$ *and*

(iii) $\#\{i \in \mathbb{N} : x(i) \notin SC_{\Phi_i}\} \leq k$.

In particular, a point $x = (x(i)) \in S(l_\Phi)$ is an extreme point if and only if $I_\Phi(x) = 1$, $\#\{i \in \mathbb{N} : |x(i)| \in [0, a_{\Phi_i})\} = 0$, and $\#\{i \in \mathbb{N} : x(i) \notin SC_{\Phi_i}\} \leq 1$. Here $\#A$ denotes the cardinality of the set A.

PROOF *Necessity.* We note that if $x = (x(i)) \in S(l_\Phi)$ is a k-extreme point, then $(|x(i)|)$ is also a k-extreme point. So we may assume that $x(i) \geq 0$ for all $i \in \mathbb{N}$. Suppose that (i) does not hold, i.e., $I_\Phi(x) < 1$. By the continuity of each Φ_i, there exists $\varepsilon > 0$ such that

$$\Phi_i(x(i) \pm \varepsilon) \leq \Phi_i(x(i)) + \frac{1 - I_\Phi(x)}{2},$$

for all $i = 1, \ldots, k+1$. Without loss of generality, we assume in addition that $x(1) > 0$. We define

$$x_1 = (x(1) + \varepsilon)e_1 + (x(k+1) - \varepsilon)e_{k+1} + \sum_{i \neq 1, k+1} x(i)e_i$$

and, for each $j \in \{2, \ldots, k+1\}$, we also define

$$x_j = (x(j-1) - \varepsilon)e_{j-1} + (x(j) + \varepsilon)e_j + \sum_{i \neq j-1, j} x(i)e_i.$$

Here e_i is the sequence which has the i-th term 1 and all other terms 0. It is easy to see that $x = \frac{x_1 + x_2 + \cdots + x_{k+1}}{k+1}$ and $\{x_1, \ldots, x_{k+1}\} \subset B(l_\Phi)$. To prove that $x_1, \ldots, x_{k+1}$ are linearly independent, let $a_1, \ldots, a_{k+1} \in \mathbb{R}$ be such that $a_1 x_1 + \cdots + a_{k+1} x_{k+1} = 0$. In particular, we have

$$\begin{aligned}
\left(\sum_{i=1}^{k+1} a_i\right) x(1) + (a_1 - a_2)\varepsilon &= 0, \\
\left(\sum_{i=1}^{k+1} a_i\right) x(2) + (a_2 - a_3)\varepsilon &= 0, \\
&\vdots \\
\left(\sum_{i=1}^{k+1} a_i\right) x(k+1) + (a_{k+1} - a_1)\varepsilon &= 0.
\end{aligned}$$

Clearly, $(\sum_{i=1}^{k+1} a_i)(\sum_{i=1}^{k+1} x(i)) = 0$. Knowing that $\sum_{i=1}^{k+1} x(i) \geq x(1) > 0$, thus $\sum_{i=1}^{k+1} a_i = 0$ and this gives $(a_1 - a_2)\varepsilon = (a_2 - a_3)\varepsilon = \cdots = (a_{k+1} - a_1)\varepsilon = 0$. Hence $a_1 = \cdots = a_{k+1} = 0$ and $x_1, \ldots, x_{k+1}$ are proved to be linearly independent. This implies that x is not a k-extreme point, a contradiction.

Suppose (iii) does not hold. Without loss of generality, we assume $x(i) \notin SC_{\Phi_i}$ for all $i \in N$ where $N = \{1, \ldots, k+1\}$. Hence $x(i) \in (b_i, c_i)$ where $[b_i, c_i]$ is a structural affine interval of Φ_i. Let $\Phi_i(u) = \alpha_i u + \beta_i$ for $u \in (b_i, c_i)$ for some constants $\alpha_i \geq 0$ and $\beta_i \in \mathbb{R}$ for each $i \in N$. We note that $\beta_i = 0$ whenever $\alpha_i = 0$. Let $A = \{i \in N : \alpha_i > 0\}$. Consider the following cases:

Case 1: $\#(A) \geq 2$. For simplicity, we assume th7at $A = \{1, \ldots, m\}$ where $2 \leq m \leq k+1$. Choose $\varepsilon_1, \ldots, \varepsilon_{k+1} > 0$ such that

$$\alpha_1\varepsilon_1 = \cdots = \alpha_m\varepsilon_m \quad \text{and} \quad x(i) \pm \varepsilon_i \in (a_i, b_i) \quad \text{for all } i \in N.$$

We define

$$\begin{aligned}
x_1 &= (x(1)+\varepsilon_1)e_1 + (x(m)-\varepsilon_m)e_m + \sum_{i\neq 1,m} x(i)e_i, \\
x_j &= (x(j-1)-\varepsilon_{j-1})e_{j-1} + (x(j)+\varepsilon_j)e_j + \sum_{i\neq j-1,j} x(i)e_i \\
&\qquad \text{for } j = 2, \ldots, m-1, \\
x_m &= (x(m-1)-\varepsilon_{m-1})e_{m-1} + (x(m)+\varepsilon_m)e_m \\
&\qquad +(x(m+1)+\varepsilon_{m+1})e_{m+1} + \sum_{i\neq m-1,m,m+1} x(i)e_i \\
x_j &= (x(j+1)+\varepsilon_{j+1})e_{j+1} + \sum_{i\neq j+1} x(i)e_i \\
&\qquad \text{for } j = m+1, \ldots, k, \text{ and finally,} \\
x_{k+1} &= \sum_{i=m+1}^{k+1} (x(i)-\varepsilon_i)e_i + \sum_{i\neq m+1,\ldots,k+1} x(i)e_i.
\end{aligned}$$

It is easy to see that $x = \frac{x_1+\cdots+x_{k+1}}{k+1}$. Moreover, $I_\Phi(x_1) = \cdots = I_\Phi(x_{k+1}) = 1$. Indeed, by the fact that $\alpha_1\varepsilon_1 = \alpha_m\varepsilon_m$, we have

$$\begin{aligned}
I_\Phi(x_1) &= \Phi_1(x(1)+\varepsilon_1) + \Phi_m(x(m)-\varepsilon_m) + \sum_{i\neq 1,m} \Phi_i(x(i)) \\
&= \alpha_1 x(1) + \alpha_1\varepsilon_1 + \beta_1 + \alpha_m x(m) - \alpha_m\varepsilon_m + \beta_m + \sum_{i\neq 1,m} \Phi_i(x(i)) \\
&= \Phi_1(x(1)) + \Phi_m(x(m)) + \sum_{i\neq 1,m} \Phi_i(x(i)) = I_\Phi(x) = 1.
\end{aligned}$$

Similarly, we also have $I_\Phi(x_j) = 1$ for all $j = 2, \ldots, k+1$. Now we prove that $x_1, \ldots, x_{k+1}$ are linearly independent and as a consequence we obtain a contradiction. Let $a_1, \ldots, a_{k+1} \in \mathbb{R}$ be such that $a_1x_1 + \cdots + a_{k+1}x_{k+1} = 0$. Hence

$$a_1x_1(i) + \cdots + a_{k+1}x_{k+1}(i) = 0$$

for all $i \in \mathbb{N}$. In particular, we have

$$\begin{aligned}\left(\sum_{i=1}^{k+1} a_i\right) x(1) + (a_1 - a_2)\varepsilon_1 &= 0,\\ \left(\sum_{i=1}^{k+1} a_i\right) x(2) + (a_2 - a_3)\varepsilon_2 &= 0,\\ &\vdots\\ \left(\sum_{i=1}^{k+1} a_i\right) x(m) + (a_m - a_1)\varepsilon_m &= 0.\end{aligned}$$

Combining all these we have

$$\left(\sum_{i=1}^{k+1} a_i\right)\left(\frac{x(1)}{\varepsilon_1} + \cdots + \frac{x(m)}{\varepsilon_m}\right) = 0.$$

Since $\frac{x(1)}{\varepsilon_1} + \cdots + \frac{x(m)}{\varepsilon_m} \neq 0$, $\sum_{i=1}^{k+1} a_i = 0$. Therefore $a_1 = \cdots = a_m$. Furthermore, for all $j = m, \ldots, k+1$, we have

$$0 = \left(\sum_{i=1}^{k+1} a_i\right) x(j) + (a_j - a_{k+1})\varepsilon_j = (a_j - a_{k+1})\varepsilon_j.$$

Again we obtain $a_m = \cdots = a_{k+1}$ and so $a_1 = \cdots = a_{k+1} = 0$.

Case 2: $\#(A) = 1$. We assume that $A = \{1\}$. Choose $\varepsilon_2, \ldots, \varepsilon_{k+1} > 0$ such that

$$x(i) \pm \varepsilon_i \in (a_i, b_i) \text{ for } i = 2, \ldots, k+1.$$

We define, for $j = 1, \ldots, k$,

$$\begin{aligned} x_j &= (x(j+1) + \varepsilon_{j+1})e_{j+1} + \sum_{i \neq j+1} x(i)e_i \text{ and}\\ x_{k+1} &= \sum_{j=2}^{k+1}(x(j) - \varepsilon_j)e_j + \sum_{i \neq 2,\ldots,k+1} x(i)e_i.\end{aligned}$$

Clearly, $x = \frac{x_1 + \cdots + x_{k+1}}{k+1}$, and $I_\Phi(x_1) = \cdots = I_\Phi(x_{k+1}) = 1$. To prove the linear independence of $x_1, \ldots, x_{k+1}$, let $a_1, \ldots, a_{k+1} \in \mathbb{R}$ be such that $a_1x_1 + \cdots + a_{k+1}x_{k+1} = 0$. Hence $a_1x(1) + \cdots + a_{k+1}x(1) = 0$. Note that $x(1) > 0$. Otherwise, $x(1) \in SC_{\Phi_1}$ which contradicts to our assumption. Thus, $\sum_{i=1}^{k+1} a_i = 0$ and so

$$0 = \left(\sum_{i=1}^{k+1} a_i\right) x(j+1) + (a_j - a_{k+1})\varepsilon_{j+1} = (a_j - a_{k+1})\varepsilon_{j+1}$$

for all $j = 1, \ldots, k$. Therefore $a_1 = \cdots = a_{k+1} = 0$.

Case 3: $\#(A) = 0$. Since $I_\Phi(x) = 1$, there exists $i_0 \notin N$ such that $\Phi_{i_0}(x(i_0)) > 0$. Let us consider the following subcases.

Subcase 3.1: $x(i_0) \notin SC_{\Phi_{i_0}}$. If we put $A' = (A \setminus \{1\}) \cup \{i_0\}$ and repeat the proof of Case 2, then we obtain a contradiction.

Subcase 3.2: $x(i_0) \in SC_{\Phi_{i_0}}$. Choose $\varepsilon_1, \dots, \varepsilon_{k+1} > 0$ such that

$$x(i) \pm \varepsilon_i \in (b_i, c_i) \text{ for } i = 1, \dots, k+1.$$

Define

$$x_1 = (x(1) + \varepsilon_1)e_1 + (x(k+1) - \varepsilon_{k+1})e_{k+1} + \sum_{i \neq 1, k+1} x(i)e_i$$

and, for each $j \in \{2, \dots, k+1\}$, we also define

$$x_j = (x(j-1) - \varepsilon_{j-1})e_{j-1} + (x(j) + \varepsilon_j)e_j + \sum_{i \neq j-1, j} x(i)e_i.$$

Again, we have $x_1, \dots, x_{k+1} \in S(l_\Phi)$ and $x = \frac{x_1 + \cdots + x_{k+1}}{k+1}$. We now prove the linear independence of $x_1, \dots, x_{k+1}$. If $a_1 x_1 + \cdots + a_{k+1} x_{k+1} = 0$ where $a_1, a_2, \dots, a_{k+1} \in \mathbb{R}$, then $a_1 x_1(i) + \cdots + a_{k+1} x_{k+1}(i) = 0$ for all $i \in \mathbb{N}$. Since $x(i_0) \in SC_{\Phi_{i_0}}$ and $I_\Phi(\frac{x_1 + \cdots + x_{k+1}}{k+1}) = I_\Phi(x) = 1$, we have $x_1(i_0) = \cdots = x_{k+1}(i_0) = x(i_0) > 0$. This implies that $a_1 + a_2 + \cdots + a_{k+1} = 0$. Now, for all $j = 1, \dots, k$,

$$a_j - a_{j+1} = (a_j - a_{j+1})\varepsilon_j + (a_1 + a_2 + \cdots + a_{k+1})x(j) = 0.$$

Hence, we get $a_1 = \cdots = a_{k+1} = 0$.

In all cases we encounter with contradictions since x is a k-extreme point and thus the necessity of (iii) is established.

To prove (ii). Suppose that $x(i) \in [0, a_{\Phi_i})$ for all $i = 1, \dots, k$. Choose $\varepsilon > 0$ so that $x(i) \pm \varepsilon \in (-a_{\Phi_i}, a_{\Phi_i})$ for all $i = 1, \dots, k$. For $j = 1, \dots, k$, we define

$$x_j = (x(j) - \varepsilon)e_j + \sum_{i \neq j} x(i)e_i$$

and

$$x_{k+1} = \sum_{i=1}^{k} (x(i) + \varepsilon)e_i + \sum_{i=k+1}^{\infty} x(i)e_i.$$

Obviously $x = \frac{x_1 + x_2 + \cdots + x_{k+1}}{k+1}$ and $\{x_1, \dots, x_{k+1}\} \subset S(l_\Phi)$. Now we prove the linear independence of these elements. If $a_1 x_1 + \cdots + a_{k+1} x_{k+1} = 0$, then $a_1 x_1(i) + \cdots + a_{k+1} x_{k+1}(i) = 0$ for all $i \in \mathbb{N}$. Since $I_\Phi(x) = 1$, there exists an index $i_0 \geq k$ such that $\Phi_{i_0}(x(i_0)) > 0$. This implies that $x(i_0) \neq 0$. It follows from (iii) that $x(i_0) \in SC_{\Phi_{i_0}}$, and then that $x_1(i_0) = \cdots = x_{k+1}(i_0) = x(i_0) > 0$. Hence $a_1 + \cdots + a_{k+1} = 0$. Moreover, we have

$$\begin{aligned} 0 &= a_1 x(1) - a_1 \varepsilon + a_2 x(1) + \cdots + a_i x(1) + a_{k+1} x(1) - a_{k+1} \varepsilon \\ &= a_1 x(1) + a_2 x(1) + \cdots + a_k x(1) + a_{k+1} x(1) + a_{k+1} \varepsilon - a_1 \varepsilon \\ &= a_{k+1} \varepsilon - a_1 \varepsilon. \end{aligned}$$

This gives $a_1 = a_{k+1}$. Similarly, we have $a_j = a_{k+1}$ for $j = 2, \dots, k$. Hence $a_1 = \cdots = a_{k+1} = 0$. Therefore x cannot be a k-extreme point.

Sufficiency. Let $x \in S(l_\Phi)$ be such that the conditions (i)-(iii) hold. Take elements $x_1, \ldots, x_{k+1}$ in the unit sphere of l_Φ with

$$x = \frac{x_1 + x_2 + \cdots + x_{k+1}}{k+1}.$$

By the condition (i) and the convexity of the modular, we obtain $I_\Phi(x_1) = \cdots = I_\Phi(x_{k+1}) = 1$. Furthermore, for each $i \in \mathbb{N}$, $\{x_j(i) : j = 1, \ldots, k+1\}$ is either a singleton or a set contained in the same SAI of Φ_i. To prove that $\{x_1, \ldots, x_{k+1}\}$ is linearly dependent, we shall find $a_1, \ldots, a_{k+1} \in \mathbb{R}$ such that $a_1x_1 + \cdots + a_{k+1}x_{k+1} = 0$ where a_i's are not all zero. It follows by the condition (iii) that, for all but k coordinates, $\{x_j(i) : j = 1, \ldots, k+1\}$ is a singleton. For the sake of convenience we assume that $\{x_j(i) : j = 1, \ldots, k+1\}$ is a singleton for all $i \geq k+1$. Then

$$(\star) \qquad I_\Phi\left(\sum_{i=1}^{k} x_1(i)e_i\right) = \cdots = I_\Phi\left(\sum_{i=1}^{k} x_{k+1}(i)e_i\right).$$

We also assume in the worst case that $\{i \in \mathbb{N} : x(i) \notin SC_{\Phi_i}\} = \{1, \ldots, k\}$. Let $\{i \in \mathbb{N} : |x(i)| \in [0, a_{\Phi_i})\} = \{1, \ldots, m\}$ where $m \leq k-1$ and let $K = \{1, \ldots, k\} \setminus \{1, \ldots, m\}$. If $x(i) = 0$ for all $i \geq k+1$, the following system of equations

$$\begin{array}{rcl} a_1x_1(1) + a_2x_2(1) + \cdots + a_{k+1}x_{k+1}(1) & = & 0, \\ a_1x_1(2) + a_2x_2(2) + \cdots + a_{k+1}x_{k+1}(2) & = & 0, \\ & \vdots & \\ a_1x_1(k) + a_2x_2(k) + \cdots + a_{k+1}x_{k+1}(k) & = & 0. \end{array}$$

always has a nontrivial solution. On the other hand, if there exists a coordinate $i \geq k+1$ such that $x(i) \neq 0$, then

$$a_1 + a_2 + \cdots + a_{k+1} = 0.$$

Consider the matrix

$$\begin{bmatrix} x_1(1) & x_2(1) & \cdots & x_{k+1}(1) \\ x_1(2) & x_2(2) & \cdots & x_{k+1}(2) \\ \vdots & \vdots & \ddots & \vdots \\ x_1(k) & x_2(k) & \cdots & x_{k+1}(k) \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

For $k \in K$, let $\Phi_k(u) = \alpha_k u + \beta_k$ when $u \in [b_k, c_k]$, where $[b_k, c_k]$ is a structural affine interval of Φ_k containing $x(k)$, $\alpha_k > 0$ and $\beta_k \in \mathbb{R}$. By $(\star)$, we have

$$\sum_{k \in K} (\alpha_k x_1(k) + \beta_k) = \sum_{k \in K} (\alpha_k x_2(k) + \beta_k) = \cdots = \sum_{k \in K} (\alpha_k x_{k+1}(k) + \beta_k).$$

This implies that the above matrix is equivalent to this following matrix

$$\begin{bmatrix} x_1(1) & x_2(1) & \cdots & x_{k+1}(1) \\ x_1(2) & x_2(2) & \cdots & x_{k+1}(2) \\ \vdots & \vdots & \ddots & \vdots \\ x_1(m) & x_2(m) & \cdots & x_{k+1}(m) \\ \Phi_{m+1}(x_1(m+1)) & \Phi_{m+1}(x_2(m+1)) & \cdots & \Phi_{m+1}(x_{k+1}(m+1)) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_k(x_1(k)) & \Phi_k(x_2(k)) & \cdots & \Phi_k(x_{k+1}(k)) \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then there exists a nontrivial solution $\{a_i : i = 1, \ldots, k+1\}$ for the above system. This implies the linear dependence of $\{x_1, \ldots, x_{k+1}\}$. ■

A Banach space X is said to be *k-strictly convex* if each point in its unit sphere is a k-extreme point (see [16]). Also, a Banach space X is said to be *strictly convex* if each point in its unit sphere is an extreme point.

Let $\sigma_i = \sup\{u \geq 0 : \Phi_i \text{ is strictly convex on } [0,u] \text{ and } \Phi_i(u) \leq 1\}$. By the previous theorem, we obtain the following characterizations.

Corollary 2.3 *The Musielak-Orlicz sequence space l_Φ is k-strictly convex if and only if the following conditions are satisfied*

1. $\Phi = (\Phi_i)$ *satisfies the δ_2-condition,*
2. *each Φ_i vanishes only at zero for all but $k-1$ indices i's and*
3. $\Phi_{i_1}(\sigma_{i_1}) + \Phi_{i_2}(\sigma_{i_2}) + \cdots + \Phi_{i_{k+1}}(\sigma_{i_{k+1}}) \geq 1$ *for all $k+1$ distinct indices* $i_1, i_2, \ldots, i_{k+1}$.

In particular, h_Φ is k-strictly convex if and only if the conditions (2) and (3) are satisfied.

3. Uniform Convexity in Every Direction. A Banach space X is said to be *uniformly convex in every direction* (UCED) if for each $\varepsilon \in (0,2]$ and any $z \in S(X)$ there is a $\delta = \delta(\varepsilon, z) > 0$ such that for any $x, y \in S(X)$ with $x - y = \alpha z$ for some scalar α, the condition $\|x - y\| \geq \varepsilon$ implies that $\|x + y\| \leq 2(1-\delta)$. Equivalently, if $x_n, z \in X$, $\|x_n\|, \|x_n + z\| \to 1$ and $\|2x_n + z\| \to 2$ imply $z = 0$ (see [17, Proposition 1, page 7]). This property was introduced by A. L. Garkavi in 1962 (see [6]). Moreover, he showed that every UCED space has weak normal structure and hence enjoys the fixed point property (see also [1, 7]).

It is easy to see that every UCED space is strictly convex.

Lemma 3.1 ([9]) *Let $v_i \in \mathbb{R}$, $i = 1, \ldots, 4$ and $v_1 < v_2 < v_3 < v_4$. If φ is strictly convex on $[v_2, v_3]$, then there exists $p \in (0,1)$ such that*

$$\varphi\left(\frac{u+v}{2}\right) \leq \frac{1-p}{2}(\varphi(u) + \varphi(v))$$

for all $u \in [v_1, v_2]$ and $v \in [v_3, v_4]$.

Lemma 3.2 ([10]) *Let φ be strictly convex on $[-a,a]$. Then for each $\varepsilon > 0$, $d_1, d_2 \in (0,a]$, $d_1 < d_2$, there exists $p \in (0,1)$ such that*

$$\varphi\left(\frac{u+v}{2}\right) \le \frac{1-p}{2}(\varphi(u)+\varphi(v))$$

if all $|u-v| \ge \varepsilon \max(|u|,|v|)$ and $\max(|u|,|v|) \in [d_1,d_2]$.

Lemma 3.3 ([11]) *If the Musielak-Orlicz function Φ satisfies the δ_2-condition and the condition $(*)$, then for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x\| \le 1-\delta$ whenever $I_\Phi(x) \le 1-\varepsilon$.*

Theorem 3.4 *The following statements are equivalent:*

1. *l_Φ is* UCED*;*
2. *h_Φ is* UCED*;*
3. *the following conditions are satisfied:*
   (a) *Φ satisfies the δ_2-condition and the condition $(*)$,*
   (b) *each Φ_i vanishes only at zero,*
   (c) *$\Phi_i(\sigma_i) + \Phi_j(\sigma_j) \ge 1$ for all $i \ne j$.*

Proof (1)⇒(2) is trivial. To prove (2)⇒(3), it suffices to prove only the necessity of (a). Because (2) implies that h_Φ is strictly convex. Suppose first that $\Phi \notin \delta_2$, then there exists $x = (x(i)) \in S(l_\Phi)$ such that $I_\Phi(x) \le \varepsilon_0 < 1$ and $I_\Phi(\lambda x) = \infty$ for all $\lambda > 1$. We can find a strictly increasing sequence $0 = i_1 < i_2 < \cdots < i_n < i_{n+1} < \cdots$ of nonnegative integers so that

$$I_\Phi\left(\frac{n+1}{n}\sum_{i=i_n+1}^{i_{n+1}} x(i)e_i\right) > 1$$

for all $n \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$,

$$1 \ge \left\|\sum_{i=i_n+1}^{i_{n+1}} x(i)e_i\right\| > \frac{n}{n+1}.$$

Define $x_n = \sum_{i=i_n+1}^{i_{n+1}} x(i)e_i$. Then $\|x_n\| \to 1$. We may assume that $x(1) \ne 0$. Put $z = x(1)e_1 \ne 0$. Then we have $\|x_n + z\| \to 1$ and $\|2x_n + z\| \to 2$. This is a contradiction.

We next prove that Φ satisfies the condition $(*)$. For an arbitrary $\varepsilon \in (0,1)$ and $i \ne 1$, let $u \in \mathbb{R}$ be such that $\Phi_i(u) \le 1-\varepsilon$. Put $z = 2ae_1$ where $\Phi_1(a) = \varepsilon$. Since h_Φ is UCED, there exists $\delta' > 0$ such that $\|x + \frac{z}{2}\| \le 1-\delta'$ for any $x \in h_\Phi$ with $\|x\|, \|x+z\| \le 1$. If we put $x = ue_n - ae_1$, then $\|x\| \le 1$, $\|x+z\| = 1$. Hence $\|ue_n\| = \|x + \frac{z}{2}\| \le 1-\delta'$. This implies that $\Phi_i\left(\frac{u}{1-\delta'}\right) \le 1$ for all $i \ne 1$. By the continuity of Φ_1, if $\Phi_1(u) \le 1-\epsilon$, there exists $\delta'' > 0$ such that $\Phi_1((1+\delta'')u) \le 1$. Put $\delta = \min\left\{\frac{\delta'}{1-\delta'}, \delta''\right\}$. Then the necessity of the condition $(*)$ is proved.

(3)$\Rightarrow$(1) Let $z = (z(i)) \in l_\Phi$ be a nonzero element. Consider the set

$$A = \{x = (x(i)) : I_\Phi(x) = 1 \text{ and } I_\Phi(x+z) \le 1\}.$$

We first consider these two following cases:

I : There exists an index k such that $\sigma_k > 0$, $z(k) \neq 0$, $|x(k)| \le \sigma_k$ and $|x(k) + z(k)| \le \sigma_k$

II : There exist an index k and numbers $t_1, t_2 \in (-\Phi_k^{-1}(1), \Phi_k^{-1}(1))$, $t_1 < t_2$, $\sigma_k > 0$ such that

(i) Φ_k is strictly convex on $[t_1, t_2]$,

(ii) $x(k) \le t_1$ and $x(k) + z(k) \ge t_2$ or $x(k) + z(k) \le t_1$ and $x(k) \ge t_2$, and

(iii) $\Phi_k(x(k)) \ge \Phi_k(\sigma_k)$ or $\Phi_k(x(k) + z(k)) \ge \Phi_k(\sigma_k)$.

We will estimate the value of $I_\Phi\left(x + \frac{z}{2}\right)$.

I : Let $n \in \mathbb{N}$ be such that $\Phi_k(z(k)) \le n$. Then

$$|z(k)| \ge \frac{\Phi_k(z(k))}{n} \max(|x(k)|, |x(k) + z(k)|).$$

Otherwise, since Φ_k vanishes only at zero,

$$\begin{aligned}
\Phi_k(z(k)) &< \frac{\Phi_k(z(k))}{n} \max(\Phi_k(x(k)), \Phi_k(x(k) + z(k))) \\
&\le \frac{\Phi_k(z(k))}{n} \Phi_k(\sigma_k) \\
&\le \frac{\Phi_k(z(k))}{n}
\end{aligned}$$

which is impossible. Moreover we also have

$$\frac{|z(k)|}{2} \le \max(|x(k)|, |x(k) + z(k)|) \le \sigma_k.$$

Now we apply Lemma 3.2 with $\frac{\Phi_k(z(k))}{n}$, $\frac{|z(k)|}{2}$, σ_k in place of ε, d_1, d_2, respectively. Then there exists $p_k \in (0,1)$ such that

$$\Phi_k\left(x(k) + \frac{z(k)}{2}\right) \le \frac{1-p_k}{2}(\Phi_k(x(k)) + \Phi_k(x(k) + z(k))).$$

This implies

$$\begin{aligned}
I_\Phi\left(x + \frac{z}{2}\right) &\le 1 - \frac{p_k}{2}(\Phi_k(x(k)) + \Phi_k(x(k) + z(k))) \\
&\le 1 - \frac{p_k}{2} \max(\Phi_k(x(k)), \Phi_k(x(k) + z(k))) \\
&\le 1 - \frac{p_k}{2} \Phi_k\left(\frac{z(k)}{2}\right).
\end{aligned}$$

II : Applying Lemma 3.1 with $-\Phi_k^{-1}(1), t_1, t_2, \Phi_k^{-1}(1)$ in place of v_i, respectively, we obtain $p_k \in (0,1)$ such that

$$\Phi_k\left(x(k)+\frac{z(k)}{2}\right) \leq \frac{1-p_k}{2}(\Phi_k(x(k))+\Phi_k(x(k)+z(k))).$$

This implies

$$\begin{aligned} I_\Phi\left(x+\frac{z}{2}\right) &\leq 1-\frac{p_k}{2}(\Phi_k(x(k))+\Phi_k(x(k)+z(k))) \\ &\leq 1-\frac{p_k}{2}\Phi_k(\sigma_k). \end{aligned}$$

Without loss of generality, we assume that

$$\Phi_1(z(1)) = \max\{\Phi_i(z(i)) : i \in \mathbb{N}\},$$

$$\Phi_2(z(2)) = \max\{\Phi_i(z(i)) : i \in \mathbb{N}, i \neq 1\},$$

and define the following sets

$$\begin{aligned} A_1 &= \{x \in A : |x(1)| \leq \sigma_1 \text{ and } |x(1)+z(1)| \leq \sigma_1\}, \\ A_2 &= \{x \in A : |x(1)| > \sigma_1 \text{ and } |x(1)+z(1)| > \sigma_1\}, \\ A_3 &= \{x \in A : |x(1)| \leq \sigma_1 \text{ and } |x(1)+z(1)| > \sigma_1\}, \text{ and} \\ A_4 &= \{x \in A : |x(1)| > \sigma_1 \text{ and } |x(1)+z(1)| \leq \sigma_1\}. \end{aligned}$$

It is evident that $A = \cup_{i=1}^4 A_i$.

We note that if $z(i) = 0$ for all $i \geq 2$, then

$$\begin{aligned} I_\Phi\left(x+\frac{z}{2}\right) &= \Phi_1\left(x(1)+\frac{z(1)}{2}\right)+\sum_{i=2}^{\infty}\Phi_i(x(i)) \\ &= \Phi_1\left(x(1)+\frac{z(1)}{2}\right)+1-\Phi_1(x(1)) \\ &\leq 1-\Phi_1\left(\frac{z(1)}{2}\right). \end{aligned}$$

■

From now on, we may assume that $z(2) \neq 0$.

First, if $\sigma_1 = 0$, then by (c) we have $\Phi_i(\sigma_i) = 1$ for all $i \geq 2$. We apply I with $k = 2$.

Secondly, if $\Phi_1(\sigma_1) = 1$, then we shall apply the case I with $k = 1$.

We now assume that $\sigma_i > 0$ for all $i \in \mathbb{N}$. In the virtue of I it is enough to consider only the sets A_2, A_3 and A_4.

Suppose that the numbers $x(1)$ and $x(1)+z(1)$ are of the different sign. Then, for such x from $A_2 \cup A_3 \cup A_4$, it falls in the case II when $k = 1$ by putting $t_1, t_2 \in \{\pm\sigma_1, 0\}$.

Now assume that $x(1)$ and $x(1)+z(1)$ are of the same sign. If $x \in A_2$ then $\Phi_2(x(2)) \leq \Phi_2(\sigma_2)$ and $\Phi_2(x(2)+z(2)) \leq \Phi_2(\sigma_2)$ since $\Phi_1(\sigma_1)+\Phi_2(\sigma_2) \geq 1$. Therefore, case I is applicable for $k = 2$. Now let $x \in A_3$. Note that the signs of $x(1)$ and $z(1)$ must be the same. Let $m \in \mathbb{N}$ such that $\sigma_1 - \frac{|z(1)|}{m} > 0$ and let

$$B_3 = \left\{x \in A_3 : |x(1)| \leq \sigma_1 - \frac{|z(1)|}{m}\right\}.$$

Putting t_1, t_2 as $\pm(\sigma_1 - \frac{|z(1)|}{m})$, $\pm\sigma_1$, so elements in B_3 satisfy the assumption of II. Denoted by $\widetilde{B_3}$ the complement of B_3 in A_3 i.e.

$$\widetilde{B_3} = \left\{x \in A_3 : |x(1)| > \sigma_1 - \frac{|z(1)|}{m}\right\}.$$

Then $|x(1) + z(1)| = |x(1)| + |z(1)| > \sigma_1 - \frac{|z(1)|}{m} + |z(1)| = \sigma_1 + \frac{(m-1)|z(1)|}{m}$. Therefore, $\Phi_1(x(1) + z(1)) > \Phi_1\left(\sigma_1 + \frac{(m-1)|z(1)|}{m}\right)$, which implies $\Phi_2(x(2) + z(2)) \le 1 - \Phi_1\left(\sigma_1 + \frac{(m-1)|z(1)|}{m}\right) < \Phi_2(\sigma_2)$. If $|x(2)| \le \sigma_2$ then we are in case I with $k = 2$. If $|x(2)| > \sigma_2$ then we are in case II for $k = 2$ with t_1, t_2 are chosen respectively from

$$\pm\left|\Phi_2^{-1}\left(1 - \Phi_1\left(\sigma_1 + \frac{(m-1)|z(1)|}{m}\right)\right)\right|, \pm\sigma_2.$$

For $x \in A_4$ we also make analogous considerations. Note that $x(1)$ and $z(1)$ must have the different signs. However $x(1)$ and $x(1) + z(1)$ have the same sign. Thus $|x(1)| > |z(1)|$ and $|x(1) + z(1)| = |x(1)| - |z(1)|$. Let

$$B_4 = \left\{x \in A_4 : |x(1) + z(1)| \le \sigma_1 - \frac{|z(1)|}{m}\right\}.$$

If $x \in B_4$ then the conditions of Case II are satisfied. Put

$$\widetilde{B_4} = \left\{x \in A_4 : |x(1) + z(1)| > \sigma_1 - \frac{|z(1)|}{m}\right\}.$$

Therefore $|x(1)| - |z(1)| = |x(1) + z(1)| > \sigma_1 - \frac{|z(1)|}{m}$ which implies $|x(1)| > \sigma_1 + \frac{(m-1)|z(1)|}{m}$. Hence $\Phi_2(x(2)) \le 1 - \Phi_1\left(\sigma_1 + \frac{(m-1)|z(1)|}{m}\right) < \Phi_2(\sigma_2)$. If $|x(2) + z(2)| \le \sigma_2$ then we are in case I with $k = 2$. If $|x(2) + z(2)| > \sigma_2$, then we are in case II for $k = 2$ with t_1, t_2 are chosen respectively from

$$\pm\left|\Phi_2^{-1}\left(1 - \Phi_1\left(\sigma_1 + \frac{(m-1)|z(1)|}{m}\right)\right)\right|, \pm\sigma_2.$$

Thus, for all $x \in A$, we have that $I_\Phi\left(x + \frac{z}{2}\right) \le \frac{1-p}{2}$ for some $p > 0$. The number p depends only on z. Indeed, p depends on the numbers

$$\pm\sigma_1, \pm\sigma_2, 0, \pm\Phi_2^{-1}\left(1 - \Phi_1\left(\sigma_1 + \frac{(m-1)|z(1)|}{m}\right)\right),$$

and

$$\pm\left(\sigma_1 - \frac{|z(1)|}{m}\right).$$

Hence by the condition $(*)$ and the δ_2-condition there exists $\delta > 0$ such that $\left\|x + \frac{z}{2}\right\| \le 1 - \delta$ for all $x \in S(l_\Phi)$ with $\|x + z\| \le 1$. The proof is now complete.

4. Property K, Property H and Property G. A point $x \in S(X)$ is called an *H-point* if $x_n \to x$ whenever $(x_n) \subset X$ such that $\|x_n\| \to 1$ and $x_n \xrightarrow{w} x$. A point $x \in S(X)$ is called a *PC-point* if the identity map $id : B(X) \to B(X)$ is weak-to-norm continuous at x. Equivalently, for any $\varepsilon > 0$ there exist $\delta > 0$ and finitely many linear functionals $x_1^*, x_2^*, \ldots, x_n^* \in X^*$ such that

$$\|y - x\| < \varepsilon$$

whenever $\|y\| \leq 1$ and $|x_i^*(y - x)| < \delta$ for all $i = 1, 2, \ldots, n$.

It is easy to see that every PC-point is an H-point. Moreover, if X is reflexive, both notions are the same ([1]).

Lemma 4.1 ([11]) *If the Musielak-Orlicz function $\Phi = (\Phi_i)$ satisfies the δ_2-condition and the condition $(*)$ and each Φ_i vanishes only at zero, then for each $\varepsilon > 0$, there exist $\delta > 0$ such that $|I_\Phi(x) - I_\Phi(y)| < \varepsilon$ whenever $I_\Phi(x) \leq 1$, $I_\Phi(y) \leq 1$ and $I_\Phi(x - y) \leq \delta$.*

Lemma 4.2 ([15]) *If Φ does not satisfy the δ_2-condition, then $S(l_\Phi)$ contains no H-points.*

Theorem 4.3 *Suppose that a Musielak-Orlicz function Φ satisfies the condition $(*)$ and each Φ_i vanishes only at zero. Then the following statements are equivalent:*

1. *$x \in S(l_\Phi)$ is a PC-point;*
2. *x is an H-point;*
3. *$\Phi \in \delta_2$.*

Proof (1)⇒(2) is obvious. See [15] for a proof of the implication (2)⇒(3).

(3)⇒(1) Suppose $\Phi \in \delta_2$. Given $\varepsilon > 0$. There exists $\delta \in (0, \varepsilon)$ such that

$$\|y\| < \frac{\varepsilon}{2} \text{ whenever } I_\Phi(y) \leq 2\delta$$

and there exists $\delta' \in (0, \delta)$ such that

$$|I_\Phi(y) - I_\Phi(z)| < \delta \text{ whenever } I_\Phi(y - z) \leq \delta', I_\Phi(y) \leq 1, I_\Phi(z) \leq 1.$$

Choose $i_0 \in \mathbb{N}$ so that $\sum_{i=i_0+1}^{\infty} \Phi_i(x(i)) < \delta$. Note that

$$\alpha := \min_{i=1,\ldots,i_0} \Phi_i^{-1}(\frac{\delta'}{i_0}) > 0.$$

Put $A_\delta = \{y \in B(l_\Phi) : |\langle y - x, e_i\rangle| = |y(i) - x(i)| < \alpha \text{ for all } i = 1, \ldots, i_0\}$. For any $y \in A_\delta$, we have

$$\sum_{i=1}^{i_0} \Phi_i(y(i) - x(i)) < \sum_{i=1}^{i_0} \Phi_i(\alpha) \leq \delta'.$$

Moreover, for $y \in A_\delta$, we also have

$$\begin{aligned}\sum_{i=i_0+1}^{\infty} \Phi_i(y(i)) &\le 1 - \sum_{i=1}^{i_0} \Phi_i(y(i)) \\ &= \sum_{i=1}^{i_0} \Phi_i(x(i)) - \sum_{i=1}^{i_0} \Phi_i(y(i)) + \sum_{i=i_0+1}^{\infty} \Phi_i(x(i)) \le 2\delta.\end{aligned}$$

These yield

$$\begin{aligned}I_\Phi\big(\frac{y-x}{2}\big) &\le \sum_{i=1}^{i_0} \Phi_i\left(\frac{y(i)-x(i)}{2}\right) + \frac{1}{2}\left(\sum_{i=i_0+1}^{\infty} \Phi_i(y(i)) + \Phi_i(x(i))\right) \\ &\le 2\delta.\end{aligned}$$

■

Hence $\|y - x\| < \varepsilon$, i.e. $A_\delta \subset x + \varepsilon B(l_\Phi)$. Therefore x is a PC-point.

A Banach space X is said to have *property H (property K, resp.)* if each point in its unit sphere is an H-point (PC-point, resp.). Sometimes, the property H is also called the *Randon-Riesz property* or the *Kadets-Klee property*. See [12] for references concerning the history and related results.

COROLLARY 4.4 *Suppose that a Musielak-Orlicz function Φ satisfies the condition $(*)$ and each Φ_i vanishes only at zero. Then the following statements are equivalent:*

1. *l_Φ has property K;*
2. *h_Φ has property K;*
3. *l_Φ has property H;*
4. *h_Φ has property H;*
5. *$\Phi \in \delta_2$.*

A point $x \in S(X)$ is called a *denting point* if for any $\varepsilon > 0$, $x \notin \overline{\text{co}}\{B(X)\backslash(x + \varepsilon B(X))\}$. Recall that $\overline{\text{co}}(A)$ denotes the closed convex hull of A. If each point in $S(X)$ is a denting point, we say that X has *property G*. The reader who is interested in a discussion of the relevance of denting points in connection with the Randon-Nikodym property is referred to the monographs [2] and [4].

Recently, B.-L. Lin, et al. ([14]) proved that $x \in S(X)$ is a denting point if and only if it is a PC-point and an extreme point (see [14]). This gives the following characterizations:

THEOREM 4.5 *Suppose that a Musielak-Orlicz function Φ satisfies the condition $(*)$ and each Φ_i vanishes only at zero. Then $x = (x(i)) \in S(l_\Phi)$ is a denting point if and only if $\Phi \in \delta_2$ and $\#\{i \in \mathbb{N} : x(i) \notin SC_{\Phi_i}\} \le 1$.*

In particular, the following statements are equivalent:

1. *l_Φ has property G;*

2. *h_Φ has property G;*

3. *l_Φ is strictly convex.*

5. Convexity Properties in Nakano Sequence Spaces. In this section, we give the characterizations of properties in the previous sections for Nakano sequence spaces. Recall that a *Nakano sequence space* $l^{\{p_i\}}$ is a Musielak-Orlicz sequence space with

$$\Phi_i(u) = |u|^{p_i}$$

where $1 \leq p_i < \infty$.

THEOREM 5.1 *For the Nakano sequence space $l^{\{p_i\}}$, we have*

1. ([5, Theorem 3]) *$l^{\{p_i\}}$ is k-strictly convex if and only if $\#\{i \in \mathbb{N} : p_i = 1\} \leq k$ and $\limsup_{i\to\infty} p_i < \infty$,*

2. ([5, Theorem 22 and Final remark]) *$l^{\{p_i\}}$ is* UCED *if and only if $l^{\{p_i\}}$ has property G; if and only if $\#\{i \in \mathbb{N} : p_i = 1\} \leq 1$ and $\limsup_{i\to\infty} p_i < \infty$, and*

3. ([5, Theorem 6 and Final remark]) *$l^{\{p_i\}}$ has property K if and only if $l^{\{p_i\}}$ has property H; if and only if $\limsup_{i\to\infty} p_i < \infty$.*

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References

[1] A. G. Aksoy and M. A. Khamsi, *Nonstandard methods in fixed point theory*, Springer-Verlag, New York 1990.

[2] R. D. Bourgin, *Geometric aspects of convex sets with the Radon-Nikodým property*, Lecture Notes in Math. No. **993**, Springer-Verlag, Berlin 1983.

[3] S. Chen, *Geometry of Orlicz spaces*, Dissertationes Math.(Rozprawy Matematyczne) **356**, 1996.

[4] J. Diestel and J. J. Uhl, *Vector measures*, Mathematical Surveys, No. **15**, American Mathematical Society, Providence, R.I. 1977.

[5] S. Dhompongsa, *Convexity properties of Nakano spaces*, Science Asia **26** (2000), 21-31.

[6] A. L. Garkavi, *On the optimal net and best cross-section of a set in a normed space.* (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **26** (1962) 87–106.

[7] K. Goebel and W. A. Kirk, *Topics in metric fixed point theory*, Cambridge Studies in Advanced Mathematics, 28. Cambridge University Press, Cambridge 1990.

[8] A. Kamińska, *Rotundity of sequence Musielak-Orlicz spaces*, Bull. Polish Acad. Sci. Math. **29** (1981), 137-144.

[9] A. Kamińska, *On uniform convexity of Orlicz spaces*, Nederl. Akad. Wetensch. Indag. Math. **44** (1982), 27-36.

[10] A. Kamińska, *The criteria for local uniform rotundity of Orlicz spaces*, Studia Math., **79** (1984), 201-215.

[11] A. Kamińska, *Uniform rotundity of Musielak-Orlicz sequene spaces*, J. Approx. Theory, **47** (1986), 302-322.

[12] R. E. Megginson, *An intorduction to Banach space theory*, Graduate Texts in Mathematics, **183**, Springer-Verlag, New York 1998.

[13] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Math. No. textbf1034, Springer-Verlag, Berlin 1983.

[14] B.-L. Lin, P.-K. Lin and S. L. Troyanski, *Characterizations of denting points*, Proc. Amer. Math. Soc., **102** (1988), 526-528.

[15] S. Saejung and S. Dhompongsa, *Extreme points in Musielak-Orlicz sequence spaces*, Acta Math. Vietnamica, **7:2** (2002), 219-229.

[16] I. Singer, *On the set of the best approximations of an element in a normed linear space*, Rev. Math. Pures Appl. **5** (1960), 383-402.

[17] V. Zizler, *On some rotundity and smoothness properties of Banach spaces*, Dissertationes Math.(Rozprawy Matematyczne) **87** (1971).

Satit Saejung

Department of Mathematics, Faculty of Science, Khon Kaen University
Khon Kaen, 40002, Thailand
E-mail: satitz@yahoo.com

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