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On the first Fourier problem for a linear parabolic equation without compatibility condition

Abstract. We consider the first Fourier problem for a linear parabolic equation in a bounded domain. We obtain some theorems on estimates and existence and uniqueness of solution to this problem. These theorems do not require a certain compatibility condition between the initial and boundary values.

1. Introduction. Let $D \subset \mathbf{R}^n \times (0, T)$ (T being a positive constant) be a bounded domain whose boundary consists of sets

$$B = B' \times \{0\} \quad \text{and} \quad B_T = B'_T \times \{T\}$$

(B' and B'_T being domains in $\mathbf{R}^n$), and of a manifold $S \subset \mathbf{R}^n \times [0, T]$.

We consider the following first Fourier problem for the linear parabolic equation

$$(Lu)(x, t) = f(x, t), \quad \forall (x, t) \in D \cup B_T, \tag{1.1}$$

$$u(x, t) = \varphi(x, t), \quad \forall (x, t) \in \Gamma = B \cup S, \tag{1.2}$$

where

$$(Lu)(x, t) = \sum_{i,j=1}^{n} a_{ij}(x, t) u_{x_i x_j}(x, t) + \sum_{i=1}^{n} b_i(x, t) u_{x_i}(x, t) + c(x, t) u(x, t) - u_t(x, t).$$

In [1] a certain boundary estimate of $(1+\delta)$-type for the solution to the above problem is derived which requires the compatibility condition

$$(L\varphi)(x, 0) = f(x, 0), \quad \forall (x, 0) \in \partial B = \partial B' \times \{0\}, \tag{1.3}$$

where $\partial B'$ is the boundary of B'. We show that this compatibility condition is superfluous. Next we obtain certain existence and uniqueness results and estimates for problem (1.1), (1.2) which do not require condition (1.3). The fact that the results of this paper do not need (1.3) is very important for their application in solving the first Fourier problem for semilinear parabolic equations.

The results similar to ours were formulated in [5] (Theorem 0.1) for problem (1.1), (1.2) in the particular case

$$b_i = 0, \quad c = 0, \quad D \text{ is a cylindrical domain.} \tag{1.4}$$

Note that the results of [5] do not need condition (1.3) either. However, our results for the case (1.4) are better than those in [5].

2. The boundary estimate of $(1+\delta)$**-type.** We use the following notation of [1]: $d(P, Q)$, $|u|^D_0$, $\overline{|u|}^D_\alpha$, $\bar{C}_\alpha(D)$, $\overline{|u|}^D_{2+\alpha}$, $\bar{C}_{2+\alpha}(D)$ (Sec. 3.2), condition $\bar{\bar{E}}$ (Sec. 3.3), $\overline{|u|}^D_{1+\delta}$, $\bar{C}_{1+\delta}(D)$, $\bar{C}_p$ (Sec. 7.2). As in [1] we need the following assumptions.

(2.I) *S satisfies condition $\bar{\bar{E}}$ and $S\in\bar{C}_{2-0}$.*

(2.II) *The coefficients a_{ij} are uniformly Hölder continuous with exponent $\alpha\in(0, 1)$ in the closure $\bar{D}$ of D and $a_{ij}\in\bar{C}_{1-0}(S)$, whereas b_i and c are continuous in $\bar{D}$.*

(2.III) *The operator L is uniformly parabolic in $\bar{D}$, i.e.*

$$\sum_{i,j=1}^{n} a_{ij}(x, t)s_i s_j \geqslant N_0|s|^2, \qquad \forall (x, t)\in\bar{D},\ s = (s_1, \ldots, s_n)\in\mathbf{R}^n,$$

N_0 being a positive constant.

It follows from (2.II) that

$$\sum_{i,j=1}^{n} \overline{|a_{ij}|}^D_\alpha + \sum_{i=1}^{n} |b_i|^D_0 + |c|^D_0 \leqslant N_1, \qquad \sum_{i,j=1}^{n} |a_{ij}|^S_{1-0} \leqslant N_2,$$

where N_1, N_2 are positive constants.

First consider problem (1.1), (1.2) in the case $\varphi = 0$, i.e.

$$(Lu)(x, t) = f(x, t), \qquad \forall (x, t)\in D\cup B_T, \tag{2.1}$$

$$u(x, t) = 0, \qquad \forall (x, t)\in\Gamma. \tag{2.2}$$

THEOREM 1 (cf. Theorem 4 [1], Sec. 7.2). *Let assumptions* (2.I)–(2.III) *be satisfied and let f be continuous in $\bar{D}$. If u is a solution to problem* (2.1), (2.2), *then for any $\delta\in(0, 1)$ there exists a constant $N > 0$ depending only on δ, N_0, N_1, N_2 and D such that*

$$\overline{|u|}^D_{1+\delta} \leqslant N|f|^D_0. \tag{2.3}$$

Proof. We proceed like in the proof of Lemma 1 of [1] (Sec. 7.2). Namely, by the Weierstrass theorem there exist sequences of polynomials (b_{im}), (c_m) and (f_m) such that

$$\lim_{m\to\infty} |b_{im} - b_i|^D_0 = 0, \qquad \lim_{m\to\infty} |c_m - c|^D_0 = 0, \tag{2.4}$$

$$\varepsilon_m \equiv \lim_{m\to\infty} |f_m - f|_0^D = 0. \tag{2.5}$$

Moreover, we introduce the functions

$$F_m(x, t) = g_m(t) f_m(x, t), \quad \forall (x, t) \in \bar{D},\ m = 1, 2, \dots, \tag{2.6}$$

where

$$g_m(t) = \begin{cases} mt, & \forall t \in [0, 1/m], \\ 1, & \forall t > 1/m. \end{cases} \tag{2.7}$$

It is clear that b_{im}, c_m, $F_m \in \bar{C}_\alpha(\bar{D})$ $(i = 1, \dots, n;\ m = 1, 2, \dots)$,

$$F_m(x, 0) = 0, \quad \forall x \in \partial B',\ m = 1, 2, \dots$$

Hence, by Theorem 7 of [1] (Sec. 3.3), for any m there exists a unique solution $u_m \in \bar{C}_{2+\alpha}(D)$ to the problem

$$\begin{aligned} L_m u_m \equiv & \sum_{i,j=1}^{n} a_{ij} u_{mx_ix_j} + \sum_{i=1}^{n} b_{mi} u_{mx_i} \\ & + c_m u_m - u_{mt} = F_m, \quad \forall (x, t) \in D \cup B_T, \end{aligned} \tag{2.8}$$

$$u_m(x, t) = 0, \quad \forall (x, t) \in \Gamma. \tag{2.9}$$

Taking into account the inequality

$$|F_m|_0^D \leqslant |f_m|_0^D \leqslant |f|_0^D + \varepsilon_m \tag{2.10}$$

(following from (2.5)–(2.7)) and (2.4), and applying to problem (2.8), (2.9) Theorem 4 of [1] (Sec. 7.2), we obtain the estimate

$$\overline{|m_m|}{}_{1+\delta}^D \leqslant N(|f|_0^D + \varepsilon_m), \quad m = 1, 2, \dots \tag{2.11}$$

Consequently, there exist a subsequence of (u_m) (denoted again by (u_m)) and a function $v \in \bar{C}_{1+\delta}(D)$ such that

$$\lim_{m\to\infty} |u_m - v|_0^D = 0, \quad \lim_{m\to\infty} |u_{mx_i} - v_{x_i}|_0^D = 0. \tag{2.12}$$

Hence, by (2.5) and (2.11), we have

$$\overline{|v|}{}_{1+\delta}^D \leqslant N|f|_0^D. \tag{2.13}$$

To complete the proof it remains to show that $v = u$. For this purpose observe that (2.1), (2.2), (2.8) and (2.9) imply that

$$L(u_m - u) = (L - L_m)u_m + F_m - f, \quad \forall (x, t) \in D \cup B_T, \tag{2.14}$$

$$(u_m - u)(x, t) = 0, \quad \forall (x, t) \in \Gamma. \tag{2.15}$$

In view of (2.4), (2.5), (2.10) and (2.11) we have

$$\varepsilon'_m \equiv |(L - L_m)u_m|_0^D \to 0 \text{ as } m \to \infty, \quad |F_m - f|_0^D \leqslant 2|f|_0^D + \varepsilon_m. \tag{2.16}$$

Hence, applying to problem (2.14), (2.15) the estimate (1.9) of [4], we find that

$$(2.17)\qquad |u_m(x,t)-u(x,t)| \leqslant tN_3(\varepsilon'_m+\varepsilon_m+2|f|_0^D), \qquad \forall (x,t)\in D,$$

where $N_3 = \exp(N_1 T)$. Take any $\varepsilon > 0$ and let $s\in(0, T)$ be a fixed number such that

$$s \leqslant \varepsilon[2N_3^2(\varepsilon'_m+\varepsilon_m+2|f|_0^D)]^{-1}, \qquad m = 1, 2, \ldots$$

Then, by (2.17) and (2.15), we get the inequalities

$$(2.18)\qquad |u_m-u|_0^{D_s} \leqslant \varepsilon(2N_3)^{-1}, \qquad m = 1, 2, \ldots,$$

$$(2.19)\qquad |u_m-u|_0^{\Gamma_{s,T}} \leqslant \varepsilon(2N_3)^{-1}, \qquad m = 1, 2, \ldots,$$

where $D_s = \{(x, t)\in D\colon t < s\}$ and $\Gamma_{s,T}$ is the parabolic boundary of $D_{s,T} = \{(x, t)\in D\colon t > s\}$. It follows from (2.5)–(2.7) and (2.16) that there exists an integer $m_0 > 1/s$ such that

$$(2.20)\qquad \varepsilon'_m+|F_m-f|_0^{D_{s,T}} \leqslant \varepsilon(2TN_3)^{-1}, \qquad \forall m \geqslant m_0.$$

(2.16), (2.19), (2.20) and the estimate (1.9) of [4] applied to (2.14) in $D_{s,T}\cup B_T$ imply

$$|u_m-u|_0^{D_{s,T}} \leqslant \varepsilon, \qquad \forall m \geqslant m_0.$$

Hence, by (2.18), we have

$$|u_m-u|_0^D \leqslant \varepsilon, \qquad \forall m \geqslant m_0.$$

We have thus proved that $\lim_{m\to\infty}|u_m-u|_0^D = 0$ and consequently $u = v$. In view of (2.13) this completes the proof.

Notice that the difference between Theorem 1 and Theorem 4 of [1] consists in the fact that Theorem 1 does not require the compatibility condition

$$(2.21)\qquad f(x, 0) = 0, \qquad \forall x\in\partial B'.$$

This condition is the particular case of (1.3) for $\varphi = 0$.

Now introduce the following assumption.

(2.IV) *There exists an extension* $\Phi\in\bar{C}_{1+\delta}(D)$ ($\delta\in(0, 1)$ *being a constant*) *of* φ *such that* $\Phi_{x_ix_j}$, Φ_t *are continuous in* $\bar{D}$.

Theorem 1 implies in the standard manner the following.

THEOREM 2. *Let assumptions* (2.I)–(2.IV) *be satisfied and let* f *be continuous in* $\bar{D}$. *Suppose that there exists a solution* u *to problem* (1.1), (1.2). *Then*

$$(2.22)\qquad \overline{|u|}_{1+\delta}^D \leqslant N(|f|_0^D+|L\Phi|_0^D)+|\bar{\Phi}|_{1+\delta}^D,$$

where N *is the constant occurring in Theorem* 1.

Notice that for problem (1.1), (1.2) in the case (1.4) the estimate (0.9) in [5] does not need condition (1.3) and is similar (in some sense) to (2.22). However, it involves $\overline{|f|}{}_\delta^D$ and is more complicated. Therefore (2.22) is more convenient for applications.

3. Existence and uniqueness results and estimates of $(2+\alpha)$**-type.** For any $a \in (0, \sqrt{T})$ define

$$D^a = \{(x, t) \in D\colon\ d(x, \partial B') > a \ \text{ or } \ t > a^2\},$$

where $d(x, A) = \inf\{|y-x|\colon\ y \in A\}$, $A \subset \mathbf{R}^n$. Retaining the assumptions of Sec. 2 we additionally introduce the following one.

(3.I) *The coefficients* b_i, c *are uniformly Hölder continuous with exponent* α *in* $\bar{D}$. *Thus we have*

$$\overline{|b_i|}{}_\alpha^D, \overline{|c|}{}_\alpha^D \leqslant N_4, \qquad N_4 > 0 \ \textit{being a constant.}$$

THEOREM 3. *Let assumptions* (2.I)–(2.III), (3.I) *be satisfied and let* $f \in \bar{C}_\alpha(D)$. *Then there exists a unique solution* u *to problem* (2.1), (2.2). *This solution satisfies* (2.3) *and the derivatives* $u_{x_i x_j}$, u_t *are continuous in* $\bar{D} \setminus \partial B$. *Moreover, there is a constant* $K > 0$ *depending only on* N_0, N_1, N_2, N_4, α *and* D *such that for any* $a \in (0, \sqrt{T}/3)$

$$\overline{|u|}{}_{2+\alpha}^{D^a} \leqslant K a^{-2-\alpha} \overline{|f|}{}_\alpha^D. \tag{3.1}$$

Proof. According to Theorems 8′ and 9 of [1] (Sec. 3.4) there exists a unique solution u to problem (2.1), (2.2). By Theorem 1 this solution satisfies (2.3). For any $a > 0$ set

$$F_a = h_a f - u\Big(\sum_{i,j=1}^{n} a_{ij} h_{a x_i x_j} + \sum_{i=1}^{n} b_i h_{a x_i} - h_{at}\Big) - \sum_{i,j=1}^{n} a_{ij}(h_{a x_i} u_{x_j} + h_{a x_j} u_{x_i}), \tag{3.2}$$

where h_a is defined as in Sec. 5 of this paper with $A = \partial B'$. It is clear that

$$F_a \in \bar{C}_\alpha(D), \qquad F_a(x, 0) = 0, \qquad \forall (x, 0) \in \partial B. \tag{3.3}$$

Now consider the problem

$$(L v_a)(x, t) = F_a(x, t), \qquad \forall (x, t) \in D \cup B_T, \tag{3.4}$$

$$v_a(x, t) = 0, \qquad \forall (x, t) \in \Gamma. \tag{3.5}$$

(3.3) and Theorem 7 of [1] (Sec. 3.3) imply the existence of a unique solution $v_a \in \bar{C}_{2+\alpha}(D)$ to problem (3.4), (3.5). By Theorem 6 of [1] (Sec. 3.2) there is a constant $N' > 0$ depending only on N_0, N_1, N_4, α and D such that

$$\overline{|v_a|}{}_{2+\alpha}^D \leqslant N' \overline{|F_a|}{}_\alpha^D. \tag{3.6}$$

(3.2) and (2.3) with $\delta = \alpha$, and properties (5.III), (5.IV) (Sec. 5) yield the estimate

$$\overline{|F_a|}^D_\alpha \leqslant N'' a^{-2-\alpha} \overline{|f|}^D_\alpha, \tag{3.7}$$

where $N'' > 0$ is a constant depending only on N_0, N_1, N_2, N_4, α and D. It results from (3.6) and (3.7) that

$$\overline{|v_a|}^D_{2+\alpha} \leqslant N' N'' a^{-2-\alpha} \overline{|f|}^D_\alpha. \tag{3.8}$$

Taking into account that $h_a u$ is a solution to problem (3.4), (3.5) it follows from the uniqueness of solution that $h_a u = v_a$. Hence, by property (5.II), we have $u = v_a$ in $\overline{D^{3a}}$. (3.8) now implies that

$$\overline{|u|}^{D^{3a}}_{2+\alpha} \leqslant N' N'' a^{-2-\alpha} \overline{|f|}^D_\alpha.$$

Substituting $a/3$ for a and setting $K = 27N'N''$ we obtain (3.1).

Finally, the continuity of $u_{x_i x_j}$, u_t in $\bar{D} \backslash \partial B$ follows immediately from (3.1). This completes the proof.

Theorems 3 and 2 imply in the standard manner the following.

THEOREM 4. *Let assumptions* (2.I)–(2.III), (3.I) *be satisfied and let* $f \in \bar{C}_\alpha(D)$. *Suppose there exists an extension*

$$\Phi \in \bar{C}_{1+\delta}(D) \cap \bar{C}_{2+\alpha}(D)$$

of φ, $\delta \in [\alpha, 1)$ *being a constant. Then there exists a unique solution u to problem* (1.1), (1.2). *This solution satisfies* (2.10) *and the derivatives* $u_{x_i x_j}$, u_t *are continuous in* $\bar{D} \backslash \partial B$. *Moreover, for any* $a \in (0, \sqrt{T}/3)$

$$\overline{|u|}^{D^a}_{2+\alpha} \leqslant K a^{-2-\alpha} (\overline{|f|}^D_\alpha + \overline{|L\Phi|}^D_\alpha) + \overline{|\Phi|}^{D^a}_{2+\alpha}, \tag{3.9}$$

where K is the constant occurring in Theorem 3.

Notice that for problem (1.1), (1.2) in the case (1.4) Theorem 0.1 of [5] is similar to Theorem 4. However, the estimate (0.8) in [5] is more complicated than (3.9).

4. Some remarks. We make some remarks concerning the necessity of compatibility conditions for some estimates and existence theorems appearing in the literature. First consider problem (1.1), (1.2) under the following assumptions.

(4.I) *L is a uniformly parabolic operator in* $\bar{D}$ *with coefficients belonging to* $\bar{C}_\alpha(\bar{D})$.

(4.II) *S has property* $\bar{E}$ (see [1], Sec. 3.2).

(4.III) *There exists an extension* $\Phi \in \bar{C}_{2+\alpha}(D)$ *of* φ.

(4.IV) *There exists a solution* $u \in \bar{C}_{2+\alpha}(D)$ *to problem* (1.1), (1.2).

It follows from Sec. 3.3 of [1] (see the remark which precedes assumption $(\bar{A})$) that the function $(L\Phi)(x, 0)$, $(x, 0) \in \partial B$, is independent of the extension Φ. But u is an extension of φ as well and $(Lu)(x, 0) = f(x, 0)$, $\forall x \in \partial B'$. This implies (1.3). At the same time we have shown that under assumptions (4.I)–(4.III) condition (1.3) in necessary for (4.IV).

In [2], theorems on the boundary estimate of $(2+\alpha)$-type (Theorem 3) and on the existence and uniqueness of solution in $\bar{C}_{2+\alpha}(D)$ (Theorem 4) for problem (1.1), (1.2) were proved. In those theorems condition (1.3) was not assumed. However, it follows from the above considerations that this condition is necessary. Note that in [1], Theorem 4 of [2] is formulated under condition (1.3).

Now consider the boundary estimate of $(1+\delta)$-type for solution to problem (1.1), (1.2). In [3] this estimate was derived for the case $\varphi = 0$ without assuming condition (2.21). However, Theorem 4 of [2] was used (which requires (2.21)). Consequently, (2.21) (in general (1.3)) is necessary for the estimate of $(1+\delta)$-type in [3] to hold. In [1] the above estimate is formulated properly, i.e. (2.21) is assumed. In Sec. 2 of this paper, using the result of [1], we have shown that (1.3) is in fact superfluous for the boundary estimate of $(1+\delta)$-type. As a consequence of this result and appropriate theorems of [1] we have obtained in Sec. 3 certain estimates and existence results for problem (1.1), (1.2) which do not need condition (1.3) either. Comparing these results with appropriate ones of [1] (Theorems 3.6 and 3.7) we see that the removal of (1.3) has weakened the regularity of the derivatives $u_{x_i x_j}$ and u_t near ∂B.

Now we discuss the necessity of compatibility conditions in [6]. There the quasi-linear parabolic equation

$$\tag{4.1} \sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_i x_j}(x,t) + \sum_{i=1}^{n} b_i(x,t,u(x,t))u_{x_i}(x,t) - u_t(x,t) = f\big(x,t,u(x,t),u_{x_1}(x,t),\ldots,u_{x_n}(x,t)\big)$$

with condition (1.2) was considered. Following [2], [3] the authors stated the boundary estimates of type $1+\delta$ and $2+\alpha$, and the existence theorem for problem (1.1), (1.2) without assuming (1.3). With the aid of these results they obtained estimates and existence theorems for problem (4.1), (1.2) (Theorems 5 and 8) without compatibility conditions. However, it follows from our remarks on [2], [3] that Theorems 5 and 8 of [6] do require suitable compatibility conditions on ∂B.

Finally, it should be pointed out that in [7], [8] intermediate Schauder theory for problem (1.1), (1.2) was presented under assumptions weaker than in this paper. This theory includes, in particular, estimates of some Hölder norms for the solution u to this problem which also depend on Hölder norms of f and φ (¹). Notice that the results of the present paper do not follow from those of [7], [8].

(¹) By a Hölder norm we mean here, roughly speaking, a sum of sup norms and Hölder constants.

5. Auxiliary function. We give the definition and properties of the function h_a which have been used in Sec. 3.

Let $A \subset \mathbf{R}^n$ be a bounded closed set. For any $a > 0$ define

$$A_a = \{(x, t) \in \mathbf{R}^{n+1}: d(x, A) \geqslant a \text{ or } |t| \geqslant a^2\},$$

$$K_a = \{(x, t) \in \mathbf{R}^{n+1}: |x|^4 + t^2 < a^4\},$$

$$g_a(x, t) = \begin{cases} Ma^{-n-2} \exp[a^4(|x|^4 + t^2 - a^4)^{-1}], & \forall (x, t) \in K_a, \\ 0, \quad \forall (x, t) \in \mathbf{R}^{n+1} \setminus K_a, \end{cases}$$

$$h_a(x, t) = \iint\limits_{A_{2a}} g_a(x-y, t-s)\,dy\,ds,$$

where

$$M = \{\iint\limits_{K_1} \exp[(|y|^4 + s^2 - 1)^{-1}]\,dy\,ds\}^{-1}.$$

The function h_a has the following properties:

(5.I) $h_a \in C^\infty(\mathbf{R}^{n+1})$, $0 \leqslant h_a(x, t) \leqslant 1$, $\quad \forall (x, t) \in \mathbf{R}^{n+1}$;

(5.II) $h_a(x, t) = 0$, $\forall (x, t) \in \mathbf{R}^{n+1} \setminus A_a$; $h_a(x, t) = 1$, $\forall (x, t) \in A_{3a}$;

(5.III) $|D_x^k D_t^m h_a(x, t)| \leqslant M_1 a^{-k-2m}$, $\forall (x, t) \in \mathbf{R}^{n+1}$, $k + 2m \leqslant 4 (k, m \geqslant 0)$, $M_1 > 0$ *being a constant*;

(5.IV) $|D_x^k D_t^m h_a(P) - D_x^k D_t^m h_a(Q)| \leqslant 2M_1(n+1)a^{-k-2m-\alpha}[d(P, Q)]^\alpha$, $\forall P, Q \in \mathbf{R}^{n+1}$, $k + 2m \leqslant 2$ $(k, m \geqslant 0)$, $\alpha \in (0, 1]$ *being a constant.*

We prove only the last property. Consider the case $k = 2$, $m = 0$. Using (5.III) and the mean value theorem we get

$$\begin{aligned} &|D_x^2 h_a(x, t) - D_x^2 h_a(y, t)| \\ &\qquad \leqslant [|D_x^2 h_a(x, t)| + |D_x^2 h_a(y, t)|]^{1-\alpha} |D_x^2 h_a(x, t) - D_x^2 h_a(y, t)|^\alpha \\ &\qquad \leqslant 2M_1 n a^{-2-\alpha} |x - y|^\alpha, \\ &|D_x^2 h_a(x, t) - D_x^2 h_a(x, s)| \\ &\qquad \leqslant [|D_x^2 h_a(x, t)| + |D_x^2 h_a(x, s)|]^{(2-\alpha)/2} |D_x^2 h_a(x, t) - D_x^2 h_a(x, s)|^{\alpha/2} \\ &\qquad \leqslant 2M_1 a^{-2-\alpha} |t - s|^{\alpha/2}. \end{aligned}$$

Hence (5.IV) follows for the case considered. The remaining cases can be proved analogously.

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