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On the degree of almost strong convergence of Fourier series

Abstract. We estimate the rate of various types of almost strong convergence of the Fourier series of functions belonging to the spaces L^p $(1 \leqslant p \leqslant \infty)$ and C, by using matrix means. As corollaries, norm and pointwise approximation of functions from Hölder type classes is examined. An almost convergence criterion is also obtained.

1. Introduction. Let $X = L^p$ $(1 \leqslant p \leqslant \infty)$ [resp. $X = C$] be the space of all 2π-periodic measurable real-valued functions $f = f(\cdot)$ on the real line which are p-integrable [resp. continuous] on $[-\pi, \pi]$, with the usual norm $\|\cdot\|_X$.

Consider the Fourier series

$$S[f](x) = \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} (a_k(f)\cos kx + b_k(f)\sin kx)$$

and denote by $S_\mu[f]$ the partial sums of $S[f]$, and by

$$\sigma_{m,n}[f] = \frac{1}{m+1} \sum_{\mu=n}^{n+m} S_\mu[f]$$

the generalized de la Vallée Poussin means of the sequence $\{S_\mu[f]\}$.

The aim of this paper is to estimate the quantities

$$H^r_{k,m}[f]_{X,T} = \Big\{ \sum_{n=0}^{\infty} t_{k,n} \|\sigma_{m,n}[f]-f\|^r_X \Big\}^{1/r},$$

$$H^{q,r}_{k,m}[f]_{X,T} = \left\{ \sum_{n=0}^{\infty} t_{k,n} \left\| \left(\frac{1}{m+1} \sum_{\mu=n}^{n+m} |S_\mu[f]-f|^q \right)^{1/q} \right\|^r_X \right\}^{1/r},$$

$$H_{k,m}[f]_{X,T} = \Big\| \sum_{n=0}^{\infty} t_{k,n} \{\sigma_{m,n}[f]-f\} \Big\|_X,$$

$$H^r_{k,m}[f]_T(x) = \Big\{ \sum_{n=0}^{\infty} t_{k,n} |\sigma_{m,n}[f](x)-f(x)|^r \Big\}^{1/r},$$

$$H_{k,m}^{q,r}[f]_T(x) = \left\{ \sum_{n=0}^{\infty} t_{k,n} \left(\frac{1}{m+1} \sum_{\mu=n}^{m+n} |S_\mu[f](x) - f(x)|^q \right)^{r/q} \right\}^{1/r},$$

$$H_{k,m}[f]_T(x) = \left| \sum_{n=0}^{\infty} t_{k,n} \{\sigma_{m,n}[f](x) - f(x)\} \right|,$$

where $T = (t_{k,n})_{k,n=0}^{\infty}$ is an arbitrary non-negative matrix, and $q, r > 0$. As a measure of these deviations we take

$$h_{k,m,\lambda}^{q,r,s}[g]_{T,\alpha,\beta} = \left\{ \sum_{n=0}^{\infty} t_{k,n} \left(\sum_{\mu=0}^{\lambda(n)} \frac{(g(\mu+\alpha m+\beta n))^q}{(\mu+n+1)^{1-s}(\mu+m+1)^s} \right)^{r/q} \right\}^{1/r},$$

where $g = g(\cdot)$ is taken to be either the modulus of continuity $\omega\left(\frac{\pi}{\cdot+1}, f\right)_X$, the best approximation $E_\cdot(f)_X$ of f by trigonometric polynomials of degree at most "$\cdot$" in the space X, or the function $w_x\left(\frac{\pi}{\cdot+1}, f\right)_X$. Here

$$\omega(\delta, f)_X = \sup_{0<t\leqslant\delta} \|\varphi_\cdot(t)\|_X,$$

$$w_x(\delta, f)_X = \begin{cases} \sup\limits_{0<u\leqslant\delta} \{u^{-1}\int_0^u |\varphi_x(t)|^p dt\}^{1/p} & \text{if } X = L^p \ (1 \leqslant p < \infty), \\ \operatorname*{ess\,sup}\limits_{0<u\leqslant\delta} |\varphi_x(u)| & \text{if } X = L^\infty, \\ \sup\limits_{0<u\leqslant\delta} |\varphi_x(u)| & \text{if } X = C, \end{cases}$$

with $\varphi_x(t) = f(x+t) + f(x-t) - 2f(x)$.

We also show how some earlier results follow from ours.

By convention, the letter M will mean either an absolute constant or a constant depending on the parameters q, r, p, not necessarily the same at each occurrence. We denote by T^Δ and T_∇ the two triangular matrices corresponding to T, i.e. $(T^\Delta)_{k,n} = t_{k,n}$ if $k \geqslant n$ and zero otherwise, and $(T_\nabla)_{k,n} = t_{k,k-n}$ if $k \geqslant n$ and zero otherwise. We set

$$T_k(u) = T_k([u]) = \sum_{n=0}^{[u]} t_{k,k-n} \quad \text{for } 0 \leqslant u < k \quad \text{and} \quad T_k(k) = 1.$$

2. Norm approximation. In this section we will investigate the first three quantities.

In the following theorem we present the general estimate basing on the well-known results of Dahmen [1] and Stechkin [7].

THEOREM 1. *If $f \in X$, then*

$$H_{k,m}^r[f]_{X,T} \leqslant M h_{k,m,m+\cdot}^{1,r,1}[E_\cdot(f)_X]_{T,0,1}.$$

Proof. The mentioned results of Dahmen and Stechkin may be written in the form

$$\|\sigma_{m,n}[f]-f\|_X \leqslant M \sum_{\mu=0}^{n+m} (\mu+m+1)^{-1} E_{\mu+n}(f)_X,$$

and hence our assertion follows.

The next theorem concerns a stronger quantity.

THEOREM 2. *If* $f \in X$ *(here* $X = C$ *when* $p = \infty$*) and* $q(q-1)^{-1} \leqslant p \leqslant q$ $(q \geqslant 2)$, *then*

$$H^{q,r}_{k,m}[f]_{X,T} \leqslant Mh^{p,r,p/q}_{k,m,\cdot}\left[\omega\left(\frac{\pi}{\cdot+1},f\right)_X\right]_{T,0,0} + Mh^{1,r,1}_{k,m,m}\left[\omega\left(\frac{\pi}{\cdot+1},f\right)_X\right]_{T,1,0}.$$

Proof. We only give a sketch of proof, because it is similar to that in [3] for almost strong summability. It is clear that

$$\left\{\frac{1}{m+1}\sum_{\mu=n}^{m+n} |S_\mu[f](x)-f(x)|^q\right\}^{1/q}$$

$$= \left\{\frac{1}{m+1}\sum_{\mu=n}^{m+n}\left|\frac{1}{\pi}\Big(\int_0^{\pi/(m+n+1)} + \int_{\pi/(m+n+1)}^{\pi/(m+1)} + \int_{\pi/(m+1)}^{\pi}\Big)\varphi_x(t)D_\mu(t)\,dt\right|^q\right\}^{1/q},$$

and the estimates of the first and second integrals follow at once from the following inequalities for the Dirichlet kernel $D_\mu(t)$ $(n \leqslant \mu \leqslant n+m)$:

$$|D_\mu(t)| \leqslant n+m+1, \qquad |D_\mu(t)| \leqslant (\pi/2)|t|^{-1} \qquad (0 < |t| \leqslant \pi).$$

The last integral may be treated as the Fourier coefficient of some function, and using the Hardy–Littlewood inequality (cf. [8, (5.20 II), p. 126]),

$$\left\{\frac{|a_0(f)|^q}{2} + \sum_{k=1}^{\infty} \left(|a_k(f)|^q + |b_k(f)|^q\right)\right\}^{1/q} \leqslant M\|f_\varrho(\cdot)\|_X,$$

where $f_\varrho(t) = t^{-\varrho}f(t)$, $f_\varrho \in X$ with $\varrho = 1/p+1/q-1$ and $q(q-1)^{-1} \leqslant p \leqslant q$ $(q \geqslant 2)$, we obtain

$$\left\{\frac{1}{m+1}\sum_{\mu=0}^{m+n}\left|\frac{1}{\pi}\int_{\pi/2(m+1)}^{\pi/2} \frac{\varphi_x(2t)}{2\sin t}\sin((2\mu+1)t)\,dt\right|^q\right\}^{1/q}$$

$$\leqslant M\{(m+1)^{-p/q}\int_{\pi/(m+1)}^{\pi} t^{-1-p/q}|\varphi_x(t)|^p\,dt\}^{1/p}.$$

Hence our assertion follows.

In the next results of this section we consider only the triangular matrices T^Δ or T_∇.

THEOREM 3. *Let $\{t_{k,n}\}$ be positive and non-decreasing with respect to n for each k. If $f \in X$, then*

$$H_{k,m}[f]_{X,T^{\Delta}} \leqslant M h^{1,1,0}_{k,m,k+m-\cdot}\left[\omega\left(\frac{\pi}{\cdot+1}, f\right)_X\right]_{T^{\nabla},0,1} + \omega\left(\frac{\pi}{k+m+1}, f\right)_X.$$

Proof. Using the Dirichlet formula we obtain

$$H_{k,m}[f]_{X,T^{\Delta}} \leqslant \frac{1}{\pi}\int_0^{\pi} \|\varphi_{\cdot}(t)\|_X L_{k,m}(t)\,dt = \frac{1}{\pi}\Big\{\int_0^{\pi/(k+m+1)} + \int_{\pi/(k+m+1)}^{\pi}\Big\} = I_1 + I_2,$$

where

$$L_{k,m}(t) = \sum_{n=0}^{k} t_{k,n} \frac{\sin(2n+m+1)\frac{t}{2}\sin(m+1)\frac{t}{2}}{2(m+1)\sin^2 t/2}.$$

As $|L_{k,m}(t)| \leqslant k+m+1$, we get

$$I_1 \leqslant \sum_{n=0}^{k} t_{k,n}\,\omega\left(\frac{\pi}{k+m+1}, f\right)_X = \omega\left(\frac{\pi}{k+m+1}, f\right)_X.$$

Next, by Abel's transformation (see [5, Lemma 5.11]), we have

$$\left|\sum_{n=0}^{k} t_{k,n}\sin(2n+m+1)\frac{t}{2}\right| \leqslant M T_k(\pi/t),$$

whence $|L_{k,m}(t)| \leqslant M t^{-1} T_k(\pi/t)$, and thus

$$\begin{aligned}
I_2 &\leqslant M \int_{\pi/(k+m+1)}^{\pi} t^{-1}\|\varphi_{\cdot}(t)\|_X T_k(\pi/t)\,dt \leqslant M \int_1^{k+m+1} \omega(\pi/u, f)_X T_k(u) u^{-1}\,du \\
&\leqslant M \sum_{\mu=1}^{k+m} \mu^{-1}\omega(\pi/\mu, f)_X T_k(\mu) = M \sum_{\mu=0}^{k+m-1}\sum_{n=0}^{\mu} \frac{1}{\mu+1}\omega\left(\frac{\pi}{\mu+1}, f\right)_X t_{k,k-n} \\
&\leqslant M \sum_{n=0}^{k+m} t_{k,k-n} \sum_{\mu=n}^{k+m} \frac{1}{\mu+1}\omega\left(\frac{\pi}{\mu+1}, f\right)_X = M h^{1,1,0}_{k,m,k+m-\cdot}\left[\omega\left(\frac{\pi}{\cdot+1}, f\right)_X\right]_{T^{\nabla},0,1}.
\end{aligned}$$

This completes the proof.

From the above theorems we can deduce some known results.

PROPOSITION 1. *Let $\{t_{k,n}\}$ be positive and non-decreasing with respect to n, for each k, and let χ be a positive function defined on $(0, \infty)$ such that, as $k \to \infty$,*

(i) $k\chi(k) = O(1)$,

(ii) $\int_1^{k+m} \chi(u) T_k(u)\,du = O(1)$ *uniformly in* $m \geqslant 0$.

Then, if $\omega(t, f)_X = o(t^{-1}\chi(\pi/t))$ as $t \to 0+$, we have

$$H_{k,m}[f]_{X,T^{\Delta}} = o(1) \quad \text{as } k \to \infty,$$

uniformly in $m \geqslant 0$.

Proof. We note that

$$\sum_{n=0}^{k} t_{k,k-n}\left\{\omega\left(\frac{\pi}{k+m+1}, f\right)_X + \sum_{\mu=n}^{m+n} \frac{1}{\mu+1}\omega\left(\frac{\pi}{\mu+1}, f\right)_X\right\}$$
$$= o((k+m+1)\chi(k+m+1)) + \sum_{n=0}^{k}\sum_{\mu=n}^{k+m} t_{k,k-n}\frac{1}{\mu+1}\omega\left(\frac{\pi}{\mu+1}, f\right)_X.$$

By (i) the first term on the right-hand side is bounded, and the second after changing the order of summation does not exceed

$$\sum_{\mu=0}^{k=m}\sum_{n=0}^{\mu} t_{k,k-n}\frac{1}{\mu+1}\omega\left(\frac{\pi}{\mu+1}, f\right)_X \leqslant 5\sum_{\mu=1}^{k+m} T_k(\mu)\int_{\mu}^{\mu+1} \omega(\pi/t, f)_X t^{-1}\,dt$$
$$= 5\int_1^{k+m+1} t^{-1}\omega(\pi/t, f)_X T_k(t)\,dt = o(1)\int_1^{k+m+1} \chi(t)T_k(t)\,dt.$$

Hence, in view of (ii), by Theorem 3 we obtain the desired relation.

Remark 1. For $m = 0$, this result is a norm analogue of Dikshit's theorem [2].

The analogue of Proposition 1 for strong means is

PROPOSITION 2. *Under the assumptions of Proposition* 1 *with* (ii) *replaced by*

(ii′) $\int_1^k \chi(u)T_k(u)\,du = O(1)$ *as* $k \to \infty$,

we have

$$H_{k,m}^{q',1}[f]_{X,T^{\Delta}} = o(1) \quad \text{as } k \to \infty,$$

uniformly in $m \geqslant 0$, *for* $0 \leqslant q' \leqslant p$, $2 \leqslant p \leqslant \infty$ *and with* $X = C$ *when* $p = \infty$.

Proof. Firstly, as in the previous proof, by (ii′), we obtain

$$\sum_{n=0}^{k} t_{k,n}\sum_{\mu=0}^{n}\frac{1}{\mu+1}\omega\left(\frac{\pi}{\mu+1}, f\right)_X = o(1)\int_1^{k+1}\chi(t)T_k(t)\,dt.$$

Secondly, by (i),

$$\sum_{n=0}^{k} t_{k,n}\left\{\sum_{\mu=0}^{m}\frac{1}{\mu+m+1}\omega^p\left(\frac{\pi}{\mu+1}, f\right)_X\right\}^{1/p} = o(1)\left\{\frac{1}{m+1}\sum_{\mu=0}^{m}((\mu+1)\chi(\mu+1))^p\right\}^{1/p}$$
$$= o(1)\left\{\frac{1}{m+1}\sum_{\mu=0}^{m} O(1)\right\}^{1/p} = o(1),$$

and thus, by Theorem 2, we have our statement.

Now, we formulate as a corollary the slightly improved result of Prem Chandra [6], which is a special case of Theorem 1.

COROLLARY 1. *Let ψ be a positive function defined on* $(0, \infty)$ *such that*

$$t^{-1} \int_{\delta(t)}^{t} \psi(u)du = O(\psi(t)) \quad \text{as } t \to 0+,$$

with some $\delta(t) \in [0, t)$. *Then, if*

$$\int_{t}^{\pi} u^{-2} \omega(u, f)_X du = O(\psi(t)) \quad \text{as } t \to 0+,$$

we have

$$\left\| \frac{1}{\lambda_k} \sum_{v=k-\lambda_k}^{k-1} S_v[f] - f \right\|_X = O(\lambda_k^{-1} \psi(\pi/\lambda_k)) \quad \text{as } k \to \infty,$$

where $\{\lambda_k\}$ *is a monotonic non-decreasing sequence of integers such that* $\lambda_1 = 1$, *and* $\lambda_{k+1} - \lambda_k \leqslant 1$.

Proof. Let $T_1^A = \text{diag}(1, 1, \ldots)$. Then, by Theorem 1 and Jackson's Theorem,

$$\left\| \frac{1}{\lambda_k} \sum_{v=k-\lambda_k}^{k-1} S_v[f] - f \right\|_X = H^r_{k-\lambda_k, \lambda_k - 1}[f]_{X, T_1^A}$$

$$\leqslant M \sum_{\mu=0}^{k-1} \frac{1}{\mu + \lambda_k} \omega\left(\frac{\pi}{\mu + k + 1}, f\right)_X = M \sum_{\mu=\lambda_k}^{k+\lambda_k-1} \frac{1}{\mu} \omega\left(\frac{\pi}{\mu + k + 1 - \lambda_k}, f\right)_X.$$

Since, for small $t > 0$,

$$t^{-1} \omega(t, f)_X \leqslant M \int_{t}^{\pi} u^{-2} \omega(u, f)_X du,$$

it follows from our assumptions that the considered expression does not exceed

$$M \int_{\lambda_k}^{k+\lambda_k+1} v^{-1} \omega\left(\frac{\pi}{v + k - \lambda_k}, f\right)_X dv = \frac{M}{\pi} \int_{\pi/(k+\lambda_k+1)}^{\pi/\lambda_k} t^{-1} \omega\left(\frac{\pi}{\pi/t + k - \lambda_k}, f\right)_X du$$

$$\leqslant \frac{M}{\pi} \int_{\pi/(k+\lambda_k+1)}^{\pi/\lambda_k} t^{-1} \omega(t, f)_X dt = O\left\{ \int_{\pi/(k+\lambda_k+1)}^{\pi/\lambda_k} \psi(t) dt \right\} = O\{\lambda_k^{-1} \psi(\pi/\lambda_k)\},$$

and the corollary follows.

3. Pointwise approximation.

THEOREM 4. *If* $f \in X$, *then*

$$H^r_{k,m}[f]_T(x) \leqslant M h^{1,r,1}_{k,m,m+\cdot} \left[w_x\left(\frac{\pi}{\cdot + 1}, f\right)_X \right]_{T,0,0}.$$

Proof. We have

$$|\sigma_{m,n}[f](x)-f(x)| \leqslant \frac{2n+m+1}{2\pi} \int_0^{\pi/(n+m+1)} |\varphi_x(t)|\,dt$$

$$+\tfrac{1}{2} \int_{\pi/(n+m+1)}^{\pi/(m+1)} t^{-1}|\varphi_x(t)|\,dt + \frac{\pi}{2(m+1)} \int_{\pi/(m+1)}^{\pi} t^{-2}|\varphi_x(t)|\,dt,$$

from which, by partial integration, our statement follows at once.

THEOREM 5. *If $f \in X$ (here $X = C$ when $p = \infty$), $q(q-1)^{-1} \leqslant p \leqslant q$ $(q \geqslant 2)$, then*

$$H^{q,r}_{k,m}[f]_T(x) \leqslant M h^{1,r,1}_{k,m,m+\cdot}\left[w_x\left(\frac{\pi}{\cdot+1}, f\right)_X\right]_{T,1,0} + h^{p,r,p/q}_{k,m,m}\left[w_x\left(\frac{\pi}{\cdot+1}, f\right)_X\right]_{T,0,0}.$$

Proof. As in the proof of Theorem 2 we have the estimate

$$\left\{\frac{1}{m+1}\sum_{\mu=n}^{n+m} |S_\mu[f](x)-f(x)|^q\right\}^{1/q} \leqslant \frac{n+m+1}{\pi} \int_0^{\pi/(n+m+1)} |\varphi_x(t)|\,dt$$

$$+\tfrac{1}{2} \int_{\pi/(n+m+1)}^{\pi/(m+1)} t^{-1}|\varphi_x(t)|\,dt + M\left\{(m+1)^{-p/q} \int_{\pi/(m+1)}^{\pi} t^{-1-p/q}|\varphi_x(t)|^p\,dt\right\}^{1/p},$$

which, by partial integration, leads to the desired result.

In the following theorem we consider the triangular non-negative matrices T^{Δ} and T_{∇}.

THEOREM 6. *Let $\{t_{k,n}\}$ be positive and non-decreasing with respect to n, for each k. If $f \in X$, then*

$$H_{k,n}[f]_{T^{\Delta}}(x) \leqslant M h^{1,1,0}_{k,m,k+m-\cdot}\left[w_x\left(\frac{\pi}{\cdot+1}, f\right)_X\right]_{T_{\nabla},0,1} + w_x\left(\frac{\pi}{k+m+1}, f\right)_X.$$

Proof. Proceeding as in the previous section, we obtain

$$H_{k,m}[f]_{T^{\Delta}}(x) \leqslant w_x\left(\frac{\pi}{k+m+1}, f\right)_X + M \int_{\pi/(k+m+1)}^{\pi} t^{-1}|\varphi_x(t)|\,T_k(\pi/t)\,dt.$$

Further, by partial integration, the above integral does not exceed

$$\sum_{\mu=1}^{k+m} \int_{\pi/(\mu+1)}^{\pi/\mu} t^{-1}|\varphi_x(t)|\,T_k(\pi/t)\,dt = \sum_{\mu=1}^{k+m} \left\{\left[T_k(\pi/t)t^{-1}\int_0^t |\varphi_x(u)|\,du\right]^{\pi/k}_{\pi/(\mu+1)}\right.$$

$$\left.+ \int_{\pi/(\mu+1)}^{\pi/\mu} \left(t^{-2}\int_0^t |\varphi_x(u)|\,du\right)dt\right\} \leqslant T_k(1)w_x(\pi, f)_X + \int_{\pi/(k+m+1)}^{\pi} t^{-1}w_x(t, f)_X\,T_k(\pi/t)\,dt.$$

Finally, by standard transformations, we have

$$H_{k,m}[f]_{T\Delta}(x) \leqslant w_x\left(\frac{\pi}{k+m+1}, f\right)_X + MT_k(1)w_x(\pi, f)_X + \pi M \int_1^{k+m+1} u^{-1} w_x(\pi/u, f)_X T_k(u)du$$

$$\leqslant w_x\left(\frac{\pi}{k+m+1}, f\right)_X + MT_k(1)w_x(\pi, f)_X + \pi M \sum_{n=0}^{k+m} \sum_{\mu=n}^{k+m} t_{k,k-n} \frac{1}{\mu+1} w_x\left(\frac{\pi}{\mu+1}, f\right)_X$$

$$\leqslant w_x\left(\frac{\pi}{k+m+1}, f\right)_X + (2+\pi)M \sum_{n=0}^{k} t_{k,k-n} \sum_{\mu=0}^{k+m-n} \frac{1}{n+\mu+1} w_x\left(\frac{\pi}{n+\mu+1}, f\right)_X,$$

and thus our proof is complete.

Remark 2. Note that our theorems remain true if we replace $w_x(\delta, f)_X$ by $\bar{w}_x(\delta, f)_X$ which arises from the definition of $w_x(\delta, f)_X$ be removing $\sup_u$ or $\operatorname{ess\,sup}_u$ and putting $u = \delta$.

PROPOSITION 3. *Let $\{t_{k,n}\}$ be positive and non-decreasing with respect to n, for each k. Let χ be a positive function defined on $(0, \infty)$ such that, as $k \to \infty$,*

(i) $k\chi(k) = O(1)$,

(ii) $\int_1^{k+m} \chi(u) T_k(u) u^{1/p-1} du = O(1)$ *uniformly in* $m \geqslant 0$.

Then if $\bar{w}_x(t, f)_X = o(t^{-1/p}\chi(\pi/t))$ as $t \to 0+$ (when $p = \infty$ or $X = C$ then $1/p = 0$), we have

$$H_{k,m}[f]_{T\Delta}(x) = o(1) \quad \text{as } k \to \infty, \text{ uniformly in } m \geqslant 0.$$

Proof. From the assumption on $\bar{w}_x(\delta, f)_X$, by (i), we get

$$\bar{w}_x\left(\frac{\pi}{k+m+1}, f\right)_X = o\big((k+m+1)^{1/p}\chi(k+m+1)\big) = o(1),$$

and by (ii) we have

$$h^{1,1,0}_{k,m,k+m-\cdot}\left[\bar{w}_x\left(\frac{\pi}{\cdot+1}, f\right)_X\right]_{T\nabla,0,1}$$

$$= \sum_{n=0}^{k} t_{k,k-n} \sum_{\mu=n}^{k+m} \frac{1}{\mu+1} \bar{w}_x\left(\frac{\pi}{\mu+1}, f\right)_X = \sum_{\mu=0}^{k+m} T_k(\mu) \frac{1}{\mu+1} \bar{w}_x\left(\frac{\pi}{\mu+1}, f\right)_X$$

$$\leqslant M \int_1^{k+m+1} T_k(u) \bar{w}_x(\pi/u, f)_X u^{-1} du = o(1) \int_1^{k+m+1} T_K(u)\chi(u) u^{1/p-1} du.$$

In view of Theorem 6 with Remark 2 the desired relation holds.

In case $p = 1$ and $m = 0$, this is a result od Dikshit [2].

For strong means we have

PROPOSITION 4. *Under the assumption of Proposition* 3 *with* (ii) *replaced by*

(ii′) $\int_1^k \chi(u)\, T_k(k-u) u^{1/p-1}\, du = O(1)$ *as* $k \to \infty$,

we have

$$H_{k,m}^{q',1}[f]_{T^\Delta}(x) = O(1) \quad \text{as } k \to \infty,$$

uniformly in $m \geqslant 0$, *for* $0 < q' \leqslant p$, $2 \leqslant p \leqslant \infty$ *and with* $X = C$ *when* $p = \infty$.

Proof. The assumption on $\bar{w}_x(\delta, f)_X$ gives

$$\sum_{n=0}^{k} t_{k,n}\left\{\left(\sum_{\mu=0}^{n} \frac{1}{\mu+1}\bar{w}_x\left(\frac{\pi}{\mu+1}, f\right)_X\right) + \left(\sum_{\mu=0}^{m} \frac{1}{\mu+m+1}\bar{w}_x^p\left(\frac{\pi}{\mu+1}, f\right)_X\right)^{1/p}\right\}$$

$$\leqslant \sum_{\mu=0}^{k} T_k(k-\mu)\frac{1}{\mu+1}\bar{w}_x\left(\frac{\pi}{\mu+1}, f\right)_X + o(1)\left\{\frac{1}{m+1}\sum_{\mu=0}^{m}(\mu+1)\chi^p(\mu+1)\right\}^{1/p};$$

whence, as before, using (i) and (ii′), by Theorem 5 with Remark 2 we have our assertion.

Now, we present the pointwise analogue of the result of Prem Chandra [5].

COROLLARY 2. *Let* ψ *be a positive function defined on* $(0, \infty)$ *such that*

$$t^{-1}\int_{\delta(t)}^{t} \psi(u)\, du = O(\psi(t)) \quad \text{as } t \to 0+,$$

with some $\delta(t) \in [0, t)$. *Then, if*

$$\int_t^{\pi} u^{-2} w_x(u, f)_X\, du = O(\psi(t)) \quad \text{as } t \to 0+,$$

we have

$$\left|\frac{1}{\lambda_k}\sum_{\mu=k-\lambda_k}^{k-1} S_\mu[f](x) - f(x)\right| = O(\lambda_k^{-1}\psi(\pi/\lambda_k)) \quad \text{as } k \to \infty,$$

where $\{\lambda_k\}$ *is the same as in Corollary* 1.

Proof. Let T_1^Δ be as in the proof of Corollary 1; then, by Theorem 4 with Remark 2,

$$\left|\frac{1}{\lambda_k}\sum_{\mu=k-\lambda_k}^{k-1} S_\mu[f](x) - f(x)\right|$$

$$= H_{k-\lambda_k,\lambda_k-1}^{r}[f]_{T_1^\Delta}(x) \leqslant M\sum_{\mu=0}^{k-1}\frac{1}{\mu+\lambda_k}\bar{w}_x\left(\frac{\pi}{\mu+1}, f\right)_X$$

$$\leqslant M\frac{1}{\lambda_k}\sum_{\mu=0}^{\lambda_k-1}\bar{w}_x\left(\frac{\pi}{\mu+1}, f\right)_X + M\sum_{\mu=\lambda_k}^{k-1}\frac{1}{\mu+1}\bar{w}_x\left(\frac{\pi}{\mu+1}, f\right)_X.$$

For the first term we use the assumption on $w_x(\delta, f)_X$ to obtain

$$\frac{1}{\lambda_k}\sum_{\mu=0}^{\lambda_k-1} w_x\left(\frac{\pi}{\mu+1}, f\right)_X \leqslant M\lambda_k^{-1}\int_{\pi/\lambda_k}^{\pi} v^{-2}w_x(v, f)_X dv = O\big(\lambda_k^{-1}\psi(\pi/\lambda_k)\big).$$

Since $\bar{w}_x(\delta, f)_X \leqslant w_x(\delta, f)_X$ and

$$t^{-1}w_x(t, f)_X \leqslant M\int_t^{\pi} u^{-2}w_x(u, f)_X du \qquad \text{for small } t>0,$$

by the assumption on ψ we obtain

$$\sum_{\mu=\lambda_k}^{k-1}\frac{1}{\mu+1}w_x\left(\frac{\pi}{\mu+1}, f\right)_X \leqslant M\int_{\pi/(k+1)}^{\pi/\lambda_k} t^{-1}w_x(t, f)_X dt$$

$$= O\Big(\int_{\pi/(k+1)}^{\pi/\lambda_k}\psi(t)dt\Big) = O\big(\lambda_k^{-1}\psi(\pi/\lambda_k)\big).$$

Hence, the second term has the desired estimate and thus the corollary follows.

Remark 3. From our results the following almost convergence criterion can be deduced analogously to [2].

Let χ be a decreasing function such that $\int_1^k \chi(u)du = O(1)$ as $k\to\infty$. If $\bar{w}_x(t, f)_{L^1} = o\big(t\chi(\pi/t)\big)$ as $t\to 0+$, then $S[f](s)$ almost converges to $f(x)$ (see definitions in [4]).

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