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Separability, duality and reflexivity of Orlicz–Besov spaces

Abstract. We prove the theorem on representation of continuous linear functionals over Orlicz–Besov spaces $B^{k,M}(\Omega)$ generated by a class of N-functions M. We give some sufficient conditions for $B^{k,M}(\Omega)$ to be separable and reflexive.

1. Preliminaries. Assume that Ω is a nonempty, open and convex set in $\mathbf{R}^n$. A function M: $\Omega\times[0, \infty)\to[0, \infty)$ is said to be a *φ-function* if

(i) $M(t, 0) = 0$ for almost every $t\in\Omega$;
(ii) $M(t, \cdot)$ is convex and continuous at zero for a.e. $t\in\Omega$;
(iii) $M(\cdot, u)$ is measurable for every $u \geqslant 0$.

A φ-function M which satisfies the condition

(iv) $M(t, u)/u\to 0$ as $u\to 0$ and $M(t, u)/u\to\infty$ as $u\to\infty$ for a.e. $t\in\Omega$.

is called an *N-function.*

Moreover, the following condition for φ-functions M will be used:

(v) $\int_A M(t, u)dt < \infty$ for every bounded set $A\subset\Omega$ and every $u\geqslant 0$.

We say that M satisfies the *condition* Δ_2 if there exists a constant $K > 0$ such that $M(t, 2u) \leqslant KM(t, u)$ for a.e. $t\in\Omega$ and every $u \geqslant 0$ (for consequences, see e.g. [5]).

Denote by X the space of all real-valued and measurable functions defined on Ω, with equality almost everywhere on Ω.

For any φ-function M we define the *Orlicz space* L^M as the set of all $f\in X$ such that $\varrho(af) < \infty$ for some $a > 0$ depending on f, where $\varrho(f) = \int_\Omega M(x, |f(x)|)dx$. The functional ϱ is a convex modular on X (see [5]).

With respect to the Luxemburg norm $\|\cdot\|_{L^M}$, defined on L^M by

$$\|f\|_{L^M} = \inf\{u > 0\colon \varrho(f/u) \leqslant 1\},$$

L^M is a Banach function space (see [5]).

If M is an N-function, then we define the *complementary N-function* N to M by

$$N(t, u) = \sup_{v>0}\{uv-M(t, v)\} \qquad \text{for } u\geqslant 0 \text{ and } t\in\Omega.$$

Let k be an arbitrary positive, noninteger number and $k = [k]+\lambda$, where $[k]$ denotes the integer part of k, $0 < \lambda < 1$. Then, for any φ-function M, we define on X a functional I by

$$I(f) = \sum_{|\alpha| \leqslant [k]} \left(\int_{\Omega} M(x, |D^\alpha f(x)|)dx + \int_{\Omega}\int_{\Omega} M\left(\frac{x+y}{2}, \frac{|\Delta(x, y)D^\alpha f|}{|x-y|^\lambda}\right) \frac{dxdy}{|x-y|^n} \right),$$

$\Delta(x, y)u = u(x)-u(y)$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex with $\alpha_i \geqslant 0$, $D^\alpha f = \partial^{|\alpha|} f / \partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}$ is the distributional derivative of f, $|\alpha| = \sum_{i=1}^n \alpha_i$. The functional I is a convex modular on X.

Further, for any fixed k, we define

$$B^{k,M}(\Omega) = \{f \in X\colon I(af) < \infty \text{ for some } a > 0\}.$$

The vector space $B^{k,M}(\Omega)$ is called the *Orlicz–Besov space* (see [4]). The space $B^{k,M}(\Omega)$ with the Luxemburg norm $\|\cdot\|_{B^{k,M}}$ generated by the convex modular I is a Banach function space ([4]).

2. Lemmas. Define

$$B = \{(x, y) \in \Omega \times \Omega\colon x = y\}.$$

For any set A in the σ-algebra Σ of Lebesgue measurable subsets of $\Omega \times \Omega$, $\Omega \subset \mathbf{R}^n$, we define the nonnegative measure ν by

$$\nu(A) = \iint_A \frac{dxdy}{|x-y|^n} \quad \text{and} \quad \nu(B) = 0. \tag{1}$$

In the following, $L^M(\Omega \times \Omega, \nu)$ is the Orlicz space of all real and measurable functions F defined on $\Omega \times \Omega$, generated by the modular

$$J(F) = \int_{\Omega}\int_{\Omega} M((x+y)/2, |F(x, y)|) d\nu(x, y).$$

The functional J is a convex modular in $L^M(\Omega \times \Omega, \nu)$ and by $\|\cdot\|_J$ we denote the Luxemburg norm generated by J.

LEMMA 2.1. *The measure ν is separable.*

Proof. Let P denote a $2n$-dimensional rectangle whose centre has rational coordinates and whose edges have rational length, and such that $\operatorname{dist}(P, B) > 0$.

Let W be a finite sum of such rectangles P. Then the family of all sets W is countable and dense in the family of sets of finite Lebesgue measure $|\cdot|$ on $\Omega \times \Omega$. Moreover, $\nu(W) < \infty$ for all W.

Let $\varepsilon > 0$. Let A be an arbitrary measurable subset of $\Omega \times \Omega$ such that $\nu(A) < \infty$. First, we suppose additionally that A is bounded. Then $\nu(A \cup W) < \infty$. Hence also $\nu(A \dot{-} W) < \infty$ for any W, where $A \dot{-} W$ is the symmetric difference.

Since $|A| < \infty$, by separability of $|\cdot|$ there exists a set W_1 such that $|A \dot{-} W_1| < \delta(\varepsilon)$. Obviously, ν is absolutely continuous with respect to $|\cdot|$. Accordingly,

$$\nu(A \dot{-} W_1) = \iint_{A \dot{-} W_1} \frac{dxdy}{|x-y|^n} < \varepsilon.$$

If A is unbounded then clearly A is a disjoint union of bounded measurable sets A_m, $m = 1, 2, \ldots$ Since $\nu(A) = \sum_{m=1}^{\infty} \nu(A_m) < \infty$ there exists k_0 such that

$$\nu\Big(\bigcup_{m=k_0+1}^{\infty} A_m\Big) < \tfrac{1}{2}\varepsilon.$$

We conclude that there exists a set W such that

$$\nu(A \dot{-} W) = \nu\Big(\bigcup_{m=1}^{k_0} A_m \dot{-} W\Big) + \nu\Big(\bigcup_{m=k_0+1}^{\infty} A_m\Big) < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

This shows that ν is separable.

Now, we shall provide an example of a set A such that $\operatorname{dist}(A, B) = 0$ and $\nu(A) < \infty$.

EXAMPLE. Let $(A_k)_{k=1}^{\infty}$ be a sequence of measurable and pairwise disjoint subsets of $\Omega \times \Omega$, $\Omega \subset \mathbf{R}^n$, such that

$$\operatorname{dist}(A_k, B) = \frac{1}{k}, \quad |A_k| = \frac{1}{(n+1)^k} \quad \text{for } k = 1, 2, \ldots$$

We set $A = \bigcup_{k=1}^{\infty} A_k$. It is easily observed that $\operatorname{dist}(A, B) = 0$ and

$$\nu(A) = \sum_{k=1}^{\infty} \iint_{A_k} \frac{dxdy}{|x-y|^n} \leqslant C \sum_{k=1}^{\infty} k^n \iint_{A_k} dxdy = C \sum_{k=1}^{\infty} \frac{k^n}{(n+1)^k} < \infty.$$

LEMMA 2.2. *The measure ν is σ-finite.*

Proof. Set

$$A_i = \{(x, y) \in \Omega \times \Omega \colon |x-y| = 1/i\}, \quad i = 1, 2, \ldots$$

Taking $K_i = \big(B(0, i) - A_i\big) \cap \Omega \times \Omega$, where $B(0, i)$ denotes the ball with centre at zero and radius i, we have

$$\bigcup_{i=1}^{\infty} K_i \cup B = \Omega \times \Omega \quad \text{and} \quad \nu(K_i) < \infty \quad \text{for every } i = 1, 2, \ldots$$

From Lemmas 2.1 and 2.2 and the theorem on separability of Orlicz spaces (see e.g. [5]) we obtain

LEMMA 2.3. *Let M be a φ-function satisfying* (v) *and the condition Δ_2. Then $L^M(\Omega \times \Omega, \nu)$ is separable.*

We now investigate the problem of continuous linear functionals over $L^M(\Omega\times\Omega, \nu)$ and the reflexivity of this space. We first prove

LEMMA 2.4. *Let $\Omega\subset \mathbf{R}^n$ be such that for all $x, y\in\Omega$ we have $\frac{1}{2}(x+y)\in\Omega$. If $A\subset\Omega$ and $|A| = 0$, then*

$$\left|\{(x, y)\in\Omega\times\Omega\colon \tfrac{1}{2}(x+y)\in A\}\right| = 0.$$

Proof. Set $C = \{(x, y)\in\Omega\times\Omega\colon \frac{1}{2}(x+y)\in A\}$. We have

$$|C| = \int_\Omega |C_u|du, \quad \text{where } C_u = \{v\in\Omega\colon (u, v)\in C\}.$$

Then

$$C_u = \{v\in\Omega\colon \tfrac{1}{2}(u+v)\in A\} = \{v\in\Omega\colon v\in 2A-u\} = \Omega\cap(2A-u)\subset 2A-u$$

for every $u\in\Omega$.

Hence $|C_u|\leqslant |2A| = 0$ for every $u\in\Omega$, and this implies $|C| = 0$.

Since $M(t, u)$, defined on the product $\Omega\times[0, \infty)$, is finite for a.e. $t\in\Omega$, in view of Lemma 2.4, the composition $M((x, y)/2, u)$, defined on $\Omega\times\Omega\times[0, \infty)$, is finite for ν -a.e. $(x, y)\in\Omega\times\Omega$. Hence and from [6] we obtain

LEMMA 2.5. *Let $(\Omega\times\Omega, \Sigma, \nu)$ be the measure space with ν defined by* (1). *Then the function $M((x+y)/2, u)\colon \Omega\times\Omega\times[0, \infty)\to[0, \infty)$ satisfies the condition* (B) (*see* [2], [3]).

Lemma 2.5 and Theorem 4.8 in [3] yield immediately

LEMMA 2.6. *Suppose $M((x+y)/2, u)\colon \Omega\times\Omega\times[0, \infty)\to[0, \infty)$ is an N-function. Then every continuous linear functional on $L^M(\Omega\times\Omega, \nu)$ is of the form*

$$F^*(G) = \int_\Omega\int_\Omega F(x, y)G(x, y)d\nu(x, y)$$

for $G\in L^M(\Omega\times\Omega, \nu)$, where $F\in L^N(\Omega\times\Omega, \nu)$ and N is the N-function complementary to M.

Simultaneously, Lemma 2.5 and Theorem 1.2 in [2] imply

LEMMA 2.7. *If $M((x+y)/2, u)\colon \Omega\times\Omega\times[0, \infty)\to[0, \infty)$ is an N-function, and both complementary functions M and N satisfy the condition Δ_2, then $L^M(\Omega\times\Omega, \nu)$ is reflexive.*

3. Separability, duality and reflexivity of $B^{k,M}(\Omega)$. Let

$$l = \sum_{|\alpha|\leqslant[k]} 1 \quad \text{and} \quad \mathcal{L}^M = \prod_{i=1}^{l}\left(L^M(\Omega)\times L^M(\Omega\times\Omega, \nu)\right).$$

For $f = (f_i, F_i)_{i=1}^l\in\mathcal{L}^M$ we define

$$\varrho(f) = \sum_{i=1}^{l}\left\{\int_\Omega M(x, |f_i(x)|)dx + \int_\Omega\int_\Omega M((x+y)/2, |F_i(x, y)|)d\nu(x, y)\right\}.$$

Obviously, ϱ is a convex modular in $\mathscr{L}^M$. Let $\|\cdot\|_{\mathscr{L}^M}$ denote the Luxemburg norm in $\mathscr{L}^M$. With this norm $\mathscr{L}^M$ is a Banach space.

Suppose that the l multiindices α satisfying $|\alpha| \leqslant [k]$ are linearly ordered so that with each $f \in B^{k,M}(\Omega)$ we may associate a well-defined vector Pf in $\mathscr{L}^M$ given by

$$Pf = \left(D^\alpha f, \frac{D^\alpha f(x) - D^\alpha f(y)}{|x-y|^\lambda}\right)_{|\alpha| \leqslant [k]}.$$

Then $\|f\|_{B^{k,M}} = \|Pf\|_{\mathscr{L}^M}$ for any $f \in B^{k,M}(\Omega)$. So P is an isometric isomorphism of $B^{k,M}(\Omega)$ onto a subspace of $\mathscr{L}^M$.

3.1. *Separability.* We shall prove the following

THEOREM 3.1. *If M is a φ-function satisfying conditions Δ_2 and* (v) *then $B^{k,M}(\Omega)$ is separable.*

Proof. $\mathscr{L}^M$ is separable as the product of a finite number of separable spaces. Since the operator P is an isomorphism of $B^{k,M}$ onto $P(B^{k,M}) \subset \mathscr{L}^M$ and is complete, $P(B^{k,M})$ is a closed subspace of $\mathscr{L}^M$. Thus $P(B^{k,M})$, and hence $B^{k,M}$, is separable.

3.2. *Duality.* From Theorem 4.8 in [3] and Lemma 6 we deduce immediately

LEMMA 3.2. *If M is an N-function satisfying condition Δ_2, then every continuous linear functional over $\mathscr{L}^M$ is of the form*

$$f^*(g) = \sum_{i=1}^{l} \left\{\int_\Omega f_i(x) g_i(x)\,dx + \int_\Omega \int_\Omega F_i(x, y) G_i(x, y) \frac{dx\,dy}{|x-y|^n}\right\}$$

for $g = (g_i, G_i)_{i=1}^l \in \mathscr{L}^M$, where $f = (f_i, F_i)_{i=1}^l \in \mathscr{L}^N$ and N is complementary to M.

Thus each $v \in \mathscr{L}^N$ defines a continuous linear functional over $B^{k,M}$ of the form

$$L(u) = \sum_{|\alpha| \leqslant [k]} \left\{\int_\Omega D^\alpha u(x) v_\alpha(x)\,dx + \int_\Omega \int_\Omega \frac{D^\alpha u(x) - D^\alpha u(y)}{|x-y|^\lambda} V_\alpha(x, y) \frac{dx\,dy}{|x-y|^n}\right\},$$

where $v = (v_\alpha, V_\alpha)_{|\alpha| \leqslant [k]}$, for $u \in B^{k,M}(\Omega)$.

THEOREM 3.2. *Let M be an N-function satisfying the condition Δ_2. Then for every continuous linear functional L over $B^{k,M}(\Omega)$ there exists $v \in \mathscr{L}^N$, $v = (v_\alpha, V_\alpha)_{|\alpha| \leqslant [k]}$, such that*

$$L(u) = \sum_{|\alpha| \leqslant [k]} \left\{\int_\Omega D^\alpha u(x) v_\alpha(x)\,dx + \int_\Omega \int_\Omega \frac{D^\alpha u(x) - D^\alpha u(y)}{|x-y|^\lambda} V_\alpha(x, y) \frac{dx\,dy}{|x-y|^n}\right\}$$

for $u \in B^{k,M}(\Omega)$.

Proof. By using Lemma 3.2, the proof is analogous to the proof of the respective theorem for the Sobolev space $W_p^k(\Omega)$, $1 < p < \infty$ (Theorem 3.8 in [1]).

3.3. *Reflexivity.*

THEOREM 3.3. *Let M be an N-function. If M and its complementary function satisfy the condition Δ_2, then $B^{k,M}(\Omega)$ is reflexive.*

Proof. $\mathscr{L}^M$ is reflexive, by Lemma 2.7. Since $B^{k,M}(\Omega)$ is isomorphic to the closed subspace $P(B^{k,M})$ of $\mathscr{L}^M$, it is also reflexive.

References

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, 1975.
[2] H. Hudzik, *On some equivalent conditions in Musielak–Orlicz spaces*, Comment. Math. 24 (1984), 57–64.
[3] A. Kozek, *Orlicz spaces of functions with values in Banach spaces*, ibid. 19 (1977), 259–288.
[4] M. Liskowski, *On generalized Besov spaces $B^{k,M}(\Omega)$*, Funct. Approx. Comment. Math. 15 (1986), 175–184.
[5] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Math. 1034, Springer, Berlin 1983.
[6] M. Wisła, *Some remarks on the Kozek Condition B*, Bull. Polish Acad. Sci. Math. 32 (1984), 407–415.

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