ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XXX (1991)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XXX (1991)

ANDRZEJ KASPERSKI (Gliwice)

Modular approximation in X^1_φ by a filtered family of sublinear operators and convex operators

Abstract. We introduce the notion of boundedness of a filtered family (T_v) of operators in a space X^1_φ of multifunctions. This notion is used to get convergence theorems for families of sublinear operators and families of convex operators.

1. Introduction. Let $L^\varphi(\Omega, \Sigma, \mu)$ be the Musielak–Orlicz function space generated by a modular $\varrho(x) = \int_\Omega \varphi(t, |x(t)|)d\mu$. Let X be the space of all multifunctions F from Ω to $2^{\mathbf{R}}$ such that $F(t)$ is nonempty and compact for every $t \in \Omega$. Two multifunctions $F, G \in X$ such that $F(t) = G(t)$ for μ-a.e. $t \in \Omega$ will be treated as the same element of X. Let $F \in X$. Define

$$\underline{f}(F)(t) = \min_{x \in F(t)} x, \qquad \overline{f}(F)(t) = \max_{x \in F(t)} x \quad \text{for } t \in \Omega,$$

$$X^1_\varphi = \{F \in X\colon F(t) \text{ is convex for every } t \in \Omega \text{ and } \underline{f}(F), \overline{f}(F) \in L^\varphi(\Omega, \Sigma, \mu)\}.$$

2. A general theorem. Let V be an abstract nonempty set and let $\mathscr{V}$ be a filter of subsets of V.

DEFINITION 1. A function $g\colon V \to \mathbf{R}$ tends to zero with respect to $\mathscr{V}$, written $g(v) \xrightarrow{\mathscr{V}} 0$, if for every $\varepsilon > 0$ there is a set $V \in \mathscr{V}$ such that $|g(v)| < \varepsilon$ for all $v \in V$.

DEFINITION 2. An operator $A\colon X^1_\varphi \to X^1_\varphi$ will be called *q-sublinear* if for both $f = \underline{f}$ and $f = \overline{f}$ we have

$$|f(A(aF+bG))(t)| \leqslant |a\underline{f}(A(F))(t)| + |a\overline{f}(A(F))(t)| + |b\underline{f}(A(G))(t)| + |b\overline{f}(A(G))(t)|$$

for all $F, G \in X^1_\varphi$ and $a, b \in \mathbf{R}$ and every $t \in \Omega$.

DEFINITION 3. A family $T = (T_v)_{v \in V}$ of operators $T_v\colon X^1_\varphi \to X^1_\varphi$ will be called *$\mathscr{V}$-bounded* if there exist positive constants $k_1, \ldots, k_8$ and a function $g\colon V \to \mathbf{R}_+$ such that $g(v) \xrightarrow{\mathscr{V}} 0$ and for all $F, G \in X^1_\varphi$ there is a set $V_{F,G} \in \mathscr{V}$ for which

$$\varrho\big(a(\underline{f}(T_v(F))-\underline{f}(T_v(G)))\big) \leqslant k_1\varrho\big(ak_2(\underline{f}(F)-\underline{f}(G))\big)$$
$$+k_3\varrho\big(ak_4(\bar{f}(F)-\bar{f}(G))\big)+g(v),$$
$$\varrho\big(a(\bar{f}(T_v(F))-\bar{f}(T_v(G)))\big) \leqslant k_5\varrho\big(ak_6(\underline{f}(F)-\underline{f}(G))\big)$$
$$+k_7\varrho\big(ak_8(\bar{f}(F)-\bar{f}(G))\big)+g(v)$$

for all $v\in V_{F,G}$ and every $a>0$.

Remark 1. If ϱ is convex, then k_1, k_3, k_5, k_7 may always be taken to be 1.

DEFINITION 4. Let $F_v\in X^1_\varphi$ for every $v\in V$ and let $F\in X^1_\varphi$. We write $F_v \xrightarrow[\varphi,\mathscr{V}]{1} F$ if for every $\varepsilon>0$ and every $a>0$ there exists $V\in\mathscr{V}$ such that

$$\varrho\big(a(\underline{f}(F_v)-\underline{f}(F))\big)<\varepsilon \quad \text{and} \quad \varrho\big(a(\bar{f}(F_v)-\bar{f}(F))\big)<\varepsilon$$

for every $v\in V$.

DEFINITION 5. Let $S\subset X^1_\varphi$. Let

$$S_{\varphi,\mathscr{V}}=\{F\in X^1_\varphi\colon\ F_v \xrightarrow[\varphi,\mathscr{V}]{1} F \quad \text{for some } F_v\in S,\ v\in V\}.$$

The set $S_{\varphi,\mathscr{V}}$ will be called the *$(\varphi,\mathscr{V})$-closure of S in X^1_φ.*

THEOREM 1. *Suppose the family $T=(T_v)_{v\in V}$ of q-sublinear operators T_v: $X^1_\varphi\to X^1_\varphi$ is $\mathscr{V}$-bounded. Let $S_0\subset X^1_\varphi$ and let $T_v(F)\xrightarrow[\varphi,\mathscr{V}]{1} 0$ for every $F\in S_0$. Let S be the set of all finite linear combinations of elements of S_0. Then $T_v(F)\xrightarrow[\varphi,\mathscr{V}]{1} 0$ for every $F\in S_{\varphi,\mathscr{V}}$.*

Proof. First, note that the assertion holds for all $F\in S$, since supposing $F=c_1F_1+\ldots+c_nF_n$ with $F_i\in S_0$ we have, writing $c=\sum_{i=1}^n|c_i|$, for $a>0$

$$\varrho\big(a\underline{f}(T_v(F))\big)\leqslant \sum_{i=1}^n \varrho\big(2ac\underline{f}(T_v(F_i))\big)+\sum_{i=1}^n \varrho\big(2ac\bar{f}(T_v(F_i))\big)\xrightarrow{\mathscr{V}} 0.$$

Now, let a, $\varepsilon>0$ be arbitrary and let $F\in S_{\varphi,\mathscr{V}}$ be given. Then there exist $G\in S$ and $V_1\in\mathscr{V}$ such that

$$\varrho\big(2ak_2(\underline{f}(F)-\underline{f}(G))\big)<\varepsilon/(4k_1), \qquad \varrho\big(2ak_4(\bar{f}(F)-\bar{f}(G))\big)<\varepsilon/(4k_3),$$
$$\varrho\big(2a\underline{f}(T_v(G))\big)<\varepsilon/4, \qquad g(v)<\varepsilon/4$$

for every $v\in V_1$, where we may assume $k_1, k_3\geqslant 1$. Let $V_{F,G}$ be chosen for $(T_v)_{v\in V}$ and F, G according to the definition of $\mathscr{V}$-boundedness. Then we have

$$\varrho\big(a\underline{f}(T_v(F))\big)\leqslant \varrho\big(2a(\underline{f}(T_v(F)))-\underline{f}(T_v(G))\big)+\varrho\big(2a\underline{f}(T_v(G))\big)$$
$$\leqslant k_1\varrho\big(2ak_2(\underline{f}(F)-\underline{f}(G))\big)+k_3\varrho\big(2ak_4(\bar{f}(F)-\bar{f}(G))\big)$$
$$+g(v)+\varrho\big(2a\underline{f}(T_v(G))\big).$$

Taking $V = V_1 \cap V_{F,G}$, we obtain $\varrho\big(a\underline{f}(T_v(F))\big) < \varepsilon$ for all $v \in V$. We prove analogously that there exists $\underline{V} \in \mathscr{V}$ such that $\varrho\big(a\overline{f}(T_v(F))\big) < \varepsilon$ for every $v \in \underline{V}$. Hence $T_v(F) \xrightarrow[\varphi,\mathscr{V}]{1} 0$ because $V_0 = V \cap \underline{V} \in \mathscr{V}$.

DEFINITION 6. An operator $A\colon X^1_\varphi \to X^1_\varphi$ will be called *m-sublinear* if for all $F, G \in X^1_\varphi$ and $a, b \in \mathbf{R}$

$$\max\big(|\underline{f}(A(aF+bG))(t)|, |\overline{f}(A(aF+bG))(t)|\big)$$
$$\leqslant \max\big(|a\underline{f}(A(F))(t)| + |b\underline{f}(A(G))(t)|, |a\overline{f}(A(F))(t)| + |b\overline{f}(A(G))(t)|\big).$$

THEOREM 2. *Let $T = (T_v)_{v\in V}$ be a $\mathscr{V}$-bounded family of m-sublinear operators $T_v\colon X^1_\varphi \to X^1_\varphi$. Let $S_0 \subset X^1_\varphi$ and let $T_v(F) \xrightarrow[\varphi,\mathscr{V}]{1} 0$ for every $F \in S_0$. Let S be the set of all finite linear combinations of elements of S_0. Then $T_v(F) \xrightarrow[\varphi,\mathscr{V}]{1} 0$ for every $F \in S_{\varphi,\mathscr{V}}$.*

The proof is quite analogous to that of Theorem 1 and we omit it.

DEFINITION 7. An operator $C\colon X^1_\varphi \to X^1_\varphi$ will be called *convex* if for both $f = \underline{f}$ and $f = \overline{f}$ we have

$$f\big(C(aF+(1-a)G)\big)(t) \leqslant af\big(C(F)\big)(t) + (1-a)f\big(C(G)\big)(t)$$

for all $F, G \in X^1_\varphi$ and $a \in [0, 1]$ and every $t \in \Omega$.

THEOREM 3. *Let $T = (T_v)_{v\in V}$ be a $\mathscr{V}$-bounded family of convex operators $T_v\colon X^1_\varphi \to X^1_\varphi$ such that for every $F \in X^1_\varphi$, $\underline{f}(T_v(F))(t) \geqslant \underline{f}(F)(t)$ and $\overline{f}(T_v(F))(t) \geqslant \overline{f}(F)(t)$ for every $t \in \Omega$ and every $v \in V$. Let $S_0 \subset X^1_\varphi$ and let $T_v(F) \xrightarrow[\varphi,\mathscr{V}]{1} F$ for every $F \in S_0$. Let now S be the set of all finite convex combinations of elements of S_0. Then $T_v(F) \xrightarrow[\varphi,\mathscr{V}]{1} F$ for every $F \in S_{\varphi,\mathscr{V}}$.*

Proof. First, the assertion holds for all $F \in S$, since supposing $F = c_1F_1 + \ldots + c_nF_n$ with $F_i \in S_0$, $c_i \geqslant 0$, $i = 1, \ldots, n$, $c_1 + \ldots + c_n = 1$ we have for both $f = \underline{f}$ and $f = \overline{f}$

$$\varrho\big(a(f(T_v(F)) - f(F))\big) \leqslant \varrho\Big(a\Big(\sum_{i=1}^n c_i(f(T_v(F_i)))\Big) - \sum_{i=1}^n c_i(f(F_i))\Big)$$
$$= \varrho\Big(a\Big(\sum_{i=1}^n c_i\big(f(T_v(F_i)) - f(F_i)\big)\Big)\Big)$$
$$\leqslant \sum_{i=1}^n \varrho\big(a(f(T_v(F_i)) - f(F_i))\big) \xrightarrow{\mathscr{V}} 0$$

for every $a > 0$.

The remainder of the proof is quite analogous to that of Theorem 1 from [2] (see also the proof of Theorem 2 from [1]) and we omit it.

Remark 2. Let the assumptions of Theorem 3 hold and assume additionally that $F(t) = -F(t)$ for all $F \in S_0$ and $t \in \Omega$. If, moreover, $T_v(aF) = aT_v(F)$ for all $a \in \mathbf{R}$, $F \in X_\varphi^1$ and $v \in V$, then we can take for S the set of all finite linear combinations of elements of S_0.

References

[1] A. Kasperski, *Modular approximation by a filtered family of sublinear operators*, Comment. Math. 27 (1987), 109–114.

[2] —, *Modular approximation in X_φ^1 by a filtered family of "linear operators"*, this fascicle, 335–341.

[3] J. Musielak, *Orlicz Spaces and Modular Spaces*, Springer, Berlin 1983.

INSTITUTE OF MATHEMATICS, SILESIAN TECHNICAL UNIVERSITY
ZWYCIĘSTWA 42, 44-100 GLIWICE, POLAND