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Exponential bounds in cones for eigenfunctions of N-body Schrödinger operators II

Abstract. We continue our study of the asymptotic behavior of square integrable solutions to the N-body Schrödinger equation. Using the concept of the Agmon metric we obtain certain L^2 exponential bounds on the decay of those solutions in a given cone.

1. Introduction. It is well known that in most situations eigenfunctions of N-body Schrödinger operators decay exponentially in all the directions of the configuration space (see [A, CT, FH1, 2, 4, H, P, Sig, FHHO, RS4, D]). This is our second paper devoted to the study of certain aspects of the dependence of this decay on the direction.

Our result is a direct continuation of the ideas of S. Agmon [A]. Let us describe in a few words the result of [A]. Suppose that H is an N-body Schrödinger operator and E is its eigenvalue that lies below the essential spectrum of H. Then Agmon shows how to construct certain positive functions r homogeneous of degree one such that if ψ is an eigenfunction of H with energy E then

$$\exp(r)\psi \in L^2. \tag{1.1}$$

The functions r are expressed in terms of the so-called Agmon metric, which in turn depends on the infima of the spectra of cluster Hamiltonians and on the value of E. For the convenience of the reader we state the above result in Theorem 2.2 (it is incidentally a special case of our main result). It is very likely that the bound (1.1) is optimal in the sense that we cannot improve the class of functions r (although we do not know how to prove this statement).

What happens if the energy E of ψ is imbedded in the essential spectrum of H? Now we cannot define the Agmon metric in the "classically forbidden regions" of the configuration space. Nevertheless, also in this case the methods of [A] lead to a bound of the form (1.1). This time, however, we know for sure that if E is not a threshold of H then the function r obtained in this way is not optimal. It is zero in the "classically forbidden regions". But it is known that if E is not a threshold then ψ decays exponentially in all the directions of the configuration space, including the "classically forbidden" ones. In particular, the result of R. Froese and I. Herbst [FH1] says that if ψ is an eigenfunction of an N-body Schrödinger operator with energy E then $\sup\{\alpha^2+E: \alpha \geqslant 0, \exp(\alpha|x|)\psi \in L^2\}$ is either a threshold or $+\infty$. Clearly, this is not optimal either if we are interested in direction dependent bounds. The main motivation that underlied [D] and this paper is the idea that we can obtain better and better bounds if we proceed step by step and apply the ideas of both [A] and [FH1]. In [D] we modified the result of [FH1] so that it can be used to improve bounds of the form (1.1). In this paper we set the result of [A] in a form which is as similar to of that of [D] as possible. Note that in spite of a similar setup the results of [D] and this paper are completely different: neither one implies the other.

As we said earlier the results of this paper are based very closely on the ideas of [A]. In fact, Theorem 1.5 of [A] contains a discussion of the decay of eigenfunctions with an imbedded eigenvalue (see also [P]). Agmon's setting is, however, very general and the reader may find it difficult to specify it to the context of N-body Schrödinger operators, which is done in our paper. Moreover, we think that our proof is somewhat simpler than that of [A].

As we mentioned earlier, both the result of [D] and of this paper can be used to improve bounds on eigenfunctions. More precisely, suppose that r is a continuous function on the configuration space homogeneous of degree one, H is an N-body Schrödinger operator and E is a real number. Then both results show how to choose functions r_1 which are homogeneous of degree one, greater than r and such that if ψ is an eigenfunction of H with energy E and $\exp(r)\psi \in L^2$ then $\exp(r_1)\psi \in L^2$. (Note that the class of functions r_1 obtained by using [D] is in general different from the class obtained in this paper).

It will be helpful to formalize the above idea as follows. Let $\mathscr{B}$ denote the set of all continuous functions on the configuration space homogeneous of degree one and let $2^{\mathscr{B}}$ denote the set of all subsets of $\mathscr{B}$. For any $A \in 2^{\mathscr{B}}$ we define

$\Gamma_1(A) := \{r_1 \in \mathscr{B}$: there exists $r \in A$ such that by using the main theorem of [D] we can show that if ψ is an eigenfunction of H of energy E then $\exp(r)\psi \in L^2$ implies $\exp(r_1)\psi \in L^2\}$.

We define $\Gamma_2(A)$ in the same way except that "[D]" is replaced by "this paper". Now the following statement is true.

Let ψ be an eigenfunction of an N-body Schrödinger operator with eigenvalue E. Suppose that

$$r \in \bigcup_{k=0}^{\infty} \underbrace{\Gamma_1\Gamma_2 \dots \Gamma_1\Gamma_2}_{k \text{ times}}(\{0\}). \tag{1.2}$$

(Here 0 denotes the zero function on the configuration space). Then

$$\exp(r)\psi \in L^2. \tag{1.3}$$

To see this note that $0 \in \mathscr{B}$ and $\exp(0)\psi \in L^2$. After repeated successive applications of [D] and of Theorem 2.1 of this paper we obtain better and better estimates which imply (1.3).

It would be nice to have an explicit description of the set of functions that appears in (1.2). Unfortunately, we do not know how to do this. Note, however, that there exists a conjecture on possible behavior of eigenfuctions of N-body Schrödinger operators due to R. Froese and I. Herbst [FH2]. Unfortunately, the results of this paper and of [D] are too weak to prove this conjecture.

It is worth noting that under certain assumptions on the potentials the L^2-estimates of the form (1.1) can be converted into pointwise estimates. This can be done by the methods described for instance in [A] and [RS4].

2. Notation and main result. Our notation will be very similar to that of [D]. The class of Schrödinger operators that we will study will resemble that of [A, D, FH1, 3] but the assumptions on the potentials will be slightly different.

Let $X := \mathbf{R}^n$. Let $\{\pi_i\}_{i=1}^N$ be a set of nonzero orthogonal projections on X with ranges X_i. Let x_i and Δ_i resp. denote the variable and the Laplacian in X_i resp. If $x \in X$ then $|x|$ denotes the Euclidean norm of x.

Define the self-adjoint operator H on $L^2(X)$ by

$$H := -\Delta + \sum_i V_i(\pi_i(x)), \qquad Q(H) := Q(-\Delta),$$

where V_i is a real-valued function such that $|V_i|^{1/2}(-\Delta_i + 1)^{-1/2}$ is compact on $L^2(X_i)$ and $Q(H)$ and $Q(-\Delta)$ denote form domains of H and $-\Delta$ respectively (see [RS1]).

Let $\mathscr{L}$ be the set of all subspaces of X of the form $\bigvee_{i \in I} X_i$ for $I \subset \{1, \ldots, N\}$ where $\bigvee$ denotes the subspace sum.

For $Z \in \mathscr{L}$ define

$$I(Z) := \{i \in \{1, \ldots, N\} \colon X_i \subset Z\}.$$

Define also the cluster Hamiltonian corresponding to Z:

$$H_Z := -\Delta + \sum_{i \in I(Z)} V_i(\pi_i(x)), \qquad Q(H_Z) := Q(Z).$$

If $\omega \in S^{n-1}$ then

$$I_\omega := \{i \in \{1, \ldots, N\} \colon \pi_i(\omega) = 0\}, \qquad Z(\omega) := \bigvee_{i \in I_\omega} X_i.$$

$\Xi(\cdot)$ will denote the function on S^{n-1} defined by $\Xi(\omega) := \inf \operatorname{sp} H_{Z(\omega)}$. Note that by the Hunziker–Van Winter–Zhislin theorem (see [RS4]) the infimum of Ξ is equal to the infimum of the essential spectrum of H.

If Q is a measurable subset of $\mathbf{R}^n$ then $\chi(Q)$ denotes the characteristic function of Q and if $\psi \in L^2(\mathbf{R}^n)$ then $\|\psi\|_Q := \|\chi(Q)\psi\|$.

$B(r)$ will denote the ball of radius r and center 0. If $U \subset S^{n-1}$ then

$$\Omega(U) := \{x \in \mathbf{R}^n \backslash \{0\} \colon x/|x| \in U\}.$$

We will prove the following theorem.

THEOREM 2.1. *Let* $\psi \in Q(H)$ *and* $H\psi = E\psi$. *Suppose that* $W \subset U \subset S^{n-1}$ *where* U *is open and* W *is closed. Let* $\Xi - E$ *be positive on* $\bar{U}$ *and let* g *be a nonnegative function of degree* 1 *such that*

$$\|\exp(g)\psi\|_{\Omega(U\setminus W)} < \infty.$$

Let μ *be a function on* $\Omega(\bar{U})$ *which is continuous outside zero and homogeneous of degree zero with the property that* $0 \leqslant \mu < \Xi - E$. *For* $x, y \in \Omega(U)$ *we define*

$$\varrho_\mu(x, y) := \inf_\gamma \int_0^1 \sqrt{\mu(\gamma(s))}|\dot{\gamma}(s)|\, ds$$

where $[0, 1] \ni s \mapsto \gamma(s) \in \Omega(U)$ *are absolutely continuous paths from* x *to* y *contained in* $\Omega(U)$. *Define also*

$$r_{\mu,g}(x) := \inf_{y \in \Omega(U\setminus W)} \big(\varrho_\mu(y, x) + g(y)\big).$$

Then $r_{\mu,g}$ *is homogeneous of degree one and*

$$\|\exp(r_{\mu,g})\psi\|_{\Omega(W)} < \infty. \tag{2.1}$$

For comparison let us formulate the special case of the above theorem when $W = U = S^{n-1}$. It will be essentially equivalent to the result of S. Agmon contained in Theorem 4.9 of [A] (with slightly different conditions on the potentials).

THEOREM 2.2. *Let* $\psi \in Q(H)$ *and* $H\psi = E\psi$. *Let* E *lie below the essential spectrum of* H. *Suppose that* μ *is a function on* $\Omega(\bar{U})$ *which is continuous outside zero and homogeneous of degree zero such that* $0 \leqslant \mu < \Xi - E$. *For* $x, y \in \mathbf{R}^n$ *we define*

$$\varrho_\mu(x, y) := \inf_\gamma \int_0^1 \sqrt{\mu(\gamma(s))}|\dot{\gamma}(s)|\, ds$$

where $[0, 1] \ni s \mapsto \gamma(s) \in \Omega(U)$ *are absolutely continuous paths from* x *to* y. *Then* $\varrho_\mu(\cdot, 0)$ *is homogeneous of degree one and*

$$\|\exp(\varrho_\mu(\cdot, 0))\psi\| < \infty.$$

3. Geometric spectral analysis. This section is devoted to a version of Lemma 2.3 of [A].

Define

$$\lambda^\varepsilon(x) := \inf\{\inf \operatorname{sp} H_Z\colon \ \forall_{i \in I(Z)} \ |\pi_i(x/|x|)| < \varepsilon\},$$

$$\lambda^\varepsilon_R(x) := \begin{cases} \lambda^\varepsilon(x) & \text{for } |x| > R, \\ \inf \operatorname{sp} H & \text{for } |x| \leqslant R. \end{cases}$$

LEMMA 3.1.

$$\forall_{\delta>0}\ \exists_{R}\ \forall_{\phi\in Q(H)}\ (\phi, H\phi) \geqslant \int(\lambda_R^{\varepsilon}(x)-\delta)|\phi(x)|^2dx. \tag{3.1}$$

Proof. Introduce the family $\{\mathscr{U}_Z^{\varepsilon}\colon Z\in\mathscr{L}\backslash\{X\}\}$ of open subsets of S^{n-1} defined by

$$\mathscr{U}_Z^{\varepsilon} := \{\omega\in S^{n-1}\colon \forall_{i\in I(Z)} |\pi_i(\omega)| < \varepsilon \text{ and } \forall_{i\notin I(Z)} \pi_i(\omega)\neq 0\}.$$

This family covers S^{n-1} (see [D] and [FH3]). Let $\{\tilde{\eta}_Z\colon Z\in\mathscr{L}\backslash\{X\}\}$ be a collection of C^∞ functions on X which are homogeneous of degree zero outside the unit ball with the properties that $\operatorname{supp}\tilde{\eta}_Z\cap S^{n-1}\subset\mathscr{U}_Z^{\varepsilon}$ and that

$$\sum_Z \tilde{\eta}_Z^2 = 1.$$

Let $f, \tilde{f}\in C^\infty(\mathbf{R})$, $f^2+\tilde{f}^2 = 1$, $f(t) = 1$ for $|t| < \frac{1}{2}$ and $f(t) = 0$ for $|t|\geqslant 1$. Let $f_R(x) := f(|x|/R)$ and $\tilde{f}_R(x) := \tilde{f}(|x|/R)$. Define $\eta_{Z,R} := \tilde{f}_R\tilde{\eta}_Z$.

Notice that $f_R^2+\sum_Z\eta_{Z,R}^2 = 1$ and $|\nabla f_R|^2+\sum_Z|\nabla\eta_{Z,R}|^2 \leqslant c/R^2$.

Now let $\phi\in C_0^\infty(\mathbf{R}^n)$. Using the IMS equation (see e.g. [Sig] and [FH3]) we obtain

$$\begin{aligned}
(3.2)\quad &(\phi, H\phi)\\
&= (f_R\phi, Hf_R\phi)+\sum_Z(\eta_{Z,R}\phi, H\eta_{Z,R}\phi)-(\phi, |\nabla f_R|^2\phi)-\sum_Z(\phi, |\nabla\eta_{Z,R}|^2\phi)\\
&\geqslant (f_R\phi, Hf_R\phi)+\sum_Z\big[(\eta_{Z,R}\phi, H_Z\eta_{Z,R}\phi)\\
&\quad+\sum_{i\notin I(Z)}\big(\eta_{Z,R}\phi, V_i\chi(\{x\colon |\pi_i(x)| < T\})\eta_{Z,R}\big)\\
&\quad+\sum_{i\notin I(Z)}\big(\eta_{Z,R}\phi, V_i\chi(\{x\colon |\pi_i(x)| \geqslant T\})\eta_{Z,R}\big)\big]-\frac{c}{R^2}(\phi, \phi).
\end{aligned}$$

Since $|V_i|^{1/2}(1-\Delta_i)^{-1/2}$ is compact on $L^2(X_i)$ and $\chi(\{x\colon |\pi_i(x)|\geqslant T\})$ goes strongly to zero as $T\to\infty$, we have

$$\lim_{T\to\infty}\|(1-\Delta_i)^{-1/2}V_i\chi(\{x\colon |\pi_i(x)|\geqslant T\})(1-\Delta_i)^{-1/2}\| = 0.$$

Consequently, for any $\gamma > 0$ we can find T such that the fourth term on the right-hand side of (3.2) is bounded from below by

$$-\gamma\sum_Z(\eta_{Z,R}\phi, (H_Z-\inf\operatorname{sp}H_Z+1)\eta_{Z,R}\phi).$$

If we choose R large enough then the third term on the right-hand side of (3.2) is zero. Thus

$$(\phi, H\phi)\geqslant \inf\operatorname{sp}H(\phi, f_R^2\phi)+\sum_Z(\inf\operatorname{sp}H_Z-\gamma)(\phi, \eta_{Z,R}^2\phi)-\frac{c}{R^2}(\phi, \phi).$$

Next notice that $\sum_i V_i - \lambda_R^\varepsilon$ is form bounded with respect to $-\Delta$ with an arbitrarily small bound. Thus the KLMN theorem (see [RS2]) implies that $H-\lambda_R^\varepsilon$ is self-adjoint and $C_0^\infty(\mathbf{R}^n)$ is its form core. Since $H-\lambda_R^\varepsilon+\delta$ is positive on its form core it is also positive on its form domain, which means that (3.1) is true for all $\phi \in Q(-\Delta)$. ∎

4. Decay of eigenfunctions. First we study the properties of the function $r_{\mu,g}$. The following lemma is closely related to Theorem 1.4 of [A].

LEMMA 4.1. (a) *$r_{\mu,g}$ is homogeneous of degree one.*

(b) *$r_{\mu,g} \leqslant g$ on $\Omega(U \setminus W)$.*

(c) *$r_{\mu,g}$ has a derivative a.e. and at the points where it exists we have $|\nabla r_{\mu,g}|^2 \leqslant \mu$.*

Proof. (a) and (b) are obvious. We will prove (c). Let $x \in \Omega(U)$, $v \in \mathbf{R}^n$ and $v \neq 0$. Then for sufficiently small t we have $x+vt \in \Omega(U)$ and

$$(4.1)\qquad |r_{\mu,g}(x+vt)-r_{\mu,g}(x)| \leqslant \inf_{y\in\Omega(U\setminus W)} |\varrho_\mu(y, x+vt)-\varrho_\mu(y, x)|$$

$$\leqslant \varrho_\mu(x, x+vt) \leqslant \int_0^t \sqrt{\mu(x+vs)}|v|\,ds \leqslant |v|t\sup\sqrt{\mu}.$$

Thus $r_{\mu,g}$ is Lipschitz continuous, which implies that it has a derivative a.e. If we take the limit $t \to 0$ in (4.1) we obtain $v\cdot\nabla r_{\mu,g} \leqslant \sqrt{\mu}|v|$, which implies (c).

Now we are ready for the proof of the main theorem.

Proof of Theorem 2.1. By the lower semicontinuity of Ξ we can find $\varepsilon, \delta > 0$ such that $\lambda^\varepsilon - 2\delta \geqslant \mu + E$. Let h be a C^∞ function which is homogeneous of degree zero outside the unit ball, $0 \leqslant h \leqslant 1$, $\operatorname{supp} h \subset \Omega(U)$ and $h = 1$ on $\Omega(W)\setminus B(1)$. Let $F_T(x) := \min(r_{\mu,g}(x), T)$ and $\psi_T := h\exp(F_T)\psi$. It is easy to see that $\psi_T \in Q(-\Delta)$. By Lemma 3.1 applied to ψ_T we obtain

$$(\psi_T, (\lambda_R^\varepsilon-\delta)\psi_T) \leqslant (\psi_T, H\psi_T) = E\|\psi_T\|^2 + (\exp(F_T)\psi, (\nabla h + h\nabla F_T)^2\exp(F_T)\psi).$$

Next note that for some c we have

$$E+\mu+\delta \leqslant \lambda_R^\varepsilon - \delta + c\chi(B(R)).$$

Thus

$$\begin{aligned}\delta\|\psi_T\|^2 &\leqslant (\psi_T, (\mu+\delta-|\nabla F_T|^2)\psi_T)\\ &\leqslant (\exp(F_T)\psi, (|\nabla h|^2 + 2\nabla h\cdot\nabla F_T h)\exp(F_T)\psi) + c\|\psi_T\|_{B(R)}\\ &\leqslant (1+1/a)\|\nabla h\exp(F_T)\psi\|^2 + a\|\nabla F_T\psi_T\|^2 + c\|\psi_T\|_{B(R)}.\end{aligned}$$

Now note that $|F_T|$ and $|\nabla h|$ are uniformly bounded and $\operatorname{supp}\nabla h \subset \Omega(U\setminus W) \cup B(1)$. We choose $a > 0$ such that $a|\nabla F_T|^2 < \frac{1}{2}\delta$. Then

$$\tfrac{1}{2}\delta\|\psi_T\|^2 \leqslant (1+1/a)\|\exp(F_T)\psi\|^2_{\Omega(U\setminus W)} + c\|\exp(F_T)\psi\|^2_{B(R)}.$$

If we let $T \to \infty$ we obtain

$$\tfrac{1}{2}\delta \|\exp(r_{\mu,g})\psi\|^2_{\Omega(W)\setminus B(1)} \leqslant (1+1/a)\|\exp(g)\psi\|^2_{\Omega(U\setminus W)} + c\|\exp(r_{\mu,g})\psi\|^2_{B(R)},$$

which implies (2.1). ∎

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