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Sequences in the range of a vector measure

Abstract. Let X be a Banach space with dual X^* . We give some necessary and some sufficient conditions for a sequence (x_n) in X to lie in the (closed convex hull of the) range of an X -valued countably additive measure. As an improvement of a theorem of Diestel and Seifert we prove that any weakly null sequence in the range of a measure admits a subsequence that has norm null Cesaro means, with any choice of \pm signs. The sequence (x_n) is said to be in $l^2_{\text{weak}}(X)$ in case the sequence $(x^*(x_n))$ of scalars is in l^2 for each x^* in X^* . We prove that this condition on (x_n) is sufficient for it to lie in the range of an X -valued measure. Although the converse is false in general even if $X = \mathbf{R}$, we prove that if X has a normalized unconditional basis that lies in the range of an X -valued measure, then the basis is in $l^2_{\text{weak}}(X)$. Our tools are: taking averages with respect to the sequence of Rademacher functions, and a classical theorem due to Orlicz on unconditionally convergent series in L^1 . In case $X = c_0, l^p$, or L^p with $2 \leq p < \infty$, every weakly null sequence in X admits a subsequence that is in $l^2_{\text{weak}}(X)$. Finally, we give a number of norm (or weakly) null sequences in some of the classical Banach spaces that cannot lie in the range of any measure; e.g. the sequence (e_n/\sqrt{n}) in l^1 .

Questions regarding the finer structure of the range of a vector-valued measure have found interest (and their answers, application) since Lyapunoff's discovery of his everintriguing convexity theorem. It is an important fact, first established by Bartle, Dunford and Schwartz in the early fifties, that the range of a vector-valued measure is always relatively weakly compact; here we might as well settle on the phrase "range of a vector-valued measure" as meaning a set of the form $\{F(E): E \in \Sigma\}$, where Σ is a σ -field of sets and F is a countably additive measure on Σ with values in an appropriate Banach space X with dual X^* .

Among the relatively weakly compact subsets of Banach spaces, those that're the range of a vector-valued measure occupy a special place; a remarkable similarity to the relatively norm compact sets is evidenced. For instance, any sequence in the range of a vector-valued measure admits a subsequence with norm convergent arithmetic means, a phenomenon not shared by all weakly compact sets. ([DS]; see also [A]). Again, the closed convex hull and absolutely closed convex hull of the range of a vector-valued measure are each, themselves, the range of a vector-valued measure. A closed convex set that's the range of a vector-valued measure has extreme points, denting points, exposed points and strongly exposed points, as does any weakly compact convex set;

however, any closed convex range of a vector-valued measure enjoys the coincidence of extreme points and denting points, as well as that of exposed points and strongly exposed points, an enjoyment shared with norm compact convex sets but not with all weakly compact convex sets ([DU1, Ch. IX], [KK]). Finally, any extreme point of the closed convex hull of the range of a vector-valued measure lies inside the range of the measure ([DU1], [KK]) – a stunning improvement on Milman’s converse to the Krein–Milman theorem in the case of vector-valued measures.

Any intuition gained by noting the similarities between relatively norm compact sets and sets arising as ranges of vector-valued measures must be tempered by the fact that the closed unit ball of an infinite dimensional Banach space *can* be the range of a vector-valued measure [KK]. Naturally, this is a special occurrence and a few choice words will be expanded on it.

Suppose the closed unit ball B_X is the range of the vector-valued measure $F: \Sigma \rightarrow X$. Following the lead of Bartle, Dunford and Schwartz, [Du–S, Ch. IV], we can find a probability measure μ on Σ , having the same null sets as does F ; in a straightforward fashion we can define the integration operator $I: L^\infty(\mu) \rightarrow X$ by $I f = \int f dF$ and quickly establish I as a weak*-weak continuous linear operator taking $B_{L^\infty(\mu)}$ onto B_X , i.e., I is a weakly compact (metric) quotient operator of $L^\infty(\mu)$ onto X . X must be reflexive. More can be said: X^* is (isometrically isomorphic to) a reflexive subspace of $L^\infty(\mu)^*$; in fact, I^* takes X^* into $L^1(\mu)$ isometrically with $I^* x^* = dx^* F/d\mu$, where $x^* F$ is the scalar measure $(x^* F)(E) = x^*(F(E))$ and $dx^* F/d\mu$ denotes the Radon–Nikodym derivative of $x^* F$ with respect to μ . On the other hand, if X^* is (isometrically isomorphic to) a reflexive subspace of $L^1(\mu)$ for some probability measure μ , then $X = X^{**}$ is a (metric) quotient of $L^\infty(\mu)$. Of course, the quotient map is weak*-weak continuous and so defining $F(E)$ to be the image in X of $\chi_E \in L^\infty(\mu)$ produces a countably additive measure F on Σ ; while F ’s range need not be the closed unit ball of X , the absolutely closed convex hull of F ’s range is the closed unit ball of X and, as mentioned earlier, is the range of a vector-valued measure. So, we have the

PROPOSITION A (See e.g. [AG]). *The closed unit ball B_X of X is the range of a measure iff X^* (with the dual norm) is isometrically isomorphic to a reflexive subspace of $L^1(\mu)$ for some probability measure μ .*

It is known (e.g. [R2]) that L^q embeds isometrically into L^1 for $1 < q \leq 2$; so by duality we get ([DU1], [KK]): *the closed unit ball of L^p is the range of a measure if $2 \leq p < \infty$* ; in fact, this is “iff” ([DU1], [KK]), but we need only the “if” part.

It is, of course, a celebrated result of H. P. Rosenthal [R2] that classifies the reflexive subspace of $L^1(\mu)$ ’s as isomorphs of subspaces of $L^p(v)$ for some $1 < p \leq 2$ and some v .

Actually the geometry of the range of a vector measure has long been known to be in the image and likeness of sets in L^p spaces for $p \geq 2$; indeed, I. Kluvánek [K] gave an incisive description of those absolutely closed convex sets that arise as the range of a vector measure involving negative-definite functionals reminiscent of the work of Schoenberg [S] and, of course, Bretagnolle, Dacunha-Castelle and Krivine [BDK].

Though much more than mentioned above is known about the range of a vector measure, much remains mysterious concerning these sets. One old chestnut [DU2, p. 192] that has remained enigmatic concerns the recognition of sequences whose values are in the range of a measure. In this note we give some necessary and some sufficient results for a sequence to be in the range of a measure. In addition, we provide a number of examples that illustrate, to some extent, constraints that being in the range of a vector measure place on a norm or weakly null sequence in special classical spaces. Our method is the one used in [KK] exploiting averages with respect to incomplete O.N.S. in $L^2[0, 1]$, especially the sequence (r_n) of Rademacher functions; we employ it as follows.

Recall that a sequence (x_n) of members of a Banach space X is said to belong to $l_{\text{weak}}^2(X)$ provided for each $x^* \in X^*$, the dual of X , we have

$$\sum_n |x^*(x_n)|^2 < \infty;$$

it is easy to see that if $(x_n) \in l_{\text{weak}}^2(X)$ then

$$\|(x_n)\|_{l_{\text{weak}}^2} = \sup \left\{ \left(\sum_n |x^*(x_n)|^2 \right)^{1/2} : \|x^*\| \leq 1 \right\} < \infty$$

and $(l_{\text{weak}}^2(X), \|\cdot\|_{l_{\text{weak}}^2})$ is itself a Banach space.

We need the next two propositions that list some of the privileges that membership in $l_{\text{weak}}^2(X)$ entails.

PROPOSITION B. *The following are equivalent:*

- (i) $(x_n) \in l_{\text{weak}}^2(X)$.
- (ii) *The series $\sum_{n=1}^{\infty} \alpha_n x_n$ converges unconditionally for every sequence $(\alpha_n) \in l^2$.*
- (iii) *The map $(\alpha_n) \rightarrow \sum_{n=1}^{\infty} \alpha_n x_n$, $(\alpha_n) \in l^2$, defines a bounded operator $T: l^2 \rightarrow X$.*

The proofs of (i) \Rightarrow (ii) \Rightarrow (iii) may be carried out (without circularity) analogously to the first part of Theorem 3 below. To see that (iii) \Rightarrow (i) we observe that for every $x^* \in X^*$ the series $\sum_{n=1}^{\infty} \alpha_n x^*(x_n) = x^* \circ T((\alpha_n))$ converges (by assumption) for every sequence $(\alpha_n) \in l^2$, whence (i).

Let us denote by (e_n) the standard basis in l^2 ; since the sequence $(n^{-1} \sum_{k=1}^n e_k)_n$ of arithmetic means of (e_n) converges in norm to zero, we get from part (iii)

COROLLARY C. *If $(x_n) \in l_{\text{weak}}^2(X)$ then the sequence (σ_n) of arithmetic means of (x_n) converges in norm to zero.*

For the next proposition, we note that if $X = c_0$ or l^p with $1 \leq p < \infty$, then the standard basis $(e_n) \in l_{\text{weak}}^1(X)$ or $l_{\text{weak}}^q(X)$, respectively, where q is, as usual, the index conjugate to p ; $l_{\text{weak}}^q(X)$ has a definition analogous to that of $l_{\text{weak}}^2(X)$. The following may be compared with [D, p. 116, Ex. 16] which it strengthens.

PROPOSITION D. *Let $X = c_0, l^p$ or L^p with $2 \leq p < \infty$. Then every weakly null sequence in X admits an $l_{\text{weak}}^2(X)$ subsequence.*

Proof. Let (x_n) be a weakly null sequence in X ; if (x_n) has a norm null subsequence then the latter clearly satisfies the conclusion. Hence, by passing to a subsequence if necessary, let us assume $\inf\{\|x_n\|: n \in \mathbf{N}\} > 0$. We may now apply the Bessaga–Pełczyński Selection Principle [D, p. 46] to extract a subsequence (x_{n_k}) of (x_n) that is equivalent to a block basic sequence (u_k) taken with respect to (e_n) . By another classical theorem [LT, p. 53], every such (u_k) is always equivalent to (e_n) for $X = c_0$ or l^p ($1 \leq p < \infty$). Thus, (x_{n_k}) is equivalent to (e_n) , and as noted (before this proposition) for our spaces X , (e_n) is then in $l_{\text{weak}}^r(X)$ with $r = 1$ or $1 < r \leq 2$, i.e. $(e_n) \in l_{\text{weak}}^2(X)$; so is the subsequence (x_{n_k}) .

Finally, let $X = L^p$, and assume as we may, that the (block basic) sequence (u_k) is normalized. Then, according to a theorem due to Kadec and Pełczyński [KP], either (u_k) is equivalent to the basis (e_n) in l^2 or there is a subsequence (v_i) of (u_k) that is equivalent to the basis (e_n) in l^p ; as $p \geq 2$, the proof may be completed (as before) by observing that $(e_n) \in l_{\text{weak}}^2$ in both cases.

Our first result provides a strong necessary condition for a sequence to be inside the range of a vector-valued measure. It constitutes an improvement on an observation of Diestel and Seifert [DS] though its proof follows more closely the pattern of Anantharaman [A].

THEOREM 1. *Let Σ be a σ -field, X be a Banach space and $F: \Sigma \rightarrow X$ be countably additive. Then any weakly null sequence (x_n) in $F(\Sigma)$ has a subsequence (y_n) each subsequence (z_n) of which enjoys*

$$\text{norm} \lim_n \frac{1}{n} \left(\sum_{k=1}^n \varepsilon_k z_k \right) \text{ is zero}$$

regardless of the choice of signs $\varepsilon_k = \pm 1$.

Proof. Applying the Bartle–Dunford–Schwartz theorem [DU1, p. 14] there exists a probability measure λ on Σ such that F is λ -continuous. Let $T: L^\infty(\lambda) \rightarrow X$ be the integration map $Tf = \int f dF$, K be the closed convex hull of $F(\Sigma)$ and P be the set $\{f \in L^\infty(\lambda): 0 \leq f \leq 1, \lambda\text{-a.e.}\}$. It is easily seen that $T(P) = K$.

Now take the weakly null sequence (x_n) in K and find a sequence (f_n) in

P such that $Tf_n = x_n$. P is both weak* compact in $L^\infty(\lambda)$ and weakly compact in $L^1(\lambda)$ with these topologies coinciding on P ; therefore (f_{n_k}) has a weak* convergent subsequence with weak* limit $f \in P$ thanks to the Eberlein–Shmul'yan theorem. Of course, T is weak* to weak continuous and so $Tf = T(\text{weak}^*\text{-}\lim_k f_{n_k}) = \text{weak}\lim_k Tf_{n_k} = \text{weak}\lim_k x_{n_k} = 0$. Again, the weak topology of $L^2(\lambda)$ and the weak* topology of $L^\infty(\lambda)$ coincide on $P-f$, so the weak* null sequence $(f_{n_k} - f)$ is weakly null when viewed in $L^2(\lambda)$. But in Hilbert spaces weakly null sequences always have l^2_{weak} -ly summable subsequences (as follows from Proposition D, for instance). Therefore, there's a subsequence $(f_{n_{k_j}} - f)$ of $(f_{n_k} - f)$ that's in $l^2_{\text{weak}}(L^2(\lambda))$. Plainly, given any signs $\varepsilon_j = \pm 1$ we have $(\varepsilon_j [f_{n_{k_j}} - f])$ in $l^2_{\text{weak}}(L^2(\lambda))$, too; by Corollary C the arithmetic means are norm convergent in $L^2(\lambda)$ and, further, every one of the subsequences also belongs to $l^2_{\text{weak}}(L^2(\lambda))$ and so each of them has norm convergent arithmetic means in $L^2(\lambda)$. Now T 's restriction to $B_{L^\infty(\lambda)}$ is continuous with respect to the $L^2(\lambda)$ norm topology thanks to Egorov's theorem and so if (σ_m) denotes the arithmetic means of any subsequence of $(\varepsilon_j [f_{n_{k_j}} - f])$, $\lim_m \|\sigma_m\|_2 = 0$. Now $\lim_m \|T\sigma_m\| = 0$, too. If we let $y_j = T(f_{n_{k_j}} - f)$, then (y_j) is a subsequence of (x_n) —after all $Tf = 0$ —with the advertised properties.

EXAMPLE 2. For $1 \leq p < \infty$, the sequence (r_n) of Rademacher functions, viewed in $L^p[0, 1]$ lies in the range of a measure.

Indeed, let $S = [-1, 1]$, Σ be the σ -field of Lebesgue measurable subsets of S and λ denote Lebesgue measure; define $F: \Sigma \rightarrow L^p[0, 1]$ by

$$F(E) = \chi_{E \cap [0, 1]} - \chi_{-(E \cap [-1, 0])}$$

where χ_A denotes the indicator function of $A \in \Sigma$ and $-A = \{-t: t \in A\}$. F is plainly additive and, since for $E \in \Sigma$

$$\|F(E)\|_p \leq [\lambda(E \cap [0, 1]) + \lambda(E \cap [-1, 0])]^{1/p} = [\lambda(E)]^{1/p},$$

F is λ -continuous and countably additive, too. Letting B_n be the set $\{t \in [0, 1]: r_n(t) = +1\}$ and C_n be $\{t \in [0, 1]: r_n(t) = -1\}$ we see that

$$F(B_n \cup -C_n) = \chi_{\{t \in [0, 1]: r_n(t) = +1\}} - \chi_{\{t \in [0, 1]: r_n(t) = -1\}} = r_n.$$

The above example is, in a sense, the prototype for the best criteria we have been able to uncover regarding sequences that're in the range of a measure.

Our next result builds on our prototype in a more or less natural manner. The measure F below is the composition of T in Proposition B with the measure $E \rightarrow (2 \int_E r_n d\lambda)$ into L^2 .

THEOREM 3. Let $(x_n) \in l^2_{\text{weak}}(X)$. Then (x_n) lies in the range of an X -valued countably additive measure defined on a suitable σ -field.

More precisely, if Σ denotes the σ -field of Lebesgue measurable subsets of $[0, 1]$ and λ denotes Lebesgue measure, then for any sequence $(x_n) \in l^2_{\text{weak}}(X)$ there is an X -valued countably additive λ -continuous measure defined on Σ whose range contains (x_n) .

Proof. Define the signed measures λ_k on Σ by

$$\lambda_k(E) = \int_E r_k(t) d\lambda(t), \quad *$$

where (r_k) is the sequence of Rademacher functions and Σ, λ are as cited in the statement of Theorem 3. Now define $F: \Sigma \rightarrow X$ by

$$F(E) = 2 \sum_{k \geq 1} \lambda_k(E) x_k.$$

For $E \in \Sigma, \chi_E \in L^2[0, 1]$ and so $(\lambda_k(E)) \in l^2$. But $(x_k) \in l^2_{\text{weak}}(X)$ and so if we let m, n be given with $m < n$, then for any $E \in \Sigma$,

$$\begin{aligned} \left\| \sum_{k=m}^n \lambda_k(E) x_k \right\| &= \sup_{x^* \in B_{X^*}} \left\{ x^* \left(\sum_{k=m}^n \lambda_k(E) x_k \right) \right\} \\ &\leq \sup_{x^* \in B_{X^*}} \left\{ \sum_{k=m}^n |\lambda_k(E)| |x^* x_k| \right\} \\ &\leq \sup_{x^* \in B_{X^*}} \left\{ \left(\sum_{k=m}^n |\lambda_k(E)|^2 \right)^{1/2} \left(\sum_{k=m}^n |x^* x_k|^2 \right)^{1/2} \right\} \\ &\leq \| (x_k) \|_{l^2_{\text{weak}}} \left(\sum_{k=m}^n |\lambda_k(E)|^2 \right)^{1/2} \rightarrow 0 \end{aligned}$$

as $m, n \rightarrow \infty$. It follows that F is indeed X -valued. Plainly F is additive. By estimating like above we see that for any $E \in \Sigma$,

$$\|F(E)\| \leq 2 \| (x_k) \|_{l^2_{\text{weak}}} \| (\lambda_k(E)) \|_{l^2} \leq 2 \| (x_k) \|_{l^2_{\text{weak}}} \sqrt{\lambda(E)}.$$

It follows that F is countably additive and λ -continuous.

Finally,

$$F([r_n = +1]) = 2 \sum_{k \geq 1} \lambda_k([r_n = +1]) x_k = 2 \sum_{k \geq 1} \int_{[r_n = +1]} r_k(t) d\lambda(t) x_k = 2 \cdot \frac{1}{2} x_n$$

with the help of (r_n) 's independence.

A remark ought to be made here, namely, on realizing that any $(x_n) \in l^2_{\text{weak}}$ is in the range of some measure, we found the pertinent measure F satisfied an inequality of the form $\|F(E)\| \leq k \sqrt{\lambda(E)}$, an inequality that ensures $\int \cdot dF$ to be a bounded linear operator not just from $L^\infty(\lambda)$ to X but even from the Lorentz function space $L_{2,1}$ to X (see [H]).

As our next example indicates, Theorem 3 does not tell the whole story.

EXAMPLE 4. Consider the sequence (x_n) of norm-one vectors in l^2 given by

$$x_n = n^{-1/2} \sum_{k=n+1}^{2n} e_k,$$

where (e_k) is the usual unit coordinate vector basis of l^2 .

As we have noted after Proposition A, B_{l^2} is the range of a countably additive measure defined on a suitable σ -field and so (x_n) lies in the range of such a measure.

It is easy to see that (x_n) is weakly null. In fact if $x^* \in l^2$, then using the Cauchy-Schwarz inequality we see that

$$|x^* x_n| = n^{-1/2} \left| \sum_{k=n+1}^{2n} x_k^* \right| \leq n^{-1/2} \left(\sum_{k=n+1}^{2n} x_k^{*2} \right)^{1/2} n^{1/2},$$

which implies $\lim_n x^*(x_n) = 0$, since $(x_k^*) \in l^2$.

However, (x_n) is not in $l_{\text{weak}}^2(l^2)$; indeed, (x_n) is not in $l_{\text{weak}}^p(l^2)$, for any $p \geq 1$. To be sure, let $x^* \in l^2$ be given by

$$x_k^* = \begin{cases} 0 & \text{if } k = 1, \\ 1/(k^{1/2} \ln k) & \text{if } k > 1, \end{cases}$$

and check that

$$x^*(x_n) = n^{-1/2} \sum_{k=n+1}^{2n} 1/(k^{1/2} \ln k) \geq n^{-1/2} \cdot n \cdot (2n)^{-1/2} (\ln(2n))^{-1} = 2^{-1/2} / \ln(2n);$$

it follows that $\sum_n |x^* x_n|^p$ diverges for any $p \geq 1$.

Finally we note that if $\sigma_n = n^{-1} \sum_{k=1}^n x_k$, then $\|\sigma_n\|_2 \geq 1/4$ for all n , i.e., while (x_n) has a subsequence which has norm null arithmetic means, (x_n) itself does not have norm null arithmetic means.

THEOREM 5. *If X has an unconditional basis (e_n) that is in the range of a measure, then (e_n) is in $l_{\text{weak}}^2(X)$.*

Proof. Let $F: \Sigma \rightarrow X$ be c.a. Then there exists a sequence (μ_n) of scalar measures on Σ such that

$$(1) \quad F(E) = \sum_{n=1}^{\infty} \mu_n(E) e_n, \quad E \in \Sigma.$$

For each $x^* \in X^*$ we have

$$(2) \quad x^* \circ F(E) = \sum x^*(e_n) \mu_n(E), \quad E \in \Sigma.$$

From the unconditional convergence of the series (1) we obtain that of (2) for every $E \in \Sigma$, and so it follows that for any fixed $x^* \in X^*$ the series

$$(3) \quad \sum x^*(e_n) \mu_n$$

is weakly unconditionally convergent in the Banach space $\mathcal{X} = ca(\Omega, \Sigma)$ of all scalar measures on Σ with the total variation norm $|\mu|$. By the Orlicz–Pettis Theorem, (3) is unconditionally convergent in the norm of \mathcal{X} .

Now we may apply a theorem due to Orlicz ([D, p. 105 for L^1]) which is also valid in \mathcal{X} , to obtain

$$(4) \quad C(x^*) = \sum x^*(e_n)^2 |\mu_n|^2 < \infty.$$

By assumption, for each $n \in \mathbb{N}$ there exists a set $A_n \in \Sigma$ such that $F(A_n) = e_n$ and

$$\mu_k(A_n) = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{if } k \neq n, \end{cases}$$

whence we get $|\mu_n| \geq \mu_n(A_n) = 1$ for each n . Combining this with (4) we obtain

$$\sum x^*(e_n)^2 \leq C(x^*) < \infty,$$

and as this holds for every $x^* \in X^*$, the basis (e_n) is in $l_{\text{weak}}^2(X)$, completing thereby the proof.

Remark. Aldous and Fremlin [AF] have proved the following remarkable theorem: if (f_n) is any norm bounded sequence in $L^1([0, 1])$ that satisfies $\|f_n - f_m\| \geq 1$ for $n \neq m$, then there is a subsequence (f_{n_k}) that is “2-colacunary”, viz. there exists a constant C such that

$$(5) \quad \left\| \sum_{k=1}^N \alpha_k f_{n_k} \right\| \geq C \cdot (\sum \alpha_k^2)^{1/2} \quad \text{for all } N,$$

for any sequence (α_n) of scalars; the norm is that in L^1 . Applying this theorem instead of the one due to Orlicz, we conclude from the hypotheses of Theorem 5 that there is a subsequence of (e_n) that is in $l_{\text{weak}}^2(X)$ (although this is a weaker conclusion, we think that this theorem of [AF] can be exploited to answer some of these “sequential” questions considered here and indicate the proof). We let $f_n = d\mu_n/d\lambda$ in eq. (1) where λ is a “control measure” for F . The assumption about (e_n) and the boundedness of the range of F imply

$$(6) \quad \left| \sum_{n=1}^m t_n \mu_n(E) x^*(e_n) \right| = \left| \int_E \sum_{n=1}^m t_n x^*(e_n) f_n d\lambda \right| \leq C$$

for every $E \in \Sigma$, and any aequence (t_n) in B_{l^∞} and for $x^* \in B_{X^*}$. Applying this inequality first to the set E of points where the integrand g_m in (6) is ≥ 0 then letting E be the set where g_m is < 0 we see that $\|g_m\|_1 \leq 2C$. It follows that the sequence (f_n) is norm bounded in $L^1(\lambda)$. Now we use the assumption as in the above proof after eq. (4), getting this time:

$$\|f_n - f_m\|_1 \geq \int_{A_n} |f_n(t) - f_m(t)| d\lambda(t) = \int_{A_n} (f_n(t) - f_m(t)) d\lambda(t) = \mu_n(A_n) = 1$$

if $n \neq m$, and so by the theorem of Aldous and Fremlin [AF] we obtain a subsequence (f_{n_k}) of (f_n) such that

$$\left\| \sum_{k=1}^n \alpha_k f_{n_k} \right\|_1 \geq C_1 (\sum \alpha_k^2)^{1/2}$$

for some constant $C_1 > 0$ and for all scalars (α_k) . We look at any $x^* \in B_{X^*}$:

$$2C \geq \int \left| \sum x^*(e_{n_k}) f_{n_k} \right| d\lambda \geq C_1 (\sum x^*(e_{n_k})^2)^{1/2},$$

and so the subsequence (e_{n_k}) of the basis is in $l_{\text{weak}}^2(X)$ as stated.

We do not know if the hypotheses of Theorem 5 imply in general: every weakly null sequence (x_n) in X has a subsequence (x_{n_k}) that is in $l_{\text{weak}}^2(X)$. Our "proof" of this was incomplete, as kindly pointed out to us by Professor Lech Drewnowski. Let us observe, however, that this is true for $X = c_0$, l^p or L^p with $p \geq 2$ (by Prop. D).

EXAMPLE 6. If $1 \leq p < 2$, then the sequence (e_n) of unit coordinate basis vectors in l^p is not in the range of a measure.

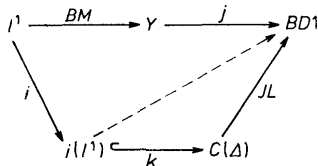
In fact, for such p it is plain that (e_n) is not a member of $l_{\text{weak}}^2(l^p)$.

This also follows from a result due to Rosenthal [R1, Theorem A2] to the effect that every bounded linear operator from l^q to L^1 is compact if $q > 2$, and by duality.

EXAMPLE 7. Those spaces whose closed unit ball is the range of a vector-valued measure obviously have the property that every relatively weakly compact subset is contained in the range of a vector-valued measure.

This phenomenon of every weakly compact subset of a Banach space X being a subset of the range of a vector-valued measure is not restricted to reflexive spaces.

Indeed, the separable \mathcal{L}_∞ space $BD1$ of Bourgain and Delbaen [BD] that has the weakly compact extension property enjoys having each of its weakly compact subsets a subset of a $BD1$ -valued measure. In fact, if K is an absolutely convex weakly compact subset of $X = BD1$, then by a result of Davis, Figiel, Johnson, and Pełczyński ([DFJP], p. 314, Cor. 2), there exists a separable reflexive space Y and a continuous linear injection $j: Y \rightarrow X$ such that $j(B_Y) \supset K$. By a classical theorem due to Banach and Mazur there exists a continuous linear surjection BM from l^1 onto the separable space Y , so that $j \circ BM: l^1 \rightarrow X$ is weakly compact with $K \subset j \circ BM(B_{l^1})$. Now l^1 is (isometrically) isomorphic to a subspace of $C(\Delta)$, where Δ is the Cantor discontinuum, and so (by the weakly compact extension property of $BD1$) there is a weakly compact operator $JL: C(\Delta) \rightarrow BD1$ such that the following diagram commutes:



Of course, $K \subset j \circ BM(B_{l^1}) \subset JL \circ k \circ i(B_{l^1})$; but the weakly compact operator JL is representable by means of a regular Borel $BD1$ -valued measure whose absolutely convex closed hull coincides with $JL(B_{C(\Delta)})$. The result: every weakly compact subset of $BD1$ lies inside the range of a vector-valued measure.

QUESTION. Which Banach spaces X have the property that every weakly compact subset of X lies inside the range of an X -valued measure?

More particularly, does every weakly compact subset of c_0 lie inside the range of c_0 -valued measure?

Our next three examples show that even norm null sequences in X do not always lie in the range of a measure; the condition on p in Ex. 9, 10 seems curious to us.

EXAMPLE 8. In l^1 , the norm null sequence (e_n/\sqrt{n}) is plainly not in $l^2_{\text{weak}}(l^1)$ and so, by Theorem 5, in l^1 the sequence (e_n/\sqrt{n}) is a norm null sequence that is not in the range of a measure. The phenomenon of Ex. 8 occurs in reflexive spaces as well.

EXAMPLE 9. Suppose $1 < p \leq 2$ and let q be the index conjugate to p . Then the sequence $(e_n/n^{1/q})$ in l^p lies in the range of a measure if and only if $4/3 < p \leq 2$.

In fact, suppose $1 < p \leq 4/3$, forcing $q \geq 4$. Let $\epsilon > 0$ be so judiciously chosen that $2 + 2\epsilon < q$. Consider the scalar sequence (t_n) given by

$$t_n = \begin{cases} 0, & n = 1, \\ n^{-1/q}(\log n)^{-(1+\epsilon)/q}, & n > 1. \end{cases}$$

It is plain that $(t_n) \in l^q$ and that viewed as a member of $(l^p)^*$, $(t_n) = x^*$ satisfies

$$\sum_n |x^*(e_n/n^{1/q})|^2 = \infty.$$

It follows that if $1 < p \leq 4/3$, then $(e_n/n^{1/q})$ is a norm null sequence in l^p that is not in the range of a measure. On the other hand if $4/3 < p \leq 2$, then $(e_n/n^{1/q}) \in l^2_{\text{weak}}(l^p)$ (and so lies in the range of a measure) as an easy application of Cauchy-Schwarz shows. The next example illustrates the phenomenon of Ex. 9 in L^p spaces as well, and the condition on p .

EXAMPLE 10. Suppose $1 < p \leq 2$ and let q be the index conjugate to p . Let $(h_{n,k})_{1 \leq k \leq 2^n, n \geq 1}$ be the sequence of Haar functions normalized in $L^p[0, 1]$. Then the sequence $(h_{n,k}/(2^n + k)^{1/q})_{1 \leq k \leq 2^n, n \geq 1}$ is in the range of an $L^p[0, 1]$ -valued measure if and only if $4/3 < p \leq 2$.

Indeed, by our earlier remarks it suffices to show that $(h_{n,k}/(2^n + k)^{1/q}) \in l^2_{\text{weak}}(L^p[0, 1])$ if and only if $4/3 < p < 2$.

Recall that

$$h_{n,k} = 2^{n/p}(\chi_{[(2k-2)/2^{n+1}, (2k-1)/2^{n+1})} - \chi_{[(2k-1)/2^{n+1}, 2k/2^{n+1})})$$

and let $x_{n,k} = h_{n,k}/(2^n + k)^{1/q}$ for $1 < k \leq 2^n$ and $n \geq 1$. Further, denote by $(h'_{n,k})$ the sequence of Haar functions normalized in $L^q[0, 1]$; so

$$h'_{n,k} = 2^{n/q}(\chi_{[(2k-2)/2^{n+1}, (2k-1)/2^{n+1})} - \chi_{[(2k-1)/2^{n+1}, 2k/2^{n+1})}).$$

Notation set, let us check out our claims. To start, let $\varphi \in L^q[0, 1] = L^p[0, 1]^*$. Then we can represent φ in the form

$$\varphi = \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} a_{n,k} h'_{n,k}$$

in precisely one unconditionally convergent manner since $(h'_{n,k})$ is an unconditional basis for $L^q[0, 1]$. Again we can appeal to the theorem of Orlicz to conclude that

$$\sum_n \sum_k |a_{n,k}|^q = \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} |a_{n,k}|^q \|h'_{n,k}\|_q^q < \infty;$$

so $((a_{n,k})_{1 \leq k \leq 2^n})_{n \geq 1} \in l^q$. Returning our attention to the vectors $x_{n,k}$ we see that

$$\varphi(x_{n,k}) = a_{n,k}/(2^n + k)^{1/q}.$$

If $4/3 < p < 2$, then $2 < q < 4$ and it follows that $((2^n + k)^{-2/q})_{n,k} \in l^r$, where $r = q/(q-2)$ is conjugate to $q/2$, and we see from the Cauchy-Schwarz inequality that $([\varphi(x_{n,k})]^2)_{n,k} \in l^1$ and $(x_{n,k}) \in l^2_{\text{weak}}(L^p[0, 1])$. An appeal to Theorem 3 tells us that $(x_{n,k})$ lies in the range of an $L^p[0, 1]$ -valued measure. Of course, if $p = 2$ then the boundedness of $(x_{n,k})$ already is enough to make the same conclusion.

Should $1 < p \leq 4/3$ then the sequence defined by

$$a_{n,k} = \frac{1}{2^{k/q} n^{2/q}}$$

belongs to l^q but not l^2 and it is easy to verify that

$$\varphi = \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} a_{n,k} h'_{n,k} \in L^q[0, 1]$$

satisfies $(\varphi(x_{n,k})) \notin l^2$. As before this denies $(x_{n,k})$ the enjoyment of being in the range of a measure.

Remark. Relatively norm compact sets K in $c_0(\Gamma)$ or l^p for $p \geq 2$ are contained in the ranges of countably additive measures.

We have observed earlier that the closed unit ball of such l^p spaces are ranges of measures. In case of $c_0(\Gamma)$, K is contained in some order interval $\{x = (x(\gamma))_{\gamma \in c_0} : |x(\gamma)| \leq y(\gamma)\}$ defined by a fixed element $y = (y(\gamma))$, and it is well known (e.g., [DS]) that weakly compact order intervals are themselves ranges of countably additive measures on some σ -fields. Thus we have the

PROBLEM. *Which Banach spaces X share the above property with $c_0(\Gamma)$?*

It is an easy consequence of the celebrated Dvoretzky–Rogers theorem that given an infinite dimensional Banach space X , there is an X -valued measure that does not have finite variation; not so easy, but true nonetheless, is another consequence of the Dvoretzky–Rogers theorem drawn by Thomas [T, p. 80] (see also Janicka and Kalton [JK]) to the effect that given any infinite dimensional Banach space X there is an X -valued measure defined on the Borel subsets of $[0, 1]$ continuous with respect to Lebesgue measure whose variation on any set of positive Lebesgue measure is infinite. In light of the above example it seems appropriate to repeat the next two old chestnuts.

QUESTION. *When is a sequence in the range of a measure having finite variation? σ -finite variation?*

EXAMPLE 11. *In l^1 , the sequence (e_n/n) is in the range of a measure but not in the range of a measure of finite variation.*

It is plain that the sequence (e_n/n) is in $l^2_{\text{weak}}(l^1)$ so Theorem 1 applies putting $\{e_n/n\}$ inside the range of a suitable measure. Suppose F is an l^1 -valued countably additive measure of finite variation defined in the σ -field Σ and $F(\Sigma)$ contains each vector (e_n/n) . Define $j: l^1 \rightarrow L^1[0, 1]$ by

$$j(x) = \sum_{n=1}^{\infty} x_n 2^n \chi_{[2^{-n}, 2^{-n+1})};$$

j is an order-preserving isometric embedding of l^1 into $L^1[0, 1]$. Therefore, $j \circ F: \Sigma \rightarrow L^1[0, 1]$ is a countably-additive $L^1[0, 1]$ -valued measure of bounded variation and as such $j \circ F(\Sigma)$ is an order bounded subset of $L^1[0, 1]$ —after all, integration against measures of bounded variation defines integral operators on spaces of bounded measurable functions [DU1] and Grothendieck has shown [G] that an operator into $L^1[0, 1]$ is integral precisely when it maps bounded sets into order bounded sets. Let $g \in L^1[0, 1]$ be the least upper bound of $j \circ F(\Sigma)$. Since $j(n^{-1}e_n) \in jF(\Sigma)$ we get

$$n^{-1} 2^n \chi_{[2^{-n}, 2^{-n+1})} = j(n^{-1}e_n) \leq g \quad \text{a.e.}$$

So, using the disjointness of the dyadic intervals, we see that

$$\sum_n n^{-1} 2^n \chi_{[2^{-n}, 2^{-n+1})} = \sup_n n^{-1} 2^n \chi_{[2^{-n}, 2^{-n+1})} \leq g \quad \text{a.e.}$$

But this contradicts $g \in L^1[0, 1]$ since clearly $\sum_n n^{-1} 2^n \chi_{[2^{-n}, 2^{-n+1})}$ does not belong to $L^1[0, 1]$.

Actually, Ex. 11 can be seen directly, i.e., without using [G], as kindly pointed out by Professor Lech Drewnowski; we suspect that the sequence in Ex. 11 cannot be in the range of a measure with σ -finite variation either.

EXAMPLE 12. L^1 -spaces, be they modelled on finite, σ -finite or arbitrary measure spaces, enjoy a remarkable property discovered by Grothendieck en route to his topological theory of tensor products, namely, if F and G are vector-valued measures (on perhaps different σ -fields) such that F has finite variation and F 's range contains G 's range then G , too, has finite variation. This follows from Grothendieck's characterization of integral operators into L^1 -spaces as those that take bounded sets into lattice bounded sets.

L^1 -spaces also have the pleasure of discriminating in a most rational way between general measures and measures having finite variation that are Bochner–Radon–Nikodym differentiable with respect to their variation. Again, the culprit responsible for first (and last) noting this is Grothendieck who showed that a bounded linear operator into an L^1 -space is nuclear if and only if the operator takes bounded sets into lattice bounded, equimeasurable sets. The upshot of this is that if F and G are L^1 -space-valued measures (on perhaps different σ -fields) with F having finite variation and $dF/d|F|$ existing and if the range of G is contained in the range of F , then G has finite variation and $dG/d|G|$ exists.

We know of no other space sharing the above property. We know, however, that c_0 does not enjoy such pleasure. In fact, if we let (r_n) be the Rademacher sequence, and define $F(E) = (\int_E r_n(t) dt)$ for Lebesgue measurable subsets E of $[0, 1]$, then F is a c_0 -valued countably additive measure whose range is contained in B_{l^2} ; a bit more is so: as a c_0 -valued measure, F has bounded variation since $\|F(E)\| \leq \lambda(E)$, where λ is Lebesgue measure, and we see with the help of [KK] and [DU1, p. 274] that B_{l^2} is the range of a c_0 -valued vector measure of finite variation. To complete this remark, we let $I_n = [2^{-n}, 2^{-n+1})$ for $n \geq 1$ and define $G(E)$ to be the member of c_0 given by $(2^n n^{-1} \lambda(E \cap I_n))$; it is plain that G 's range is bounded in l^2 and that G is countably additive into l^2 . It is clear that G does not have bounded variation as a c_0 -valued measure. So, a scalar multiple of $i \times G$ ($\pi/\sqrt{6}$ will do) is a countably additive c_0 -valued measure defined on the σ -field of Lebesgue measurable subsets of $[0, 1]$ such that the range of G lies inside that of a c_0 -valued measure having finite variation yet G fails to have bounded variation.

QUESTION. Can G, G' be distinct X -valued measures with the same range with exactly one of them having finite variation?

QUESTION. If a Banach space X has the property (enjoyed by subspaces of L^1 -spaces) that any X -valued measure whose range is contained inside that of an X -valued measure of finite variation is necessarily itself of finite variation, then need X be a subspace of an L^1 -space?

EXAMPLE. There is a sequence in L^1 that is in the range of a measure of finite variation but not in the range of any measure with a Bochner integrable derivative with respect to any finite positive measure.

For define F from the σ -algebra Σ of Borel subsets of $[0, 1]$ into L^1 by $F(E) = \chi_E, E \in \Sigma$. This F clearly has finite variation, and $F(\Sigma)$ contains the non-relatively compact sequence $\{\chi_{E_n}\}_n$, where $E_n = \{t: r_n(t) = 1\}$. Since the range of every measure that is a Bochner integral is relatively (norm) compact, the above sequence is no such a measure's range.

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Added in proof (September 1990). The answers to the last two Questions are “No” and “Yes”; see a recent paper by Luis Rodriguez Piazza, to appear in Proc. Amer. Math. Soc.