ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XXX (1990)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XXX (1990)

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# On the variation of an indefinite integral in Banach spaces*

**Abstract.** For a finitely additive measure $\lambda$ that extends the Lebesgue measure $\mu$ on $\Omega = [0, 1]$, we prove the existence of a $\Gamma$-integral $G_f\colon \Sigma_\lambda \to l^\infty$ with respect to $\lambda$, which is $\lambda$-continuous, strongly additive and such that its variation $|G_f|(E) = \infty$ for every $E \in \Sigma_\lambda$ with $\lambda(E) > 0$. In consequence, on the $\sigma$-algebra of Lebesgue measurable sets, $G_f$ is $\mu$-continuous and countably additive.

Let us note that in the case of countably additive measures an analogous result is not true since Musiał [3] has proved that every indefinite $\Gamma$-integral has always $\sigma$-finite variation. Earlier Rybakov [4] proved the same for the Pettis integral. (See also [1] and [5] for this and other related topics.)

We shall denote by $(\Omega, \Sigma, \mu)$ the Lebesgue measure space on $\Omega = [0, 1]$ and by $\mu_*$ and $\mu^*$ the respective inner and onter measures.

Let $\Gamma$ be a vector subspace of $X^*$ total on $X$, where $X$ is a Banach space and $X^*$ its dual. A function $f\colon \Omega \to X$ is called *$\Gamma$-integrable* if $x^*f$ is $\mu$-integrable for every $x^* \in \Gamma$, and for every $E \in \Sigma$ there exists $G_f(E) \in X$ such that

$$x^* G_f(E) = \int_E x^* f \, d\mu \qquad (x^* \in \Gamma^*).$$

LEMMA 1. *There exists a sequence $(A_n)$ of disjoint subsets of $\Omega$ such that $\bigcup_1^\infty A_n = \Omega$ and $\mu^*(A_n) = 1$ for every $n$.*

Proof. By using the axiom of choice, one can construct a sequence $(Q_n)$ of pairwise disjoint, nonmeasurable subsets of $\Omega$ such that $\bigcup_1^\infty Q_n = \Omega$ and

$$Q_n \equiv Q_1 + r_n \qquad (\text{mod. } 1),$$

where $(r_n)$ is the sequence of rational numbers in $[0, 1)$ $(r_1 = 0)$ ([2], p. 142). In the usual way, it can be proved that $\mu_*(\bigcup_1^n Q_k) = 0$ for every $n \in N$ and thus $\mu^*(\bigcup_n^\infty Q_k) = 1 - \mu_*(\bigcup_1^{n-1} Q_k) = 1$ $(n \in N)$.

As $\lim_n \mu^*(E_n) = \mu^*(\lim_n E_n)$ for a nondecreasing sequence $(E_n)$ of subsets of $\Omega$, we can prove inductively the existence of a double sequence $(A_{nk})$ of pairwise disjoint subsets of $\Omega$ $((n, k) \in N \times N)$ such that every $A_{nk}$ is a finite union of $Q_i$'s, $\bigcup_{n,k} A_{nk} = \Omega$ and $\mu^*(A_{nk}) \geqslant 1 - 1/k$ for every $(n, k) \in N \times N$. Let $A_n = \bigcup_{k=1}^\infty A_{nk}$;

* Supported in part by CAYCIT grant 0338-84 (Spain).

then the $A_n$ are pairwise disjoint, $\bigcup_1^\infty A_n = \Omega$ and

$$\mu^*(A_n) \geqslant \mu^*(A_{nk}) \geqslant 1-1/k$$

for every $k$. Therefore, $\mu^*(A_n) = 1$.

LEMMA 2. *Let $U$ be a nontrivial ultrafilter on $\mathbf{N}$ and*

$$\lambda^*(E) = \lim_{n,U} \frac{1}{n} \sum_{k=1}^{n} \mu^*(E \cap A_k) \qquad (\leqslant \mu^*(E))$$

*for $E \subset \Omega$, where $(A_n)$ is the sequence in Lemma 1. Then $\lambda^*$ is a finitely subadditive outer measure such that its restriction $\lambda$ to the algebra $\Sigma_\lambda$ of sets $E$ satisfying*

$$\lambda^*(X) = \lambda^*(X \cap E) + \lambda^*(X - E) \quad \text{for every } X \subset \Omega,$$

*is a finitely additive measure on $\Sigma_\lambda$. Moreover, $\Sigma_\lambda$ contains $\Sigma$ and $(A_n)$, $\lambda(E) = \mu(E)$ for every $E \in \Sigma$ and $\lambda(A_n) = 0$ for every $n$.*

Proof. This is easy, on taking into account the properties of "lim"$_{n,U}$ and the fact that $\mu^*(E \cap A_n) = \mu(E)$ for every $E \in \Sigma$ and $n \in \mathbf{N}$ (see [2], p. 87).

Note that we can use a generalized Banach limit instead of $\lim_{n,U}$ (see [2], p. 58).

The integral of a real bounded $f$ with respect to a finitely additive measure $\lambda$ is defined by uniform approximation of $f$ by simple functions, like in the countably additive measure case. For unbounded functions $f \geqslant 0$, $\int f d\lambda$ means $\lim_n \int f_n d\lambda$ where $f_n(x) = f(x)$ if $f(x) \leqslant n$ and $f_n(x) = n$ if $f(x) > n$, and in general $\int f d\lambda = \int (f^+ + g) d\lambda - \int (f^- + g) d\lambda$ for a function $g \geqslant 0$, when these integrals exist.

THEOREM. *With the notations of Lemma 2, there exists a $\Gamma$-integrable function $f$: $\Omega \to l^\infty(\Omega)$ (resp. $\Omega \to l^\infty$) with respect to $\lambda$ such that its indefinite integral $G_f$: $\Sigma_\lambda \to l^\infty(\Omega)$ (resp. $G_f$: $\Sigma_\lambda \to l^\infty$) is $\lambda$-continuous and strongly additive, but not of $\sigma$-finite variation. Even more, $|G_f|(E) = \infty$ for every $E \in \Sigma_\lambda$ with $\lambda(E) > 0$, and $\Gamma$ can be taken as the subspace of $l^1(\Omega)$ (resp. $l^1$) spanned by the unit vectors.*

Proof. Let $0 < \alpha < 1$, $\varepsilon(x) = 1/n$ for $x \in A_n$, and let $f_t$: $\Omega \to \mathbf{R}$ $(t \in \Omega)$ be the function defined by

$$f_t(x) = |x-t|^{-\alpha} \quad \text{for } |x-t| \geqslant \varepsilon(x),$$

$$f_t(x) = \varepsilon(x)^{-\alpha} \quad \text{for } |x-t| < \varepsilon(x).$$

It is obvious that $f = (f_t)_{t \in \Omega}$: $\Omega \to l^\infty(\Omega)$. We shall prove that every $f_t$ is "$\Sigma_\lambda$-measurable": In fact, if $(n-1)^\alpha \leqslant a < n^\alpha$ and $E_a = E_a^t = \{x \in \Omega: |x-t|^{-\alpha} \leqslant a\}$, we have

$$\{x: f_t(x) \leqslant a\} = \Big(\bigcup_1^{n-1} A_k\Big) \cup \Big(\bigcup_n^\infty A_k \cap E_a\Big) = \Big(\Big(\bigcup_1^{n-1} A_k\Big) \cap E_a^c\Big) \cup E_a,$$

which is a $\Sigma_\lambda$-measurable set, $\lambda$-equivalent to $E_a$. Furthermore,

$$\int_\Omega |x-t|^{-\alpha} d\lambda = \int_\Omega |x-t|^{-\alpha} d\mu \leqslant 2\int_\Omega x^{-\alpha} dx = 2/(1-\alpha) < \infty.$$

It follows that every $f_t$ is integrable and

$$\int_E f_t\, d\lambda = \int_E |x-t|^{-\alpha} d\lambda$$

for every $E \in \Sigma_\lambda$.

For each $E \in \Sigma_\lambda$, let us write

$$G(E) = \Big(\int_E f_t\, d\lambda\Big)_{t\in\Omega} = \Big(\int_E |x-t|^{-\alpha}\, d\lambda\Big)_{t\in\Omega}.$$

Since $\int_\Omega |x-t|^{-\alpha} d\lambda \leqslant 2/(1-\alpha)$, we have $G(E) \in l^\infty(\Omega)$ and $\|G(E)\| = \|G(E)\|_\infty \leqslant 2/(1-\alpha)$ for every $E \in \Sigma_\lambda$.

It is evident that $G\colon \Sigma_\lambda \to l^\infty(\Omega)$ is finitely additive. Let us prove that it is $\lambda$-continuous and, in consequence, strongly additive and countably additive on $\Sigma$. Given $\varepsilon > 0$, take $0 < 3\delta^{1-\alpha}/(1-\alpha) < \varepsilon$ and $\lambda(E) < \delta$ $(E \in \Sigma_\lambda)$. Then

$$\begin{aligned}\int_E |x-t|^{-\alpha} d\lambda &\leqslant \int_{E\cap E_a} |x-t|^{-\alpha} d\lambda + \int_{E_a^c} |x-t|^{-\alpha} d\lambda \\ &\leqslant a\lambda(E) + \int_{E_a^c} |x-t|^{-\alpha} d\mu \leqslant a\delta + 2a^{1-1/\alpha}/(1-\alpha).\end{aligned}$$

Thus, the choice $a = \delta^{-\alpha}$ yields

$$\int_E |x-t|^{-\alpha} d\lambda \leqslant \frac{3}{1-\alpha}\delta^{1-\alpha} < \varepsilon \qquad (t \in \Omega)$$

and so $\|G(E)\| < \varepsilon$.

Let $a = \inf E$ and $b = \sup E$ $(E \in \Sigma_\lambda)$. Then

$$\|G(E)\| \geqslant \int_E f_a\, d\lambda = \int_E |x-t|^{-\alpha} d\lambda \geqslant \frac{\lambda(E)}{(b-a)^\alpha}.$$

Let $a_k = a+(k-1)(b-a)/n$ and $E_k = E \cap [a_k, a_{k+1})$ $(k = 1, 2, \ldots, n)$. Then $\|G(E_k)\| \geqslant n^\alpha \lambda(E_k)/(b-a)^\alpha$ and

$$|G|(E) \geqslant \sum_1^n \|G(E_k)\| \geqslant n^\alpha \sum_l^n \lambda(E_k)/(b-a)^\alpha = n^\alpha \lambda(E)/(b-a)^\alpha$$

for every $E \in \Sigma_\lambda$ and $n \in N$. Therefore, $|G|(E) = \infty$ for every $E \in \Sigma_\lambda$ with $\lambda(E) > 0$.

Let now $\Gamma$ be the vector subspace of $l^1(\Omega)$ spanned by the unit vectors. Then

$$\int_E x^* f\, d\lambda = \sum_{t\in\Omega} x_t \int_E f_t\, d\lambda = x^* G(E)$$

for every $E \in \Sigma_\lambda$ and $x^* \in \Gamma$, which proves that $G$ is the $\Gamma$-indefinite integral $G_f$ of $f$.

Finally, in order to prove the theorem when $f\colon \Omega \to l^\infty$ it suffices to take $f = (f_{r_n})$, where $(r_n)$ is the sequence of rational numbers in $[0, 1)$.

Note that $G_f$ is a $\lambda$-continuous vector measure and so its range $G_f(\Sigma_\lambda)$ is relatively weakly compact ([1], I.5.3.).

Remark. We can take $f = (h_t)_{t\in\Omega}$ with $h_t = f_t$ except for a finite set $T$ of values of $t$ for which $h_t = g_t$, where $g_t(x) = |x-t|^{-\alpha}$ $(g_t(t) = 0)$. Note that $h_t$ is then $\mu$-measurable for $t \in T$.

## References

[1] J. Diestel and J. J. Uhl, Jr., *Vector Measures.* Math. Surveys 15, Amer. Math. Soc. Providence, R.I., 1977.

[2] M. E. Munroe, *Introduction to Measure and Integration*, Addison-Wesley, 1959.

[3] K. Musiał, *The weak Radon–Nikodym property in Banach spaces*, Studia Math. 64 (1979), 151–173.

[4] V. I. Rybakov, *On the completeness of the space of Pettis integrable functions*, Uchen. Zap. Mosk. Gos. Ped. Inst. 277 (1971), 58–64 (in Russian).

[5] B. Rodríguez-Salinas, *Integration of Banach-valued functions* (preprint).

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