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Non-convex integral functionals on Musielak–Orlicz spaces

Abstract. Non-convex integral functionals on Musielak–Orlicz spaces are discussed. A representation for the generalized gradient of an integral functional is given under some conditions. A sufficient condition for the lower semicontinuity of a functional is given.

Introduction and preliminaries. It is well known that integral functionals on a Banach space are important non-linear functionals, which can be applied in optimization theory. In 1972 A. D. Ioffe and V. L. Levin (see [1]) discussed the subdifferential of a convex integral functional on L^∞. In 1979 A. Kozek extended these results and studied convex integral functionals on Orlicz spaces of vector-valued functions. In this paper we discuss integral functionals, without the convexity assumption, on Musielak–Orlicz spaces of vector-valued functions.

Let (T, Σ, μ) be a measure space with a non-negative, complete, non-atomic and σ-finite measure; $(X, \|\cdot\|_X)$ denotes a separable real Banach space and $(Y, \|\cdot\|_Y)$ its dual space. We denote by $M(X) = M(T, \Sigma, X)$ the linear space obtained from the set of all strongly μ-measurable functions $x\colon T\to X$ by identifying the functions which are equal μ-almost everywhere, and similarly for $M(Y)$. Moreover, let $\langle y, x\rangle$ stand for the value of the functional $y\in Y$ at the point $x\in X$. Obviously, for $x\in M(X)$ and $y\in M(Y)$ the function $\langle y(\cdot), x(\cdot)\rangle$ is μ-measurable.

A function $\Phi\colon T\times[0, \infty)\to[0, \infty)$ is called a *Musielak–Orlicz function* iff

(i) for a.e. $t\in T$ the function $\Phi(t, \cdot)\colon [0, \infty)\to[0, \infty)$ is convex, left-continuous on $(0, \infty)$, $\Phi(t, 0) = 0$, $\lim_{u\to\infty}\Phi(t, u) = \infty$ and there exists $u_0 > 0$ such that $\Phi(t, u_0) < \infty$;

(ii) $\Phi(\cdot, u)\colon T\to[0, \infty)$ is μ-measurable for every $u\in[0, \infty)$.

The functional $I_\Phi\colon M(X)\to[0, \infty)$ defined by

$$I_\Phi(x, X) = \int_T \Phi(t, \|x(t)\|_X)\, d\mu(t)$$

is a convex pseudomodular.

The *Musielak–Orlicz space* is defined by

$$L_\Phi(T, X) = \{x\in M(X)\colon I_\Phi(rx, X) < \infty \text{ for some } r > 0\}.$$

The set

$$\operatorname{dom} I_\Phi = \{x \in M(X):\ I_\Phi(x, X) < \infty\}$$

is called the *Musielak–Orlicz class.*

For every Musielak–Orlicz function Φ, we define the *complementary function* Ψ: $T \times [0, \infty) \to [0, \infty)$ by

$$\Psi(t, v) = \sup\{uv - \Phi(t, u);\ u \in [0, \infty)\}$$

for every $t \in T$, $v \in [0, \infty)$. The function Ψ is a Musielak–Orlicz function, too. Moreover,

$$\Phi(t, u) = \int_0^u \varphi(t, s)ds \quad \text{and} \quad \Psi(t, v) = \int_0^v \psi(t, s)ds,$$

where $\varphi(t, \cdot)$ and $\psi(t, \cdot)$ are mutually right inverse functions. Both $\varphi(t, \cdot)$ and $\psi(t, \cdot)$ are increasing for a.e. $t \in T$.

The Luxemburg and Orlicz norms are introduced as follows:

$$\|x\|_{(\Phi)} = \inf\{r > 0:\ I_\Phi(x/r, X) \leqslant 1\}, \quad \|x\|_\Phi = \sup_{I_\Phi(y, Y) \leqslant 1} \left|\int_T \langle y(t), x(t)\rangle d\mu(t)\right|.$$

For any $x \in L_\Phi(T, X)$ and $y \in L_\Psi(T, Y)$ the function $\langle y(\cdot), x(\cdot)\rangle$: $T \to (-\infty, \infty)$ is integrable and

$$\left|\int_T \langle y(t), x(t)\rangle d\mu(t)\right| \leqslant \|x\|_\Phi \|y\|_{(\Psi)}, \quad \left|\int_T \langle y(t), x(t)\rangle d\mu(t)\right| \leqslant \|x\|_{(\Phi)} \|y\|_\Psi$$

(see [3]).

CONDITION Δ. A Musielak–Orlicz function $\Phi(t, u)$ is said to satisfy *Condition* Δ if there exist $K \geqslant 1$ and a positive μ-measurable function $\delta(t)$ on T such that

$$\int_T \Phi(t, \delta(t))d\mu(t) < \infty \quad \text{and} \quad \Phi(t, 2u) \leqslant K\Phi(t, u)$$

for a.e. $t \in T$ and $u \geqslant \delta(t)$ (see [4]).

Let V be a subset of a Banach space U. We say that a function f: $V \to \mathbf{R}$ satisfies the *Lipschitz condition* if there exists a constant $K > 0$ such that $|f(y) - f(y')| \leqslant K\|y - y'\|_U$ for every $y, y' \in V$.

We say that f satisfies the *Lipschitz condition near* $x \in U$ if there exists $\varepsilon > 0$ such that f satisfies the Lipschitz condition on $x + \varepsilon B_1$, where B_1 is the unit ball of U.

We say that f satisfies the *local Lipschitz condition* if it satisfies the Lipschitz condition near every $x \in U$.

DEFINITION 1. Let f: $U \to \mathbf{R}$ satisfy the Lipschitz condition near $x \in U$. The expression

$$f^0(x, v) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t}, \quad \text{for } v \in U,$$

is called the *generalized directional derivative* at x along v. The expression

$$\partial f(x) = \{\xi \in U^*: f^0(x, v) \geqslant \langle \xi, v\rangle, \text{ for all } v \in U\},$$

where U^* is the dual of U, is called the *generalized gradient* of f at x (cf. [5]).

DEFINITION 2. A function $f\colon U \to \boldsymbol{R}$ is said to be *regular* at x if the Gateaux derivative $f'(x, v)$ of f at x along v exists and $f'(x, v) = f^0(x, v)$ for every $v \in U$ (cf. [5]).

An integral functional on $L_\Phi(T, X)$ is defined by

$$f(x) = \int_T f_t(x(t))\,d\mu(t)$$

for every $x \in L_\Phi(T, X)$, where $f_t(\cdot)\colon X \to \boldsymbol{R}$ $(t \in T)$ is a class of functions on X and $t \to f_t(x)$ is measurable for every $x \in X$.

Results. For simplicity, we introduce

CONDITION (M). The function $t \to f_t(x)$ is measurable for every $x \in X$.

CONDITION (A). There exists a function $K(\cdot) \in L_\Psi(T, \boldsymbol{R})$ such that

$$|f_t(x_1) - f_t(x_2)| \leqslant |K(t)|\, \|x_1 - x_2\|_X$$

for every $t \in T$ and for all $x_1, x_2 \in X$.

CONDITION (B). For every $t \in T$, the function $f_t(\cdot)$ satisfies the Lipschitz condition on X and there exists $c > 0$ such that

$$\xi \in \partial f_t(x) \Rightarrow \|\xi\|_Y \leqslant c\{1 + \varphi(t, \|x\|_X)\}$$

for all $t \in T$ and $x \in X$.

LEMMA 1. *Let $\varphi(t, \cdot)$ be the right derivative of $\Phi(t, \cdot)$, for any fixed $t \in T$ and $u \in L_\Phi(T, \boldsymbol{R})$.*

(i) $\|u\|_\Phi \leqslant 1$ *implies* $\varphi(\cdot, |u(\cdot)|) \in \operatorname{dom} I_\Psi$ *and* $\|\varphi(\cdot, |u(\cdot)|)\|_{(\Psi)} \leqslant 1$.

(ii) $\|u\|_\Phi \leqslant 1$ *implies* $\int_T \Phi(t, |u(t)|)\,d\mu(t) \leqslant \|u\|_\Phi$.

(iii) *If Φ satisfies Condition Δ, then for any $m > 0$, there exists $M > 0$ such that* $\|u\|_\Phi \leqslant m$ *implies* $\varphi(\cdot, |u(\cdot)|) \in L_\Psi(T, \boldsymbol{R})$ *and* $\|\varphi(\cdot, |u(\cdot)|)\|_{(\Psi)} \leqslant M$.

Proof. (i) By the definition of Luxemburg norm, it is easy to show that $\|u\|_\Phi \leqslant 1$ implies $\varphi(\cdot, |u(\cdot)|) \in \operatorname{dom} I_\Psi$ and

$$I_\Psi[\varphi(\cdot, |u(\cdot)|), \boldsymbol{R}] \leqslant 1.$$

If not, we would have

$$1 < \int_T \Psi[t, \varphi(t, |u(t)|)]\,d\mu(t) \leqslant \infty.$$

Without loss of generality, we may suppose that the integral is finite.

Then

$$\int_T \Psi\left[t, \frac{\varphi(t,|u(t)|)}{I_\Psi[\varphi(\cdot,|u(\cdot)|), R]}\right] d\mu(t)$$

$$\leqslant \frac{1}{I_\Psi[\varphi(\cdot,|u(\cdot)|), R]} \int_T \Psi[t, \varphi(t,|u(t)|)]\, d\mu(t) = 1.$$

Therefore, we get a contradiction as follows:

$$1 = \frac{1}{I_\Psi[\varphi(\cdot,|u(\cdot)|), R]} \int_T \Psi[t, \varphi(t,|u(t)|)]\, d\mu(t)$$

$$< \frac{1}{I_\Psi[\varphi(\cdot,|u(\cdot)|), R]} \int_T [\Phi(t,|u(t)|) + \Psi[t, \varphi(t,|u(t)|)]]\, d\mu(t)$$

$$= \frac{1}{I_\Psi[\varphi(\cdot,|u(\cdot)|), R]} \int_T |u(t)|\, \varphi(t,|u(t)|)\, d\mu(t) \leqslant \|u\|_\Phi,$$

which implies (i).

(ii) By (i) and using the estimate from the proof of (i), we have

$$\int_T \Phi(t, |u(t)|) d\mu(t) \leqslant \frac{1}{I_\Psi[\varphi(\cdot,|u(\cdot)|), R]} \int_T \Phi(t,|u(t)|) d\mu(t)$$

$$\leqslant \frac{1}{I_\Psi[\varphi(\cdot,|u(\cdot)|), R]} \int_T [\Phi(t,|u(t)|) + \Psi(t, \varphi(t,|u(t)|))]\, d\mu(t) \leqslant \|u\|_\Phi,$$

which finishes the proof of (ii).

(iii) In view of (i), we can suppose that $m > 1$. By the monotonicity of $\varphi(t,\cdot)$ for a.e. $t \in T$, we have

$$|u(t)|\, \varphi(t,|u(t)|) \leqslant \int_{|u(t)|}^{2|u(t)|} \varphi(t, s) ds$$

$$\leqslant \int_0^{2|u(t)|} \varphi(t, s) ds = \Phi(t, 2|u(t)|)$$

for a.e. $t \in T$. Hence

$$\Psi[t, \varphi(t,|u(t)|)] = |u(t)|\, \varphi(t,|u(t)|) - \Phi(t,|u(t)|)$$

$$\leqslant \Phi(t, 2|u(t)|) + \Phi(t,|u(t)|) \leqslant 2\Phi(t, 2|u(t)|)$$

for a.e. $t \in T$. Since Φ satisfies Condition Δ, there exist $K \geqslant 1$ and $\delta(t)$ defined on T such that

$$\int_T \Phi(t, \delta(t)) d\mu(t) < \infty \quad \text{and} \quad \Phi(t, 2u) \leqslant K\Phi(t, u) \text{ for a.e. } t \in T,$$

provided $u \geqslant \delta(t)$. Hence, choosing an integer $n_0 > 1$ such that $2 < 2m < 2^{n_0}$, we have $\Phi(t, 2mu) \leqslant \Phi(t, 2^{n_0}u) \leqslant K^{n_0}\Phi(t, u)$, whenever $u \geqslant \delta(t)$ for a.e. $t \in T$. Let

$$T_1 = \{t \in T: |u(t)| < m\delta(t)\} \quad \text{and} \quad T_2 = T \setminus T_1.$$

Therefore

$$\begin{aligned} I_\Psi[\varphi(\cdot, |u(\cdot)|), \mathbf{R}] &= \int_T \Psi[t, \varphi(t, |u(t)|)]\, d\mu(t) \leqslant 2 \int_T \Phi(t, 2|u(t)|)\, d\mu(t) \\ &= 2 \int_{T_1} \Phi[t, 2|u(t)|]\, d\mu(t) + 2 \int_{T_2} \Phi[t, 2m|u(t)|/m]\, d\mu(t) \\ &\leqslant 2 \int_{T_1} \Phi(t, 2m\delta(t))\, d\mu(t) + 2K^{n_0} \int_{T_2} \Phi[t, |u(t)|/m]\, d\mu(t) \\ &\leqslant 2K^{n_0}\Big\{\int_T \Phi(t, \delta(t))\, d\mu(t) + 1\Big\} = M < \infty, \end{aligned}$$

which completes the proof of (iii).

Lemma 2. *Let $x(\cdot) \in L_\Phi(T, X)$ and let $f_t(\cdot)$ satisfy the local Lipschitz condition and Condition* (M). *Define*

$$\hat{f}_t(v) = f_t^0(x(t), v)$$

for each fixed $v \in X$ and for any $t \in T$. Then the function $t \to \hat{f}_t(v)$ is measurable.

Proof. Since $f_t(\cdot)$ satisfies the local Lipschitz condition, we may express $f_t^0(x(t), v)$ as the upper limit of

$$\frac{f_t(y(t) + \lambda v) - f_t(y(t))}{\lambda} \tag{1}$$

with $\lambda \downarrow 0$ taking only rational values and $y(t) \to x(t)$ taking values in a countable dense subset $\{x_i\}_{i=1}^{\infty}$ of X. But (1) defines a measurable function of t by Condition (M). Thus $f_t^0(x(t), v)$, being a countable upper limit of measurable functions of t, is measurable in t.

Lemma 3. (i) *For $x(\cdot) \in L_\Phi(T, X)$, we have*

$$\|x\|_\Phi = \sup_{I_\Psi(\bar{y}, \mathbf{R}) \leqslant 1} \int_T |\bar{y}(t)|\, \|x(t)\|_X\, d\mu(t) \stackrel{\text{df}}{=} |||x|||_\Phi.$$

(ii) *For $x(\cdot) \in L_\Phi(T, X)$ and $\bar{y}(\cdot) \in L_\Psi(T, \mathbf{R})$, we have*

$$\int_T |\bar{y}(t)|\, \|x(t)\|_X\, d\mu(t) \leqslant |||\bar{y}|||_{(\Psi)} \|x\|_\Phi,$$

where $|||\bar{y}|||_{(\Psi)} \stackrel{\text{df}}{=} \inf\{r > 0: I_\Psi[\bar{y}/r, \mathbf{R}] \leqslant 1\}$.

Proof. (i). By the definition of $\|\cdot\|_\Phi$, it is easy to see that

$$\begin{aligned} \|x\|_\Phi &= \sup_{I_\Psi(y, Y) \leqslant 1} \int_T \langle y(t), x(t)\rangle\, d\mu(t) \leqslant \sup_{I_\Psi(y, Y) \leqslant 1} \int_T \|y(t)\|_Y \|x(t)\|_X\, d\mu(t) \\ &= \sup_{I_\Psi(\|y(\cdot)\|_Y, \mathbf{R}) \leqslant 1} \int_T \|y(t)\|_Y \|x(t)\|_X\, d\mu(t) \leqslant \sup_{I_\Psi(\bar{y}, \mathbf{R}) \leqslant 1} \int_T |\bar{y}(t)|\, \|x(t)\|_X\, d\mu(t). \end{aligned}$$

For every $\bar{y} \in L_{\Psi}(T, \mathbf{R})$ such that $I_{\Psi}(y, \mathbf{R}) \leqslant 1$ and for every $x(\cdot) \in M(X)$ choose $z(\cdot) \in M(Y)$ such that $\langle z(t), x(t) \rangle = \|x(t)\|_X$ and $\|z(t)\|_Y = 1$ for a.e. $t \in T$. Define $y(t) = z(t)|\bar{y}(t)|$ for $t \in T$. Then $I_{\Psi}(y, Y) = I_{\Psi}(\bar{y}, \mathbf{R}) \leqslant 1$ and

$$\int_T \langle y(t), x(t) \rangle d\mu(t) = \int_T |\bar{y}(t)| \langle z(t), x(t) \rangle d\mu(t) = \int_T |\bar{y}(t)| \, \|x(t)\|_X d\mu(t),$$

which implies that

$$\|x\|_{\Phi} = \sup_{I_{\Psi}(y, \mathbf{R}) \leqslant 1} \int_T |\bar{y}(t)| \, \|x(t)\|_X d\mu(t).$$

(ii) In virtue of part (i) of Lemma 3 and by the Hölder inequality we have

$$\int_T |\bar{y}(t)| \, \|x(t)\|_X d\mu(t) \leqslant |||\bar{y}|||_{(\Psi)} |||x|||_{\Phi} = |||y|||_{(\Psi)} \|x\|_{\Phi},$$

and the proof is complete.

Now, we will prove the main theorem.

THEOREM 1. *If* $f_t(\cdot)$ *satisfies Condition* (M) *and Condition* (A), *then*

(i) f *satisfies the Lipschitz condition on* $L_{\Phi}(T, X)$;

(ii) *For every* $x \in L_{\Phi}(T, X)$

$$\partial f(x) \subset \int_T \partial f_t(x(t)) d\mu(t); \tag{2}$$

(iii) *If for each* $t \in T$, $f_t(\cdot)$ *is regular at* $x(t)$, *then* f *is regular at* x, *and*

$$\partial f(x) = \int_T \partial f_t(x(t)) d\mu(t), \tag{3}$$

where

$$\partial f_t(x(t)) = \{y \in L_{\Psi}(T, Y) \colon f_t^0(x(t)) \geqslant \langle y(t), v(t) \rangle, \textit{ for all } v \in L_{\Phi}(T, X)\}.$$

Proof. (i) Let $x_1, x_2 \in L_{\Phi}(T, X)$. Then, by Lemma 3 and Condition (A), we have

$$|f(x_1) - f(x_2)| \leqslant \int_T |f_t(x_1(t)) - f_t(x_2(t))| d\mu(t)$$

$$\leqslant \int_T |K(t)| \, \|x_1(t) - x_2(t)\|_X d\mu(t) \leqslant |||K|||_{(\Psi)} \|x_1 - x_2\|_{\Phi} = L \|x_1 - x_2\|_{\Phi}.$$

(ii) For every $x \in L_{\Phi}(T, X)$, $\lambda > 0$ and $v \in L_{\Phi}(T, X)$, by Condition (A) and Lemma 3, we have

$$\left| \frac{f_t(y(t) + \lambda v(t)) - f_t(y(t))}{\lambda} \right| \leqslant |K(t)| \, \|v(t)\|_X,$$

where $y \in L_{\Phi}(T, X)$, $y \to x$ in $L_{\Phi}(T, X)$ and

$$\int_T |K(t)| \, \|v(t)\|_X d\mu(t) \leqslant |||K|||_{(\Psi)} \|v\|_{\Phi} < \infty.$$

It follows from the Fatou lemma and (i) that

$$(5)\qquad -\infty < f^0(x,v) = \limsup_{\substack{y\to x\\ \lambda\downarrow 0}} \int_T \frac{f_t(y(t)+\lambda v(t))-f_t(y(t))}{\lambda}\,d\mu(t)$$

$$\leqslant \limsup_{\substack{y(t)\to x(t)\\ \lambda\downarrow 0}} \int_T \frac{f_t(y(t)+\lambda v(t))-f_t(y(t))}{\lambda}\,d\mu(t)$$

$$= \int_T f_t^0(x(t), v(t))\,d\mu(t),$$

where in the fourth term $y(t)\to x(t)$ almost everywhere. According to the definition of the generalized gradient, in view of (5), we have

$$(6)\qquad \int_T f_t^0(x(t), v(t))\,d\mu(t) \geqslant f^0(x, v) \geqslant \langle \xi, v\rangle$$

for any $\xi\in\partial f(x)$ and for all $v\in L_\Phi(T, X)$.

Let $\hat{f}_t(v) = f_t^0(x(t), v)$ for $v\in X$. By Lemma 2, the function $t\to\hat{f}_t(v)$ is μ-measurable for every $v\in X$. Define

$$\hat{f}(v) = \int_T \hat{f}_t(v(t))\,d\mu(t)$$

for all $v\in L_\Phi(T, X)$. Then (5) can be written as follows:

$$\hat{f}(v)-\hat{f}(0) \geqslant \langle \xi, v\rangle,$$

for all $v\in L_\Phi(T, X)$. In virtue of Proposition 2.11 from [5], $\hat{f}(v)$ is a finite convex function and $\xi\in\partial\hat{f}(0)$, where $\partial\hat{f}(0)$ is the subdifferential of $\hat{f}$ at 0. By Theorem 3.1 and Corollary 3.11 in [2] there exist $\zeta\in L_\Psi(T, Y)$ and a singular functional ϑ such that

$$\langle \xi, v\rangle = \int_T \langle \zeta(t), v(t)\rangle\,d\mu(t) + \vartheta(v)$$

for all $v\in L_\Phi(T, X)$, where $\zeta\in\partial\hat{f}_t(0) = \partial f_t(x(t))$ and $\vartheta(v)\leqslant 0$ for all $v\in\operatorname{dom}\hat{f}$. Since $\hat{f}(v)$ is finite for all $v\in L_\Phi(T, X)$, we have $\vartheta = 0$. Hence

$$\langle \xi, v\rangle = \int_T \langle \zeta(t), v(t)\rangle\,d\mu(t)$$

for all $v\in L_\Phi(T, X)$, i.e.

$$\xi\in\int_T \partial f_t(x(t))\,d\mu(t)$$

and so

$$\partial f(x) \subset \int_T \partial f_t(x(t))\,d\mu(t).$$

(iii) For any $v\in L_\Phi(T, X)$, by the assumptions, we have

$$f_t'(x(t), v(t)) = f_t^0(x(t), v(t)).$$

According to (5) and the Fatou lemma, we have

$$f^0(x, v) \leqslant \int_T f_t^0(x(t), v(t))d\mu(t) = \int_T f_t'(x(t), v(t))d\mu(t)$$

$$= \int_T \lim_{\lambda\to 0} \frac{f_t(x(t)+\lambda v(t))-f_t(x(t))}{\lambda} d\mu(t)$$

$$\leqslant \liminf_{\lambda\to 0} \int_T \frac{f_t(x(t)+\lambda v(t))-f_t(x(t))}{\lambda} d\mu(t)$$

$$= \liminf_{\lambda\to 0} \frac{f(x+\lambda v)-f(x)}{\lambda} \leqslant f'(x, v),$$

i.e. $f'(x, v) = f^0(x, v)$ for all $v \in L_\Phi(T, X)$.

Let $\xi \in \int_T \partial f_t(x(t))d\mu(t)$. Then there exist $\zeta(t) \in \partial f_t(x(t))$ for a.e. $t \in T$ such that

$$\langle \xi, v\rangle = \int_T \langle \zeta(t), v(t)\rangle d\mu(t) \leqslant \int_T f_t^0(x(t), v(t))d\mu(t) = f^0(x, v),$$

so $\xi \in \partial f(x)$. From this and by part (ii) of our theorem, we have

$$\partial f(x) = \int_T \partial f_t(x(t))d\mu(t),$$

which finishes the proof of Theorem 1.

THEOREM 2. *If $\Phi(t, u)$ satisfies Condition* Δ *and $f_t(\cdot)$ satisfies Condition* (M) *and Condition* (B), *then*

(i) *f satisfies the Lipschitz condition on any bounded subset of $L_\Phi(T, X)$;*
(ii) *For every $x \in L_\Phi(T, X)$*

$$\partial f(x) \subset \int_T \partial f_t(x(t))d\mu(t);$$

(iii) *If for each $t \in T$, $f_t(\cdot)$ is regular at $x(t)$, then f is regular at x and*

$$\partial f(x) = \int_T \partial f_t(x(t))d\mu(t).$$

Proof. (i). Let $x \in L_\Phi(T, X)$ and $\|x\|_\Phi \leqslant m$. For any $u \in L_\Phi(T, X)$ such that $\|u\|_\Phi \leqslant m$, by the mean-value theorem (cf. Th. 2.3.7 in [5]), we have

$$f_t(u(t))-f_t(x(t)) = \langle \xi(t), u(t)-x(t)\rangle, \tag{7}$$

where $\xi(t) \in \partial f_t(x^*(t))$, $x^*(t) = \theta(t)u(t)+(1-\theta(t))x(t)$ and $\theta(t) \in [0, 1]$ for all $t \in T$. By the monotonicity of $\varphi(t, \cdot)$ and Condition (B), there is a $c > 0$ such that

$$\|\xi(t)\|_Y \leqslant c[1+\varphi(t, \|x^*(t)\|_X)] = c[1+\varphi(t, \|\theta(t)u(t)+(1-\theta(t))x(t)\|_X)]$$

$$\leqslant c[1+\varphi(t, \max(\|u(t)\|_X, \|x(t)\|_X))]$$

$$\leqslant c[1+\varphi(t, \|u(t)\|_X)+\varphi(t, \|x(t)\|_X)].$$

Define

$$\bar{y}(t) = c\left[1+\varphi\left(t, \|u(t)\|_X\right)+\varphi\left(t, \|x(t)\|_X\right)\right]. \tag{8}$$

By Lemmas 1 and 2 and Hölder's inequality, it follows from (7) and (8) that

$$\begin{aligned}|f(u)-f(x)| &\leqslant \int_T |f_t(u(t))-f_t(x(t))|\,d\mu(t)\\ &= \int_T |\langle \xi(t), u(t)-x(t)\rangle|\,d\mu(t) \leqslant \int_T \|\xi(t)\|_Y \|u(t)-x(t)\|_X\,d\mu(t)\\ &\leqslant \int_T |\bar{y}(t)|\,\|u(t)-x(t)\|_X\,d\mu(t) \leqslant L\|u-x\|_\Phi,\end{aligned}$$

where $L = |||\bar{y}|||_{(\Psi)}$ depends only on m.

(ii) For $x \in L_\Phi(T, X)$, $y \in L_\Phi(T, X)$, $y \to x$ and $v \in L_\Phi(T, X)$, by the mean-value theorem, we have

$$\left|f_t(y(t)+\lambda v(t))-f_t(y(t))\right| = \left|\frac{\langle \xi(t), \lambda v(t)\rangle}{\lambda}\right| = |\langle \xi(t), v(t)\rangle| \tag{9}$$

for any $\lambda > 0$, where $\xi(t) \in \partial f_t(x^*(t))$ and

$$x^*(t) = \theta(t)\left(y(t)+\lambda v(t)\right)+\left(1-\theta(t)\right)y(t) = \lambda\theta(t)v(t)+y(t).$$

By the same proof as for (8) there exists a $\bar{y} \in L_\Psi(T, \boldsymbol{R})$ such that $\|\xi(t)\|_Y \leqslant \bar{y}(t)$ and hence, by (9), we have

$$\left|f_t(y(t)+\lambda v(t))-f_t(y(t))\right| \leqslant |\bar{y}(t)|\,\|v(t)\|_X.$$

Since

$$\int_T |\bar{y}(t)|\,\|v(t)\|_X\,d\mu(t) \leqslant |||\bar{y}|||_{(\Psi)}\|v\|_\Phi < \infty,$$

using the Fatou lemma, we obtain a formula analogous to (5). Therefore, repeating the argument in the proof of Th. 1 (ii), we have

$$\partial f(x) \subset \int_T \partial f_t(x(t))\,d\mu(t).$$

(iii) The proof is almost the same as the proof of Theorem 1 (iii).

Now, we will discuss the lower semicontinuity of the functional f. To this end we will introduce some general definition concerning this fact.

DEFINITION 3. A functional $f\colon U \to \boldsymbol{R}$ on a Banach space U is called *lower semicontinuous* if and only if its epigraph

$$\operatorname{Epi} f = \{(x, \alpha) \in U \times \boldsymbol{R}\colon f(x) \leqslant \alpha\}$$

is closed in $U \times \boldsymbol{R}$.

LEMMA 4. *If* $x_n \to x$ *in* $L_\Phi(T, X)$ *and* Φ *satisfies Condition* Δ, *then* $\|x_n(t)-x(t)\|_X \to 0$ *almost everywhere in* T.

Proof. Passing to a subsequence, we may assume that $\|x_n - x\|_\Phi < 2^{-n}$ for all natural n. For each natural k, let

$$A_{n,k} = \{t \in T_k\colon\ \|x_n(t) - x(t)\|_X \geqslant 1/k\},$$

where (T_k) is an increasing sequence of measurable subsets of T such that

$$\bigcup_{k=1}^{\infty} T_k = T \quad \text{and} \quad \mu(T_k) < \infty \quad \text{for } k = 1, 2, \ldots$$

Set

$$A_{\infty,k} = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_{n,k}.$$

Since $\|\chi_{A_{n,k}}\|_\Phi \leqslant \|k(x_n - x)\|_\Phi < k/2^n$, we have

$$\|\chi_{A_{\infty,k}}\|_\Phi \leqslant \sum_{n=m}^{j} \|\chi_{A_{n,k}}\|_\Phi + \|\chi_{\bigcup\limits_{n>j} A_{n,k}}\|_\Phi < k/2^{m-1} + \|\chi_{\bigcup\limits_{n>j} A_{n,k}}\|_\Phi$$

for each $j \geqslant m \geqslant 1$. Since Φ satisfies Condition Δ, it follows that

$$\|\chi_{\bigcup\limits_{n>j} A_{n,k}}\|_\Phi \downarrow 0 \qquad \text{as } j \to \infty,$$

so $\|\chi_{A_{\infty,k}}\|_\Phi = 0$ and hence $\mu(A_{\infty,k}) = 0$. Putting $k = 1, 2, \ldots$, we obtain

$$\mu\Big[\bigcup_{k=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{t \in T_k\colon\ \|x_n(t) - x(t)\|_X \geqslant 1/k\}\Big] = 0,$$

which shows that $\|x_n(t) - x(t)\|_X \to 0$ almost everywhere in T.

THEOREM 3. *If $f_t(\cdot)$ satisfies Condition* (M) *and $f_t(\cdot)$ is lower semicontinuous on X for almost every $t \in T$, then*

(i) *f is lower semicontinuous on $L_\Phi(T, X)$;*

(ii) *For every $x \in L_\Phi(T, X)$ the set function $A \to f(\chi_A x)$ is countably additive on the σ-algebra Σ, provided $f_t(0) = 0$ and f is proper, i.e. $f(x) > -\infty$ for every $x \in L_\Phi(T, X)$.*

Proof. (i) Let (x_n, α_n) be a sequence in Epi f convergent to (x, α) in $L_\Phi(T, X) \times \boldsymbol{R}$, i.e. $\|x_n - x\|_\Phi \to 0$ and $|\alpha_n - \alpha| \to 0$ as $n \to \infty$ and $f(x_n) \leqslant \alpha_n$ for every natural n. By Definition 3 it is sufficient to show that $f(x) \leqslant \alpha$. By Lemma 4 we have $\|x_n(t) - x(t)\|_X \to 0$ a.e. in T. In view of the lower semicontinuity of $f_t(\cdot)$ for a.e. $t \in T$, we have

$$f_t(x(t)) \leqslant \liminf_{n \to \infty} f_t(x(t)) \qquad \text{a.e. in } T,$$

so by the Fatou lemma

$$f(x) = \int_T f_t(x(t))\,d\mu(t) \leqslant \int_T \liminf_{n\to\infty} f_t(x_n(t))\,d\mu(t) \leqslant \liminf_{n\to\infty} \int_T f_t(x_n(t))\,d\mu(t) \leqslant \alpha.$$

(ii) Let $x \in L_\Phi(T, X)$ and $A = \bigcup_{n=1}^{\infty} A_n$ with pairwise disjoint $A_n \in \Sigma$. Then we have

$$f(\chi_A x) = \sum_{i=1}^{n} f(\chi_{A_i} x) + f(\chi_{B_n} x) \qquad (n \geqslant 1),$$

where $B_n = \bigcup_{i>n} A_i$. By the lower semicontinuity of f, we have

$$\liminf_{n\to\infty} f(\chi_{B_n} x) \geqslant f(0) = 0.$$

Moreover, it is easy to prove that $\|\chi_{B_n} x\|_\Phi \downarrow 0$ as $n \to \infty$. It follows that

$$f(\chi_A x) \geqslant \limsup_{n\to\infty} \sum_{i=1}^{n} f(\chi_{A_i} x) + \liminf_{n\to\infty} f(\chi_{B_n} x) \geqslant \limsup_{n\to\infty} \sum_{i=1}^{n} f(\chi_{A_i} x).$$

On the other hand, since

$$\Big\| \sum_{i=1}^{n} (\chi_{A_i} x - \chi_A x) \Big\|_\Phi = \|\chi_{B_n} x\|_\Phi \downarrow 0,$$

we have

$$f(\chi_A x) \leqslant \liminf_{n\to\infty} f\Big(\sum_{i=1}^{n} \chi_{A_i} x\Big) = \liminf_{n\to\infty} \sum_{i=1}^{n} f(\chi_{A_i} x).$$

Thus

$$f(\chi_A x) = \sum_{i=1}^{\infty} f(\chi_{A_i} x).$$

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