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On some properties of linear operators on $L^{*\varphi}$ which are continuous with respect to a modular

Abstract. We investigate absolute continuity and continuity with respect to a modular, mutual relations of these two types of continuity of operators over the spaces $L^{*\varphi}$, $L_f^{*\varphi}$ (the space of finite elements). We obtain some generalizations and supplements of results in papers [5]–[7].

1. In this paper $(T, \mathscr{E}, \mu)$ denotes the measure space over a non-empty T, where $\mathscr{E}$ is a σ-algebra of μ-measurable sets, with a σ-additive, atomless measure μ, such that $0 < \mu(T) < \infty$. S denotes the space of real functions μ-measurable on T finite almost everywhere.

1.1. A nondecreasing, continuous function $\varphi\colon \langle 0, \infty) \to \langle 0, \infty)$ taking 0 only for $u = 0$ and such that $\varphi(u) \to \infty$ as $u \to \infty$ is from now on called a *φ-function.* In Sections 5, 6 of this paper we make an additional assumption that these φ-functions satisfy the conditions

$$(o_1) \qquad \varphi(u)/u \to 0 \quad \text{as } u \to 0,$$

$$(\infty_1) \qquad \varphi(u)/u \to \infty \quad \text{as } u \to \infty.$$

For such φ-functions a complementary function in the sense of Young ([1], [2]) can be defined by

$$\psi(v) = \sup\big(uv - \varphi(u)\big) \quad \text{where sup is taken over } u \geq 0.$$

ψ is a convex φ-function satisfying (o_1), (∞_1). Let $\bar{\varphi}$ denote the complementary function of ψ; $\bar{\varphi}$ satisfies the inequality $\bar{\varphi}(u) \leq \varphi(u)$ for $u \geq 0$ and it is the greatest convex φ-function satisfying this inequality; φ is convex iff $\varphi(u) = \bar{\varphi}(u)$ for $u \geq 0$.

A φ-function φ satisfies condition Δ_2 if for some $k > 0$ $\varphi(2u) \leq k\varphi(u)$ for $u \geq u_0$.

1.2. The generalized Young inequality holds:

$$uv \leq \varphi(u) + \psi(v) \quad \text{for } u, v \geq 0.$$

For every $v \geqslant 0$ there exist u_v's such that

$$u_v v = \varphi(u_v) + \psi(v).$$

If $0 < v_0 \leqslant v \leqslant v_1$ then $0 < \inf u_v \leqslant \sup u_v < \infty$, if $v \to \infty$ then $u_v \to \infty$, if $v \to 0$ then $u_v \to 0$ ([1], [2]).

1.3. In what follows we denote by $h(v)$ the smallest u_v corresponding to a given v. For $v \geqslant 0$ we have $\varphi(h(v)) = \bar{\varphi}(h(v))$.

For the proof let us observe that $h(v)v = \varphi(h(v)) + \psi(v) \leqslant \bar{\varphi}(h(v)) + \psi(v)$, so $\varphi(h(v)) \leqslant \bar{\varphi}(h(v))$ and since $\bar{\varphi}(h(v)) \leqslant \varphi(h(v))$ so $\varphi(h(v)) = \bar{\varphi}(h(v))$.

1.4. For a simple function $s \in S$ the function $h(s(t))$ is μ-measurable. Let $s = \sum_1^n a_i \chi_{e_i}$, where $e_i \in \mathscr{E}$, $e_i \cap e_j = \emptyset$, χ_{e_i} is the characteristic function of e_i. Evidently

$$h(s(t)) = \sum_1^n h(a_i) \chi_{e_i}(t).$$

2. Let us introduce the following notation for $x \in S$:

$$I(x) = \int_T x(t)\, d\mu; \qquad I_\varphi(x) = \int_T \varphi(|x(t)|)\, d\mu.$$

It is known that $I_\varphi(x)$ is a modular in S in the sense of [4]. Set

$$L^{*\varphi} = \{x \in S\colon\ I_\varphi(\lambda x) < \infty \text{ for some } \lambda > 0\},$$

$$L_f^{*\varphi} = \{x \in S\colon\ I_\varphi(\lambda x) < \infty \text{ for every } \lambda > 0\},$$

$$K^\varphi = \{x \in L^{*\varphi}\colon\ I_\varphi(x) \leqslant 1\},$$

$$K_f^\varphi = \{x \in L_f^{*\varphi}\colon\ I_\varphi(x) \leqslant 1\}.$$

It is known that $L^{*\varphi}$ is a linear space with the standard operations on functions and with the equality $x = y$ defined to be the equality $x(t) = y(t)$ μ-almost everywhere in T.

$L^{*\varphi}$ can be given a complete F-norm

$$\|x\|_\varphi = \inf\{\varepsilon > 0\colon\ I_\varphi(x/\varepsilon) \leqslant \varepsilon\}.$$

$L_f^{*\varphi}$ (the space of finite elements) is a linear subspace of $L^{*\varphi}$, closed with respect to the norm $\|x\|_\varphi$. The relation $\|x_n\|_\varphi \to 0$ is equivalent to $I_\varphi(\lambda x_n) \to 0$ for every $\lambda > 0$. Apart from the norm convergence in $L^{*\varphi}$ a modular convergence is in operation.

A sequence $(x_n) \subset L^{*\varphi}$ is called *modular convergent* to $x \in L^{*\varphi}$, in symbols $x_n \xrightarrow{m} x$, if for some $\lambda > 0$, $I_\varphi(\lambda(x_n - x)) \to 0$. Convergence of the sequence (x_n) with respect to the norm $\|\ \|_\varphi$ implies modular convergence. However, the reverse implication holds in $L^{*\varphi}$ iff φ satisfies condition Δ_2.

2.1. Next, U, U_n (ξ, ξ_n) always denote linear operators (linear functionals) in $L^{*\varphi}$ taking values in a Banach space. U is called *φ-modular continuous* (φ-m *continuous*) in $L^{*\varphi}$ ($L_f^{*\varphi}$) if $x_n \xrightarrow{m} x$, $x_n, x \in L^{*\varphi}$ ($L_f^{*\varphi}$), implies $U(x_n) \to U(x)$. Modular continuity of functionals is defined similarly. We define for U a "quasi-norm" by the formula

$$\|U\| = \sup\{\|U(x)\|: \; x \in L^{*\varphi}, \; I_\varphi(x) \leqslant 1\}, \quad \text{or}$$

$$\|U\|_f = \sup\{\|U(x)\|: \; x \in L_f^{*\varphi}, \; I_\varphi(x) \leqslant 1\}$$

and analogously $\|\xi\|$, $\|\xi\|_f$ for functionals. If φ is convex we get the classical operator norms (norms of functionals).

2.2. An operator U is called *φ-absolutely continuous in $L^{*\varphi}$* (φ-ac in $L^{*\varphi}$) if for every $x \in L^{*\varphi}$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ (in general depending on x) such that

$$(*) \qquad \|U(x\chi_e)\| < \varepsilon \quad \text{whenever } \mu(e) < \delta.$$

φ-absolute continuity of an operator on $L_f^{*\varphi}$ is defined similarly. A sequence of operators (U_n) is called *φ-absolutely equicontinuous* in $L^{*\varphi}$ ($L_f^{*\varphi}$) if $(*)$ is satisfied for $x \in L^{*\varphi}$ ($L_f^{*\varphi}$) and for $U = U_n$, $n = 1, 2, \ldots$, with some δ independent of n.

A sequence (U_n) is *φ-modular equicontinuous* for $x = 0$ (φ-m equicontinuous) if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\|U_n(x)\| < \varepsilon \quad \text{for } n = 1, 2, \ldots \text{ whenever } I_\varphi(x) < \delta.$$

2.3. *An operator U is continuous in $(L^{*\varphi}, \|\ \|_\varphi)$ $[(L_f^{*\varphi}, \|\ \|_\varphi)]$ iff $\|U\|$ $(\|U\|_f)$ is finite.*

Let $\|U\| < \infty$, $\|x_n\|_\varphi \to 0$, $(x_n) \subset L^{*\varphi}$. Pick $\varepsilon > 0$; since $I_\varphi(x_n/\varepsilon) \to 0$, for $n \geqslant n_0$, $\|U(x_n/\varepsilon)\| \leqslant \|U\|$, $\|U(x_n)\| \leqslant \varepsilon\|U\|$ and thus $\|U(x_n)\| \to 0$.

If U is continuous in $L^{*\varphi}$ with respect to the norm $\|\ \|_\varphi$ then for some $\varepsilon > 0$, $\|U(x)\| \leqslant 1$ follows from $I_\varphi(x/\varepsilon) \leqslant \varepsilon$. Consequently, $\|U(x)\| \leqslant 1/\varepsilon$ follows from $I_\varphi(x) \leqslant \varepsilon$. Choose an integer k, $k > 1/\varepsilon$. For a given x, $I_\varphi(x) \leqslant 1$, choose disjoint sets $e_i \in \mathscr{E}$, $\bigcup_1^k e_i = T$ in such a way that $I_\varphi(x\chi_{e_1}) = I_\varphi(x\chi_{e_2}) = \ldots = I_\varphi(x\chi_{e_k})$. Since $I_\varphi(x\chi_{e_i}) \leqslant \varepsilon$ we obtain $\|U(x\chi_{e_i})\| \leqslant 1/\varepsilon$, $\|U(x)\| \leqslant k/\varepsilon$ and we have $\|U\| < \infty$ because k is independent of $x \in K^\varphi$. For $L_f^{*\varphi}$ the proof is analogous.

2.4. *An operator U φ-*m *continuous in $L^{*\varphi}$ ($L_f^{*\varphi}$) is φ-*ac *in $L^{*\varphi}$ ($L_f^{*\varphi}$).*

Let $x \in L^{*\varphi}$ ($L_f^{*\varphi}$). Then $I_\varphi(\lambda x) < \infty$ for some $\lambda > 0$. If U is φ-m continuous in $L^{*\varphi}$ ($L_f^{*\varphi}$), $\mu(e_n) \to 0$ then $I_\varphi(\lambda x\chi_{e_n}) \to 0$ and consequently $\|U(x\chi_{e_n})\| \to 0$.

3. *If an operator U is φ-*ac *in $L^{*\varphi}$ then for every $r > 0$ there exists a $\delta > 0$ such that*

$$(*) \qquad \|U(x\chi_e)\| < 2r \quad \textit{for } I_\varphi(x) < \delta, \; \mu(e) < \delta.$$

If $(*)$ does not hold then there exist sequences $(x_n), (e_n)$ such that $I_\varphi(x_n) \to 0$, $\mu(e_n) \to 0$, $\|U(x_n\chi_{e_n})\| > 2r$ for $n = 1, 2, \ldots$ Taking into consideration the φ-absolute continuity of U we can define by induction a subsequence $x_{k_n} = y_n$ satisfying

(1) $$I_\varphi(y_n) \leqslant 1/2^n \quad \text{for } n = 1, 2, \ldots,$$

(2) $$\|U(y_n \chi_{a_n})\| > r, \quad \text{where } a_n = e_{k_n} \setminus \bigcup_{n+1}^{\infty} e_{k_i}.$$

Define

(3) $$y = \sum_{n=1}^{\infty} y_n \chi_{a_n}.$$

Since the sets a_n are disjoint, by (1)

$$I_\varphi(y) = \sum_{n=1}^{\infty} I_\varphi(y_n \chi_{a_n}) \leqslant 1.$$

However, $U(y\chi_{a_n}) = U(y_n \chi_{a_n})$, $\mu(a_n) \to 0$, hence $\|U(y\chi_{a_n})\| \to 0$, contrary to (2).

3.1. *If an operator U is φ-ac on $L_f^{*\varphi}$ ($L^{*\varphi}$) then $\|U\|_f < \infty$ ($\|U\| < \infty$).*

(i) Let us prove first that there exist a natural n and a $\delta > 0$ such that $\|U(x\chi_e)\| \leqslant n$ for $I_\varphi(x) < \delta$, $\mu(e) < \delta$. Otherwise, we would have for some $x_n \in L_f^{*\varphi}$ and some sets e_n

$$\|U(x_n \chi_{e_n})\| > n, \quad I_\varphi(x_n) < 1/2^n, \quad \mu(e_n) < 1/2^n, \quad n = 1, 2, \ldots$$

Put $z_n = x_n/n$; then we have

$$\|U(z_n \chi_{e_n})\| > 1, \quad I_\varphi(nz_n) < 1/2^n, \quad z_n \in K_f^\varphi.$$

Similarly to the proof of 3, using the φ-absolute continuity of U in $L_f^{*\varphi}$ we are able to define a subsequence $z_{k_n} = y_n$, $n = 1, 2, \ldots$, and a sequence of μ-measurable disjoint sets a_n so as to have $I_\varphi(k_n y_n) < 1/2^{k_n}$,

(1′) $$\|U(y_n \chi_{a_n})\| \geqslant 1.$$

For the function $y = \sum_{n=1}^{\infty} y_n \chi_{a_n}$, we have for an arbitrary $l > 0$

(2′) $$I_\varphi(ly) = \sum_{n=1}^{\infty} I_\varphi(l y_n \chi_{a_n}) < \infty,$$

which yields $y \in L_f^{*\varphi}$. However, like in 3, $\|U(y\chi_{a_n})\| = \|U(y_n \chi_{a_n})\| \to 0$ by $\mu(a_n) \to 0$, contrary to (1′).

(ii) Let n, δ be constants from the assertion of (i), $I_\varphi(x) \leqslant 1$. Choose k so as to have $1/k < \delta$, $\mu(T)/k < \delta$ and choose disjoint sets e_i, $\bigcup_1^k e_i = T$, such that $I_\varphi(x\chi_{e_1}) = I_\varphi(x\chi_{e_2}) = \ldots = I_\varphi(x\chi_{e_k})$. This implies $I_\varphi(x\chi_{e_i}) < \delta$. Every e_i can be decomposed into k μ-measurable disjoint sets e_{ij}, $e_i = \bigcup_{j=1}^k e_{ij}$. Thus, we have $\mu(e_{ij}) < \delta$, $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, k$. We get

$$U(x) = \sum_{i,j=1}^{k} U(x\chi_{e_{ij}}), \quad \|U(x)\| \leqslant nk^2 \quad \text{for } x \in K_f^\varphi.$$

For $L^{*\varphi}$ we set in (2′) $l = 1$ or we can use Proposition 3 and the reasoning analogous to the one in (ii).

3.2. *An operator U for which $\|U\|_f < \infty$ is φ-ac in $L_f^{*\varphi}$.*

Choose l so that $\|U\|_f/l < \varepsilon$. Let $\mu(e_n) \to 0$. Since $|lx\chi_{e_n}| \leqslant |lx|$, $I_\varphi(lx) < \infty$ we have $I_\varphi(lx\chi_{e_n}) \to 0$. For $n \geqslant n_0$ we obtain

$$I_\varphi(lx\chi_{e_n}) \leqslant 1 \text{ and } \|U(lx\chi_{e_n})\| \leqslant \|U\|_f, \ \|U(x\chi_{e_n})\| \leqslant \varepsilon.$$

3.2.1. A Banach space Y is said to have *property* (0) if for $y_n \in Y$ and arbitrary (η_n), $\eta_n = 0, 1$, $m = 1, 2, \ldots$, $\|\sum_1^m \eta_n y_n\| \leqslant k < \infty$ implies the convergence of the series $\sum_1^\infty \eta_n y_n$ for any sequence (η_n) of zeros and ones (i.e., the series $\sum y_n$ is subseries convergent). We will say that an operator U has property (0) if the Banach space where U takes its values does.

3.2.2. *An operator U having property* (0) *(in particular a functional ξ) and such that $\|U\|_f < \infty$ ($\|\xi\|_f < \infty$) is φ-m continuous on $L_f^{*\varphi}$.*

(i′) First, let us prove that property 3$(*)$ is satisfied. If 3$(*)$ fails then the same reasoning as in 3 shows that there exist a sequence $(y_n) \subset L_f^{*\varphi}$ and a sequence (a_n) of sets such that conditions (1), (2) from 3 are satisfied with some $r > 0$.

Let $z_k = \sum_1^k \eta_n y_n \chi_{a_n}$, where $\eta_n = 0, 1$. We have $I_\varphi(z_k) \leqslant 1$ for $k = 1, 2, \ldots$ and since $z_k \in L_f^{*\varphi}$, $\|U(z_k)\| = \|\sum_1^k \eta_n U(y_n\chi_{a_n})\| \leqslant \|U\|_f$. Thus, in virtue of property (0), $\|U(y_n\chi_{a_n})\| \to 0$, a contradiction with 3(2).

(ii′) Let $I_\varphi(x_n) \to 0$. Choose arbitrary $\varepsilon > 0$ and $\eta > 0$ in such a way that $\mu(T)\varphi(\eta) \leqslant 1$. Let $a_n = \{t: |x_n(t)| \geqslant \varepsilon\eta\}$, $n = 1, 2, \ldots$ We have

$$I_\varphi(x_n\chi_{a_n}) \geqslant \varphi(\varepsilon\eta)\mu(a_n), \quad \text{so } \mu(a_n) \to 0.$$

The inequality

$$I_\varphi\left(\frac{1}{\varepsilon}x_n\chi_{T\setminus a_n}\right) \leqslant \varphi(\eta)\mu(T) \leqslant 1,$$

is satisfied, hence

$$\left\|U\left(\frac{1}{\varepsilon}x_n\chi_{T\setminus a_n}\right)\right\| \leqslant \|U\|_f, \quad \|U(x_n\chi_{T\setminus a_n})\| \leqslant \|U\|_f\varepsilon.$$

In view of property 3$(*)$, for $n \geqslant n_0$ we have $\|U(x_n\chi_{a_n})\| < \varepsilon$, and so $\|U(x_n)\| \leqslant \|U(x_n\chi_{a_n})\| + \|U(x_n\chi_{T\setminus a_n})\| \leqslant \|U\|_f\varepsilon + \varepsilon$, consequently $\|U(x_n)\| \to 0$.

3.2.2′. In connection with 3.2.2. let us give here the following counterexample. There exists an operator U with values in c_0, φ-absolutely continuous on $L_f^{*\varphi}$, which for some φ is not φ-m continuous on $L_f^{*\varphi}$. Let φ satisfy condition (A_1) given in Section 6. In virtue of 6.8 there exists a sequence (ξ_n) of functionals over $L^{*\varphi}$ satisfying the conditions

(α) $k = \sup_n \|\xi_n\| < \infty$,
(β) $\xi_n(x) \to 0$ for $x \in L_f^{*\varphi}$,
(γ) $|\xi_n(x_n)| \geqslant 1/2$ for some sequence $(x_n) \subset K_f^\varphi$, $I_\varphi(x_n) \to 0$.

Define U on $L^{*\varphi}$, $U(x)\in c_0$ for $x\in L^{*\varphi f}$, setting $U(x) = (\xi_i(x))$. By (α) we have $\|U\|_f \leqslant k$ and thus it follows from 3.2 that U is φ-ac on $L_f^{*\varphi}$. But $\|(x_n)\| = \sup_i |\xi_i(x_n)| \geqslant {}^{1/2}$, $I_\varphi(x_n)\to 0$, which means U is not φ-m continuous in $L_f^{*\varphi}$.

3.2.3. THEOREM 1. *Consider the following properties of the operator* U:

(a) U *is* φ-ac *on* $L^{*\varphi}$,
(a′) U *is* φ-ac *on* $L_f^{*\varphi}$,
(b) U *is* φ-m *continuous on* $L^{*\varphi}$,
(b′) U *is* φ-m *continuous on* $L_f^{*\varphi}$.

Then

(i″) (a)$\Leftrightarrow$(b),
(ii″) (b′)$\Rightarrow$(a′),
(iii″) (a′)$\Rightarrow$(b′) *if* U *has property* (0).

The implication (a)$\Rightarrow$(b) follows from 3 if we apply the same reasoning as in the proof 3.2.2(ii′). The implication (b)$\Rightarrow$(a) follows from 2.4, (a′)$\Rightarrow$(b′) from 3.1 and 3.2, (b′)$\Rightarrow$(a′) from 2.4.

3.2.4. THEOREM 2. *An operator* U *having property* (0) *and continuous in* $(L^{*\varphi}, \|\ \|_\varphi)$ *is* φ-m *continuous in* $L_f^{*\varphi}$.

We have $\|U\|_f \leqslant \|U\| < \infty$ by 2.3 and it is sufficient to apply 3.2.2.

4. *Let a sequence* (U_n) *of operators be* φ*-absolutely equicontinuous in* $L^{*\varphi}$ $(L_f^{*\varphi})$. *Then* $\sup_n \|U_n\| < \infty$ $(\sup_n \|U_n\|_f < \infty)$.

First, notice that $\sup_n \|U_n(x)\| < \infty$ for every $x\in L^{*\varphi}$. Indeed, choose k so as to have $\mu(T)/k < \delta$ and take k disjoint sets e_i, $\bigcup_1^k e_i = T$, of equal μ-measure. We have $\mu(e_i) \leqslant \mu(T)/k < \delta$ and thus $\|U_n(x\chi_{e_i})\| < \varepsilon$ for $n = 1, 2, \ldots$ and consequently $U_n(x) \leqslant k\varepsilon$. Let Y be a Banach space where the values $U_n(x)$ belong. Define an operator V on $L^{*\varphi}$ by $V(x) = (U_n(x))$. The sequence $(U_n(x))$ belongs to the space Z of bounded sequences $(y_n)\subset Y$ with the norm $\sup_n \|y_n\| < \infty$. The assumption of φ-equicontinuity of $U_n(x)$ means here that the operator V is φ-ac and so $\sup_n \|U_n(x)\| \leqslant r < \infty$ for $x\in K^\varphi$, by 3.1. Consequently $\|U_n\| \leqslant r$ for $n = 1, 2, \ldots$ For $L_f^{*\varphi}$ the proof is analogous.

Observe that the assumption $\sup \|U_n\| < \infty$ need not imply φ-absolute equicontinuity. It is sufficient to consider on $L^{*\varphi}$ the operator U defined in 3.2.2′. By 3.2.2′(α), $\|U\| < \infty$, $U(x)\in l^\infty$. If U were φ-ac on $L^{*\varphi}$ then by 3.2.3 it would be φ-m continuous and this contradicts 3.2.2′(α) because $\|U(x_n)\| \geqslant 1/2$ for some sequence x_n, $I_\varphi(x_n)\to 0$.

4.1. *A sequence* (U_n) *of operators in* $L^{*\varphi}$ *is* φ-m *equicontinuous iff it is* φ*-absolutely equicontinuous in* $L^{*\varphi}$.

We apply the same reasoning as in 2.4. We have $\|U_n(x)\| < \varepsilon$, $n = 1, 2, \ldots$, when $I_\varphi(x) < \delta$. Let $x\in L^{*\varphi}$; then $I_\varphi(\lambda x) < \infty$ for some $\lambda > 0$.

When $\mu(e) < \eta$, with η sufficiently small, then $I_\varphi(\lambda x \chi_e) < \delta$, so $\|U_n(x\chi_e)\| \leqslant \varepsilon/\lambda$, $n = 1, 2, \ldots$

If (U_n) are φ-absolutely equicontinuous in $L^{*\varphi}$ then the operator V defined in 4 is φ-ac. By 3.2.3 it is φ-m continuous in $L^{*\varphi}$, which means φ-m equicontinuity of (U_n).

5. Next we shall need a known lemma whose proof we give here for the reader's convenience. We apply the Baire category method.

Let (x_{ni}) be a given matrix of elements from a Banach space $(X, \| \ \|)$. Suppose for every sequence $\eta = (\eta_i)$, $\eta_i = 0, 1$ the series

(a) $$y_n(\eta) = \sum_{i=1}^{\infty} \eta_i x_{ni}, \qquad n = 1, 2, \ldots$$

is convergent and the limit

(b) $$y(\eta) = \lim_{n\to\infty} y_n(\eta)$$

exists. Then for every $\varepsilon > 0$ there exists an $i(\varepsilon)$ such that for $i \geqslant i(\varepsilon)$

$$\sup_n \|x_{ni}\| \leqslant \varepsilon.$$

Define a metric in the space H of sequences η by

$$d(\eta', \eta'') = \sum_{i=1}^{\infty} \frac{1}{2^i} |\eta_i' - \eta_i''|, \qquad \eta' = (\eta_i'),\ \eta'' = (\eta_i'').$$

H is complete in this metric.

Define $H_k = \{\eta \in H\colon \|y_n(\eta) - y_m(\eta)\| \leqslant \varepsilon/4\}$ for $n, m = k, k+1, \ldots$ From the subseries convergence of the series (a) follows their continuity in H so the sets H_k are closed in H. Consequently, one of them, say H_k, contains a ball B, $B(\eta, \eta_0) = \{\eta\colon d(\eta, \eta_0) \leqslant \varrho\}$. Let

(1) $$\sum_{i=1}^{\infty} \frac{1}{2^i} |\eta_i| \leqslant \varrho, \qquad \eta_i' = \eta_i^0 + (\eta_i - \eta_i \eta_i^0).$$

Let $\eta_i = \eta_i' - \eta_i''$, $\eta' = (\eta_i')$, $\eta'' = (\eta_i'')$. We have $d(\eta', \eta_0) \leqslant \varrho$, $d(\eta'', \eta_0) \leqslant \varrho$, thus

(2) $$\|y_n(\eta) - y_m(\eta)\| \leqslant \varepsilon/2 \qquad \text{for } n, m = k, k+1, \ldots$$

and η satisfying (1). Hence, in view of (b) we get (2) for $n, m \geqslant l \geqslant k$, $\eta \in H$, where l is sufficiently large. From (2) and (b) we obtain

(3) $$\|y_n(\eta) - y(\eta)\| \leqslant \varepsilon \qquad \text{for } n \geqslant l,\ \eta \in H,$$

and we have in particular $\|y_l(\eta) - y(\eta)\| \leqslant \varepsilon$. Choose sequences η^i consisting of zeros everywhere but for the ith term. The last inequality gives

$$\|x_{li} - y(\eta^i)\| < \varepsilon,$$

and since it follows from assumption (a) that $\|x_{li}\| \to 0$ as $i \to \infty$ we conclude $\|y(\eta^i)\| \leqslant 2\varepsilon$ for $i \geqslant i_0$. By (3) we have

$$\|x_{ni}\| \leqslant \|x_{ni} - y(\eta^i)\| + \|y(\eta^i)\| < 3\varepsilon,$$

for $n \geqslant l$ and $i \geqslant i_0$. For $n < l$ we can find $i \geqslant i_1 \geqslant i_0$ in such a way that $\|x_{ni}\| < 3\varepsilon$ and finally $\sup_n \|x_{ni}\| \leqslant 3\varepsilon$ for $i \geqslant i_1$.

5.1. THEOREM 3. *Let the operators* U_n *be* φ-ac *in* $L^{*\varphi}$, $U_n(x) \to U(x)$ *for* $x \in L^{*\varphi}$. *Then*

(a) *the sequence* (U_n) *is* φ-m *equicontinuous in* $L^{*\varphi}$.

(b) *the limit operator is* φ-m *continuous* (cf. [6]).

Generalizing Lemma 3 we shall prove: for every $r > 0$ there exists a $\delta > 0$ such that $\|U_n(x\chi_e)\| \leqslant 2r$, $n = 1, 2, \ldots$ for $I_\varphi(x) < \delta$, $\mu(e) < \delta$.

For if not, reasoning analogously to Lemma 3, there is a sequence (y_n) and a sequence (a_n) of μ-measurable disjoint sets and an increasing sequence (l_n) of indices such that

(1) $$I_\varphi(y_n) \leqslant 1/2^n,$$

(2) $$\|U_{l_n}(y_n\chi_{a_n})\| > r.$$

For an arbitrary sequence $\eta = (\eta_n)$, $\eta_n = 0, 1$, define

$$y(\eta) = \sum_{n=1}^{\infty} \eta_n y_n \chi_{a_n}.$$

We have $I_\varphi(y(\eta)) \leqslant 1$, and since U_{l_i}, being φ-ac in $L^{*\varphi}$, is φ-m continuous, we have

$$U_{l_i}(y(\eta)) = \sum_{n=1}^{\infty} \eta_n U_{l_i}(y_n\chi_{a_n}), \qquad i = 1, 2, \ldots$$

By the assumption $U_{l_i}(y(\eta)) \to U(y(\eta))$ as $i \to \infty$, so from 5, $\|U_{l_n}(y_n\chi_{a_n})\| < r/2$ for $n \geqslant n_0$ and we get a contradiction with (2). From the preceding lemma and reasoning as in 3.1 we have $\sup_n \|U_n\| < \infty$, which by the same lemma again and the reasoning analogous to 3.2.2(ii″) gives us (a). (b) is an immediate consequence of (a).

The theorem above for the case $T = \langle a, b\rangle$ and $\mathscr{E}$ the algebra of sets Lebesgue measurable and with the proof based on an idea similar to the one presented here can be found in [6].

5.2. *If the operators* U_n *are* φ-ac *and for every* $x \in L^{*\varphi}$ *the sequence* $\|U_n(x)\|$ *is bounded then* $\sup_n \|U_n\| < \infty$.

Let (λ_n) denote an arbitrary sequence of non-negative terms tending to 0. The operators $W_n(x) = \lambda_n U_n(x)$ satisfy the assumptions of Theorem 3 and in virtue of this theorem and 4 we have

$$\|\lambda_n U_n(x)\| \leqslant \sup_n \|W_n\| = k < \infty \qquad \text{for } n = 1, 2, \ldots, \ x \in K^\varphi,$$

so $\lambda_n \|U_n(x)\| \leqslant k$, consequently $\sup_n \|U_n\| < \infty$.

6. We assume throughout this section that the φ-functions considered satisfy (o_1), (∞_1).

Let us define for φ-functions the following properties:

(A) $$\lim_{v\to\infty} \psi(v)/\varphi(h(v)/2) = \infty;$$

(A_1) $$\limsup_{v\to\infty} \psi(v)/\varphi(h(v)/2) = \infty;$$

(B) $$\lim_{v\to\infty} \varphi(h(v))/\varphi(h(v)/2) = \infty;$$

(B_1) $$\limsup_{v\to\infty} \varphi(h(v))/\varphi(h(v)/2) = \infty.$$

ψ denotes here the function complementary to φ, $h(v)$ is the function defined in 1.3.

6.1. (a) *We have the implications* (B) $\Rightarrow$ (A), (B_1) $\Rightarrow$ (A_1).

(b) *Property* (B) *is satisfied whenever*

(C) $$\lim_{u\to\infty} \varphi(2u)/\varphi(u) = \infty.$$

The proof of (a) follows from the inequalities

$$h(v)v = \varphi(h(v)) + \psi(v) \leqslant 2\big(\varphi(h(v)/2) + \psi(v)\big),$$

$$\varphi(h(v)/2)\left[\frac{\varphi(h(v))}{\varphi(h(v)/2)} - 2\right] \leqslant \psi(v) \quad \text{for } v > 0.$$

(C) $\Rightarrow$ (B) is immediate because $h(v)\to\infty$ as $v\to\infty$.

6.2. *If* φ *is a convex function which does not satisfy the* Δ_2 *condition then there exists a convex function* φ_0 *such that*

(1) $$\varphi_0(u) \leqslant \varphi(u), \quad \varphi_0(2u) \geqslant \varphi(u) \quad \textit{for } u \geqslant 0$$

and φ_0 *satisfies property* (B_1).

Define $p(t) = \varphi(t)/t$, $p(0) = 0$. p is continuous, strictly increasing and $p(t)\to 0$ as $t\to 0$, $p(t)\to\infty$ as $t\to\infty$. Let $\varphi_0(u) = \int_0^u p(t)\,dt$. It is known that this function in convex and satisfies (o_1), (∞_1) and (1). (1) implies that condition Δ_2 fails for φ_0. The complementary function of φ_0 is $\psi_0(u) = \int_0^u p_{-1}(t)\,dt$ and its corresponding $h(v)$ equals $h(v) = p_{-1}(v)$ (cf. [1]). We have for some sequence $u_n\to\infty$, $\varphi_0(2u_n)/\varphi_0(u_n)\to\infty$. Define v_n so as to have $2u_n = p_{-1}(v_n) = h(v_n)$ for $n = 1, 2, \ldots$ whence we get (B_1).

In connection with the preceding theorem let us note that we have $L^{*\varphi} = L^{*\varphi_0}$, $L_f^{*\varphi} = L_f^{*\varphi_0}$.

6.3. (a) *The following properties are equivalent for a functional* ξ: (1) $\|\xi\|_f \leqslant \infty$, (2) ξ *is* φ-ac *in* $L_f^{*\varphi}$, (3) ξ *is* φ-m *continuous in* $L_f^{*\varphi}$.

(b) *A functional* ξ φ-ac *in* $L_f^{*\varphi}$ ($L^{*\varphi}$) *is of the form*

(*) $$\xi(x) = I(xy),$$

where $I_\psi(y/r) \leqslant 1$, $r = \|\xi\|_f$ ($r = \|\xi\|$) *when* $\xi \neq 0$. *Conversely, a functional of type* $(*)$ *is* φ-ac *on* $L_f^{*\varphi}$ ($L^{*\varphi}$).

(c) *If the integral* $\xi(x) = I(xy)$, $y \neq 0$, *is defined for* $x \in L_f^{*\varphi}$ ($L^{*\varphi}$) *then* $I_\psi(y/r) \leqslant 1$.

(a) By 3.2.2 we obtain the implication $(1) \Rightarrow (3)$ for $L_f^{*\varphi}$, by 3.2 we have $(1) \Rightarrow (2)$, 3.1 implies $(3) \Rightarrow (1)$, and 2.4 gives $(3) \Rightarrow (2)$.

(b) Consider ξ over $L_f^{*\varphi}$. Let $\xi \neq 0$ and then $0 < \|\xi\|_f < \infty$. The proof can be carried out as in [7], with a slight modification. $\xi(\chi_e)$ is σ-additive, μ-absolutely continuous on $\mathscr{E}$, so $\xi(\chi_e) = I(y\chi_e)$, $e \in \mathscr{E}$, where y is integrable. Thus, for an arbitrary simple function s we have $\xi(s) = I(sy)$. Choose a sequence of simple functions y_n, $y_n(t) \geqslant 0$, $y_n(t) \leqslant |y(t)|$ for $t \in T$, $y_n(t) \to |y(t)|$ almost everywhere. Set $r = \|\xi\|_f$. We have:

$$(1) \qquad s_n(t) y_n(t)/r = \varphi(s_n(t)) + \psi(y_n(t)/r), \qquad t \in T,$$

where $s_n(t) = h(y_n(t)/r)$. By 1.4, s_n is μ-measurable. We have $\xi(s_n/r) = I(s_n y_n/r) = I_\varphi(s_n) + I_\psi(y_n/r)$. If $I_\varphi(s_n) \leqslant 1$ we have $1 \geqslant \xi(s_n/r \operatorname{sign} y) = I(s_n|y|/r) \geqslant I(s_n y_n/r)$, hence $I_\psi(y_n/r) \leqslant 1$. If $I_\varphi(s_n) > 1$ then we choose $k+1$ disjoint sets e_i such that $I_\varphi(s_n \chi_{e_i}) = 1$ for $i = 1, 2, \ldots, k$, $I_\varphi(s_n \chi_{e_{k+1}}) \leqslant 1$. From (1) we obtain

$$I(s_n y_n/r) = I_\varphi(s_n) + I_\psi(y_n/r) \leqslant I_\varphi(s_n) + 1,$$

and then $I_\psi(y_n/r) \leqslant 1$. As $y_n(t) \to |y(t)|$ almost everywhere we have $I_\psi(y/r) \leqslant 1$.

Define $\eta(x) = I(xy)$. From the Young inequality we have

$$|\eta(x)| \leqslant |I(xy)| \leqslant \big(I_\varphi(x) + I_\psi(y/r)\big)/r,$$

so η is defined on $L_f^{*\varphi}$ and $\|\eta\| < \infty$. It is then φ-m continuous on $L_f^{*\varphi}$. ξ is also φ-m continuous on $L_f^{*\varphi}$. Let $x \in K_f^{\varphi}$. Choose a sequence (s_n) of simple functions such that $I_\varphi(s_n - x) \to 0$. We have $\xi(s_n) \to \xi(x)$, $\eta(s_n) \to \eta(x)$, $\xi(s_n) = \eta(s_n)$, so $\xi(x) = \eta(x)$. For $L^{*\varphi}$ the proof is analogous.

(c) $I(xy)$ is φ-ac on $L_f^{*\varphi}$ ($L^{*\varphi}$) and then it is sufficient to apply (b).

6.4. THEOREM 4. *The general representation of functionals continuous in* $(L^{*\varphi}, \|\ \|_\varphi)$ *is*

$$(*) \qquad \xi(x) = I(xy) + \eta(x),$$

where

$$(+) \qquad I_\psi(y/\|\xi\|) \leqslant 1$$

when $\xi \neq 0$, η *is a functional continuous in* $(L^{*\varphi}, \|\ \|_\varphi)$, $\eta(x) = 0$ *for* $x \in L_f^{*\varphi}$.

From 2.3 we have $0 < \|\xi\| < \infty$. Thus, by 6.3(a), (b) on $L_f^{*\varphi}$ we have $\xi(x) = I(xy)$, condition $(+)$ being satisfied. $I(xy)$ is then defined on the whole $L^{*\varphi}$. On $L^{*\varphi}$ we have $(*)$, where $\eta(x) = \xi(x) - I(xy)$. $I(xy)$ is φ-ac (cf. 6.3(b)), so it is also φ-m continuous and in this way it is a functional continuous with respect to $\|\ \|_\varphi$ and so is η. Evidently, $\eta(x) = 0$ for $x \in L_f^{*\varphi}$.

On assumption $(+)$, as already noticed, $I(xy)$ is a functional continuous with respect to $\|\ \|_\varphi$, hence a functional of the form $(*)$ is continuous in $(L^{*\varphi}, \|\ \|_\varphi)$.

6.5. *Let* φ *satisfy condition* (A) *from* 6. *There exists a functional* η *continuous in* $(L^{*\varphi}, \|\ \|_{\varphi})$, *such that* $\eta(x) = 0$ *for* $x \in L_f^{*\varphi}$, *which is not modular continuous.*

A construction of this type of functional was given in [7] for $T = \langle a, b\rangle$, $\mathscr{E}$ the algebra of Lebesgue measurable sets and under condition 6(c). However, the proof only uses assumption (A) and one can apply the proof of existence of the functional η with the required property for the space $(L^{*\varphi}, \|\ \|_{\varphi})$ as in [7], 1.7.

6.6. *Let* φ *satisfy condition* (A) *from* 6, $\xi_n(x) = I(xy_n)$ *for* $x \in L^{*\varphi}$, $\|\xi_n\| > 0$ *at least for one* n, $\lim_{n\to\infty} \xi_n(x) = \xi(x)$ *for* $x \in L^{*\varphi}$, *Then for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$(*)\qquad I_{\psi}(y_n\chi_e/k) < \varepsilon \quad \text{for } \mu(e) < \delta,\ n = 1, 2, \ldots,$$

where $k = \sup_n \|\xi_n\| < \infty$.

If $\|\xi_n\| = 0$ then $y_n = 0$ and $(*)$ is satisfied. We can assume $\|\xi_n\| > 0$ for all n. By 5.2 and 5.1, $k = \sup_n \|\xi_n\| < \infty$. From 6.3(c) we have $I_{\psi}(y_n/k) \leqslant 1$ for $n = 1, 2, \ldots$ Let $z_n = y_n/k$; then $I_{\psi}(z_n) \leqslant 1$ for $n = 1, 2, \ldots$ Pick v_0 such that

$$(1)\qquad \varphi(h(v)/2) \leqslant \delta\psi(v) \quad \text{for } v \geqslant v_0.$$

This is possible according to (A). Define sets $a_n = \{t\colon |z_n(t)| \geqslant v_0\}$ and simple functions s_i, $s_i(t) \geqslant v_0$, $s_i(t) \leqslant |z_n(t)|$ for $t \in T$, $s_i(t) \to |z_n(t)|$ almost everywhere in T. We have

$$s_i(t)h(s_i(t))\tfrac{1}{2}\chi_{a_n} = \tfrac{1}{2}\varphi(h(s_i(t)\chi_{a_n})) + \tfrac{1}{2}\psi(s_i(t)\chi_{a_n}),$$

$$(2)\qquad I(s_ih(s_i)\tfrac{1}{2}\chi_{a_n}) = \tfrac{1}{2}I_{\varphi}(h(s_i)\chi_{a_n}) + \tfrac{1}{2}I_{\psi}(s_i\chi_{a_n}).$$

Next, we have in virtue of (1)

$$I_{\varphi}(h(s_i)\tfrac{1}{2}\chi_{a_n}) \leqslant \delta I_{\psi}(s_i\chi_{a_n}) \leqslant \delta I_{\psi}(z_n\chi_{a_n}) \leqslant \delta.$$

But the sequence (3) $I(xz_n)$ is convergent for $x \in L^{*\varphi}$ and therefore the integrals (3) are φ-m equicontinuous by 5.1. Then, if a suitably small δ is chosen in (1) we have

$$I(h(s_i)\tfrac{1}{2}s_i\chi_{a_n}) \leqslant I(h(s_i)\tfrac{1}{2}|z_n|\chi_{a_n}) \leqslant \varepsilon/2,$$

and consequently $I_{\psi}(s_i\chi_{a_n}) \leqslant \varepsilon$. Letting $i \to \infty$ we get

$$(4)\qquad I_{\psi}(z_n\chi_{a_n}) \leqslant \varepsilon.$$

Let $a_n' = T \backslash a_n$, choose a set e such that $\psi(v_0)\mu(e) < \varepsilon$. We have

$$(5)\qquad I_{\psi}(z_n\chi_{a_n'\cap e}) \leqslant I_{\psi}(v_0\chi_{a_n'\cap e}) \leqslant \psi(v_0)\mu(a_n' \cap e) < \varepsilon.$$

From (4), (5) we get

$$I_{\psi}(z_n\chi_e) = I_{\psi}(z_n\chi_{a_n\cap e}) + I_{\psi}(z_n\chi_{a_n'\cap e}) \leqslant 2\varepsilon \quad \text{for } n = 1, 2, \ldots,\ \mu(e) < \varepsilon/\psi(v_0).$$

6.7. THEOREM 5. *Let φ satisfy condition* (A) *and let ψ satisfy condition Δ_2. If the sequence of integrals $\xi_n(x) = I(xy_n)$ is convergent for $x \in L^{*\varphi}$ and $y_n(t) \to y(t)$ in measure then $\|y_n - y\|_\psi \to 0$.*

We have from 6.6 for $\mu(e) < \delta$ (δ sufficiently small)

$$I_\psi(y_n \tfrac{1}{k} \chi_e) \leqslant \varepsilon, \quad I_\psi(y \tfrac{1}{k} \chi_e) \leqslant \varepsilon,$$

k as in 6.6. Since y_n converges to y in measure it follows that $I_\psi[(y_n - y)/2k] \to 0$ and, by condition Δ_2, $\|y_n - y\|_\psi \to 0$.

6.8. In connection with Theorem 3 we give the following counterexample. Let φ satisfy condition (A_1). There exists a sequence of functionals in $L^{*\varphi}$, $\xi_n(x) = I(xy_n)$, $I_\psi(y_n) \leqslant 1$, $n = 1, 2, \ldots$, with the following properties:

(a) $\sup_n \|\xi_n\| < \infty$,

(b) $\lim_{n\to\infty} \xi_n(x) = 0$ for $x \in L_f^{*\varphi}$,

(c) the sequence (ξ_n) is not modular equicontinuous in $L_f^{*\varphi}$.

It follows from condition (A_1) that for some sequence $v_n \to \infty$, $\psi(v_n) > 1/\mu(T)$, we have

$$\varphi(h(v_n)/2)/\psi(v_n) \to 0.$$

Take $e_n \in \mathscr{E}$ such that $\mu(e_n) = 1/\psi(v_n)$ and let $y_n(t) = v_n \chi_{e_n}(t)$ for $n = 1, 2, \ldots$ Then $I_\psi(y_n) = 1$. Let $\xi_n(x) = I(xy_n)$. We have

$$|\xi_n(x)| = |I(xy_n)| \leqslant I_\varphi(x) + 1.$$

This means $\|\xi_n\| \leqslant 2$ for $n = 1, 2, \ldots$ Define $x_n(t) = h(v_n)\frac{1}{2}\chi_{e_n}(t)$, $t \in T$. We have $I_\varphi(x_n) = \varphi(h(v_n)/2)/\psi(v_n) \to 0$, which yields

$$\xi_n(x_n) = \tfrac{1}{2}\varphi(h(v_n)/2)/\psi(v_n) + \tfrac{1}{2} \geqslant \tfrac{1}{2} \quad \textit{for } n = 1, 2, \ldots$$

To prove (b) observe that $I(y_n) = v_n/\psi(v_n) \to 0$, therefore $\xi_n(x) \to 0$ for $x \in L^\infty(T)$. Since the μ-measurable bounded functions are dense in $(L_f^{*\varphi}, \|\ \|_\varphi)$ from the Mazur–Orlicz theorem [3] we get $|\xi_n(x)| < \varepsilon$ when $\|x\|_\varphi < \delta$, $x \in L_f^{*\varphi}$. Consequently $|\xi_n(x)| \to 0$ for $x \in L_f^{*\varphi}$.

7. Let Y be the Banach space where the image of U belongs. We say that the operator U is weakly φ-m continuous if for every functional η from the dual Y^* the operator $\eta(U)$ is φ-m continuous.

THEOREM 6. *Each of the following conditions is sufficient for a weakly φ-m continuous operator in $L^{*\varphi}$ to be φ-m continuous in $L^{*\varphi}$:*

(a) *$L^{*\varphi}$ is separable,*

(b) *Y is separable.*

Suppose U is not φ-m continuous in $L^{*\varphi}$. Then there exists a sequence (x_n), $I_\varphi(x_n) \to 0$, such that for some $\varepsilon > 0$, $\|U(x_n)\| \geqslant \varepsilon$. We first prove that if the functionals η_n are such that $\|\eta_n\| = 1$, $\eta_n(U(x_n)) = \|U(x_n)\|$ for $n = 1, 2, \ldots,$

then for some subsequence η_{k_n}

(1) $$\lim_{k_n\to\infty} \eta_{k_n}(U(x)) \quad \text{exists for } x\in L^{*\varphi}$$

Assume condition (a). The functionals $\eta_n(U(x))$, being φ-m continuous, are continuous in the space $(L^{*\varphi}, \| \ \|_\varphi)$ and for $x\in L^{*\varphi}$ the sequence $|\eta_n(U(x))|$ is bounded. Let L_0 be a countable dense set in $L^{*\varphi}$. Proceeding in the known way we find a subsequence $\eta_{k_n}(U(x))$ convergent for $x\in L_0$. The sequence $\eta_n(U(x))$ is bounded in the whole space $L^{*\varphi}$, therefore (1) holds by the Mazur–Orlicz theorem [3].

Assume condition (b). Let Y_0 be a countable dense set in Y. The sequence $\eta_n(y)$ is bounded for $y\in Y$, so it is possible to find a subsequence $\eta_{k_n}(y)$ convergent for $y\in Y_0$ and consequently also convergent in the whole space Y. Setting $y = U(x)$ we get (1).

It follows from (1) in virtue of Theorem 3 that the sequence of functionals $\eta_{k_n}(U(x))$ is φ-m equicontinuous in $L^{*\varphi}$. Thus, there exists a $\delta > 0$ such that

$$|\eta_{k_n}(U(x))| \leqslant \varepsilon/2 \quad \text{when } I_\varphi(x) < \delta,\ n = 1, 2, \ldots$$

For $n \geqslant n_0$ we have $I_\varphi(x_n) < \delta$, so $\|U(x_{k_n})\| = \eta_{k_n}(U(x_{k_n})) < \varepsilon/2$ and we get a contradiction.

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