ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XXX (1990)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XXX (1990)

ALEXANDER ABIAN and ANDREW D. MARTIN (Ames, Ia.)

Compact partially ordered sets

Abstract. Two definitions of compactness and tower compactness (which are equivalent for topological spaces) for partially ordered sets are introduced and results are obtained showing when one implies or is equivalent to the other.

For any topological space X, the axiom of choice implies [2, p. 163] the equivalence of the following two definitions of compactness:

(1) *Every open cover of X has a finite subcover,*

and

(2) *X is not the union of any nondecreasing* (*with respect to inclusion*) *chain* (*i.e., tower*) *of open proper subsets of X.*

The set T of all open sets of X forms a *poset* (short for *partially ordered set*) with respect to set inclusion $\subseteq$. We call such a poset $(T, \subseteq)$ a *topology poset*. A topology poset $(T, \subseteq)$ has many order-theoretic properties which are not shared by posets in general. For instance, $(T, \subseteq)$ is a complete distributive lattice. On the other hand, since $(T, \subseteq)$ is a poset to begin with, many of the topological notions pertaining to $(T, \subseteq)$ can be judiciously introduced in any poset. Accordingly, motivated by (1) and (2), we introduce two nonequivalent (even in the presence of the axiom of choice) definitions of compactness for posets and show how they are related.

DEFINITION 1. A poset P with a maximum element 1 is called *compact* iff for every subset A of P if $\sup A = 1$ then $\sup F = 1$ for some finite subset F of A.

EXAMPLE 1. Let us consider the set $P = \{1, 0.1, 0.11, 0.111, 0.1111, \ldots\}$ whose nonunit elements are all the real numbers of the form 0 followed by a finite succession of 1's. Clearly, P is a poset with respect to the usual order among the real numbers and 1 is the maximum element of P. However, P is not compact. This is because in P the sup of the subset $A = \{0.1, 0.11, 0.111, 0.1111, \ldots\}$ of P is equal to 1, whereas the sup of no finite subset of A is equal to 1.

1980 *Mathematics Subject Classification*: Primary 06F30.

EXAMPLE 2. The set $P \cup \{0.5\}$, where P is as in Example 1, is a compact poset with respect to the usual order among the real numbers. This is because if in P we have $\sup H = 1$ for a subset H of P, we must have $1 \in H$ and thus $\sup F = 1$ for the finite subset $F = \{1\}$ of H.

Let us recall [1] that a poset P is called *A-inductive* iff every nonempty well ordered subset of P has a supremum in P, and thus by Zorn's Lemma every nonempty chain of P has a supremum in P.

Based on the above, and motivated by (2), we introduce our second definition of compactness for posets.

DEFINITION 2. A poset P with a maximum element 1 is called *tower compact* iff the subset of nonmaximum elements of P is A-inductive.

Accordingly, the poset P of Example 1 is not tower compact. This is because the subset A of the nonmaximum elements of P is not A-inductive. Indeed, A is a nonempty well ordered subset of itself with no supremum in A.

However, the poset $P \cup \{0.5\}$ of Example 2 is clearly tower compact.

To show that a poset can be compact but not tower compact, let us consider the following:

EXAMPLE 3. Let $P \cup \{a, b\}$ be a poset where P and A are as in Example 1, and where a and b are uncomparable but each is an upper bound of the subset A of P and where 1 is the maximum element of $P \cup \{a, b\}$. The poset $P \cup \{a, b\}$ is compact. This is because if $\sup S = 1$ for a subset S of $P \cup \{a, b\}$, we must have $1 \in S$, or $a \in S$ and $b \in S$. Thus, $\sup E = 1$ respectively for the finite subsets $E = \{1\}$, or $E = \{a, b\}$ of $P \cup \{a, b\}$. On the other hand, the poset $P \cup \{a, b\}$ is not tower compact. This is because the subset $A \cup \{a, b\}$ of the nonmaximum elements of $P \cup \{a, b\}$ is not A-inductive. Indeed, the nonempty well ordered subset A of $A \cup \{a, b\}$ has no supremum.

To show that a poset can be tower compact but not compact, let us consider the following:

EXAMPLE 4. Let $Q = \{1\} \cup E \cup S$ where $E = \{0.76, 0.776, 0.7776, 0.77776, \ldots\}$ is the set of all the terminating decimals of the form indicated by the pattern and $S = \{0.7, 0.77, 0.777, 0.7777, 0.77777, \ldots\}$ is also the set of all the terminating decimals of the form indicated by the pattern. The order in Q is defined as follows. The elements of E are pairwise uncomparable, likewise, the elements of S are pairwise uncomparable; otherwise, any two elements of Q are compared as a pair of real numbers in their usual order. The poset Q thus defined is tower compact. This is because the subset $E \cup S$ of the nonmaximum elements of Q is A-inductive. Indeed, every nonempty well ordered subset of $E \cup S$ has $\leqslant 2$ elements. On the other hand, Q is not compact. This is because $\sup S = 1$ whereas the sup of no finite subset of S is equal to 1.

The above examples show that the notions of compactness and tower compactness are not mutually exclusive but are independent of each other.

Below we prove theorems indicating conditions under which one of these two notions of compactness implies or is equivalent to the other.

THEOREM 1. *Let P be a compact poset which is A-inductive. Then P is tower compact.*

Proof. Let P^- be the set of all the nonmaximum elements of P. Thus, $P^- = P - \{1\}$. To show that P is tower compact we must show that P^- is A-inductive. Let us assume the contrary, and let W be a nonempty well ordered subset of P^- such that W has no sup in P^-. However, since P is A-inductive $\sup W$ exists in P and so by our assumption $\sup W = 1$. But then, since P is compact, $\sup W = 1$ implies that $\sup F = 1$ for some finite subset F of the well ordered set W. Since F is a finite well ordered set, $\sup F \in F$. Consequently, $1 \in F$ contradicting that $1 \notin P^-$. Hence, our assumption is false and the theorem is proved.

The following lemma and definition are needed for our next theorem.

LEMMA. *Let P be an A-inductive poset and S a nonempty subset of P such that the supremum of every nonempty finite subset of S exists in P. Then the supremum of S exists in P.*

We do not give a proof of the Lemma since it can be found in [1]. The proof uses the axiom of choice.

A poset P is said to have the *finite supremum property* denoted by *fsp* iff every nonempty finite subset of P has a supremum in P.

THEOREM 2. *Let P be a tower compact poset which has fsp. Then P is compact.*

Proof. Let S be a subset of P such that $\sup S = 1$. To avoid trivial cases, we let $1 \notin S$ and $S \neq \emptyset$. To show that P is compact, we must show that $\sup F = 1$ for some finite subset F of S. Let us assume the contrary. But then since P has fsp, every nonempty finite subset of S has a supremum in $P - \{1\}$. However, P is tower compact and therefore $P - \{1\}$ is A-inductive. Hence, every nonempty finite subset S of the A-inductive poset $P - \{1\}$ has a supremum in $P - \{1\}$. Thus, by the above lemma, $(\sup S) \in (P - \{1\})$ which is a contradiction since $\sup S = 1$. Hence, our assumption is false and the theorem is proved.

THEOREM 3. *Let P be a complete poset. Then P is compact iff P is tower compact.*

Proof. Since P is compact, it is A-inductive and also has fsp. But then the proof of the theorem follows directly from Theorems 1 and 2.

References

[1] A. Abian, *On a fundamental property of A-inductive partially ordered sets*, AMS Abstracts 8 (1987), 257.

[2] J. Kelley, *General Topology*, Van Nostrand, New York 1965.

DEPARTMENT OF MATHEMATICS
IOWA STATE UNIVERSITY
AMES, IOWA 50011, USA