ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XXIX (1990)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XXIX (1990)

ANDRZEJ WALENDZIAK (Warszawa)

Infinite θ-decompositions in upper continuous lattices

Abstract. The present paper is a continuation of the author's paper [3]. Here we shall study infinite θ-decompositions of the unit element of an upper continuous lattice. In this paper we give a generalization of Theorem 3 from [1].

1. Basic notions. Let L be a complete lattice. Lattice join, meet, inclusion and proper inclusion are denoted respectively by the symbols $\vee$, $\wedge$, $\leqslant$ and $<$. Let 0 be the least element of L, and 1 the greatest element of L. Finally, let $K(L)$ denote the set of all compact elements of L, i.e. $c \in K(L)$ iff, for all $T \subseteq L$, $c \leqslant \bigvee T$ implies $c \leqslant \bigvee T'$ for some finite subset T' of T.

An element $u \in L$ is called *join-irreducible* iff, for all $x, y \in L$, $u = x \vee y$ implies $x = u$ or $y = u$. An element $u \in L$ is called *completely join-irreducible* iff, for all $T \subseteq L$, $u = \bigvee T$ implies $u \in T$.

A subset T of the lattice L is said to be *redundant* iff $\bigvee T = \bigvee (T-t)$(1) for some $t \in T$, otherwise it is *irredundant*. If a is an element of L and T is a subset of L, we say that a is an *irredundant join* of T, and we write $a = \tilde{\bigvee} T$, if T is irredundant and $a = \bigvee T$. If $a = \bigvee T$ and for each $t \in T$, $t \wedge \bigvee (T-t) = 0$, then we say that a is a *direct join* of T, and we write $a = \dot{\bigvee} T$.

We denote by $\Theta(L)$ the set of all congruence relations in the lattice L. Let by $0 \in \Theta(L)$. A subset T of L is said to be *θ-independent* if T is irredundant and for each $t \in T$,

$$t \wedge \bigvee (T-t) \equiv 0 \ (\theta).$$

If a is an element of L and $T = \{t_m\colon m \in M\}$ is a subset of L, then we say that a is a θ-join of T, and we write

$$a = \sum T \quad \text{or} \quad a = \sum (t_m\colon m \in M)$$

if T is θ-independent and $a = \bigvee T$. The θ-join of finitely many elements

(1) If $y \in X$, then $X - \{y\}$ is also written $X - y$.

$t_1, \ldots, t_n$ is also written $t_1 + \ldots + t_n$. A representation of an element as a θ-join of elements of the lattice L is said to be a *θ-decomposition* of the element.

A nonzero element a of L that cannot be represented as a θ-join of two elements of L, is said to be *θ-indecomposable.* An element b is called a *θ-summand* of a if $a = b + x$ for some element x. We denote by $S(L)$ the set of all θ-summands of the unit element of L.

Let ω, ι be the congruence relations in L defined by

$$x \equiv y \ (\omega) \quad \text{iff } x = y,$$

$$x \equiv y \ (\iota) \quad \text{for all } x \text{ and } y.$$

Observe that irredundant joins and direct joins are special cases of θ-joins. Indeed, if $T \subseteq L - \{0\}$, then a is the ι-join of T iff $a = \bigvee T$, and a is the ω-join of T iff $a = \bigvee T$. Furthermore, an element $x \in L - \{0\}$ is ι-indecomposable iff it is join-irreducible.

Now, we shall prove the following simple but useful lemma.

LEMMA 1. *Let L be a complete modular lattice. Let*

$$1 = a + b, \qquad a = \sum T.$$

If the set $T \cup \{b\}$ is irredundant, then $1 = \sum T + b$.

Proof. Let t be an element of T. Now compute:

$$t \wedge \left(b \vee \bigvee (T-t)\right) = t \wedge a \wedge \left(b \vee \bigvee (T-t)\right)$$

(observe $\bigvee (T-t) \leqslant a$ and apply modularity)

$$= t \wedge \left[(a \wedge b) \vee \bigvee (T-t)\right]$$

(observe $a \wedge b \equiv 0 \ (\theta)$)

$$\equiv t \wedge \bigvee (T-t) \equiv 0 \ (\theta).$$

Thus the set $T \cup \{b\}$ is θ-independent. Therefore, $1 = \sum T + b$. ∎

2. The property ($\bar{\mathrm{R}}$). Let L be a complete lattice and let $\theta \in \Theta(L)$.

DEFINITION 1. A θ-decomposition $a = \sum T$ of an element $a \in L$ is said to have the *property* (R) with respect to a θ-decomposition $a = \sum T'$ of a iff, for each $t \in T$ there is a $t' \in T'$ such that

$$a = t' + \sum (T-t) = t + \sum (T'-t').$$

DEFINITION 2. We say that a θ-decomposition $a = \sum T$ of a has the *property* ($\bar{\mathrm{R}}$) with respect to a θ-decomposition $a = \sum T'$ if it has property (R) with respect to every θ-decomposition of a of the form $a = \sum S$, where S is a subset of $T \cup T'$.

Now we shall prove two lemmas.

LEMMA 2. *Let the unit element of the lattice L have two θ-decompositions*

$$1 = \sum(a_i\colon\ i \in I), \tag{1}$$

$$1 = \sum(b_j\colon\ j \in J). \tag{2}$$

If the θ-decomposition (2) *has property* ($\bar{\mathrm{R}}$) *with respect to* (1), *then for every finite subset* $J' = \{j_1, \ldots, j_k\} \subseteq J$ *there exists a finite subset* $I' = \{i_1, \ldots, i_k\} \subseteq I$ *such that*

$$\begin{aligned} 1 &= a_{i_n} + \sum(b_j\colon\ j \neq j_n) \\ &= b_{j_n} + b_{j_{n-1}} + \ldots + b_{j_1} + \sum(a_i\colon\ i \in I - \{i_1, \ldots, i_n\}) \end{aligned} \tag{3}$$

for all $n = 1, 2, \ldots, k$.

Proof. We prove Lemma 2 by induction on the number of elements in J'. If $|J'| = 1$ ([1]), then this lemma follows from Definition 1. Let us assume this statement for every $(k-1)$-element subset of J and let $J' = \{j_1, \ldots, j_k\}$. By the induction hypothesis for the subset $\{j_1, \ldots, j_{k-1}\}$ of J' there exists $\{i_1, \ldots, i_{k-1}\} \subset I$ such that (3) holds for each $n = 1, 2, \ldots, k-1$. In particular,

$$1 = b_{j_{k-1}} + \ldots + b_{j_1} + \sum(a_i\colon\ i \in I - \{i_1, \ldots, i_{k-1}\}). \tag{4}$$

By Definition 2, the θ-decomposition (2) has property (R) with respect to (4). Then, for j_k there is an index $i_k \in I - \{i_1, \ldots, i_{k-1}\}$ such that (3) holds for $n = k$. Thus, for the subset $J' \subseteq J$ there exists $I' = \{i_1, \ldots, i_k\} \subseteq I$ such that (3) holds for all $n = 1, 2, \ldots, k$, and the proof of Lemma 2 is complete. ∎

LEMMA 3. *If the unit element of the lattice L has two θ-decompositions*: (2) *and*

$$1 = \sum(a_i\colon\ i \in I') + \sum(b_j\colon\ j \in J') \tag{5}$$

such that J' *is a proper subset of* J, a_i *is compact for each* $i \in I'$, *and* (2) *has the property* ($\bar{\mathrm{R}}$) *with respect to θ-decomposition* (5), *then there are two countable or finite (with an equal number of elements) subsets* $I_0 = \{i_1, \ldots, i_n, \ldots\} \subseteq I'$ *and* $J_0 = \{j_1, \ldots, j_n, \ldots\} \subseteq J'' = J - J'$ *such that*

$$\begin{aligned} 1 &= a_{i_n} + \sum(b_j\colon\ j \neq j_n) \\ &= b_{j_n} + \ldots + b_{j_1} + \sum(b_j\colon\ j \in J') + \sum(a_i\colon\ i \in I' - \{i_1, \ldots, i_n\}) \end{aligned} \tag{6}$$

for all $n = 1, 2, \ldots,$ *and*

$$\bigvee(a_i\colon\ i \in I_0) \leqslant \bigvee(b_j\colon\ j \in J' \cup J_0). \tag{7}$$

Proof. Let j_1 be an arbitrary element of J''. By Definition 1, since the θ-decomposition (2) has property (R) with respect to (5), we conclude that for the index j_1 there is an $i_1 \in I'$ such that

---

([2]) For each set X, $|X|$ denotes the cardinality of X.

$$1 = a_{i_1} + \sum(b_j\colon\ j \in J - j_1),$$

(8) $$1 = b_{j_1} + \sum(b_j\colon\ j \in J') + \sum(a_i\colon\ i \in I' - i_1).$$

The element a_{i_1} is compact and hence there is a finite subset $\{j_2, \ldots, j_k\} \subseteq J'' - \{j_1\}$ such that

$$a_{i_1} \leqslant \bigvee(b_j\colon\ j \in J' \cup J_1), \quad \text{where } J_1 = \{j_1\ j_2, \ldots, j_k\}.$$

By assumption, the θ-decomposition (2) has property ($\bar{\mathrm{R}}$) with respect to (5), and, therefore, it has also this property with respect to (8). Then applying Lemma 2 to the θ-decompositions (2) and (8), we conclude that for the elements $b_{j_2}, \ldots, b_{j_k}$ there exist distinct indices $i_2, \ldots, i_k \in I' - \{i_1\}$ such that (6) holds for each $n = 2, \ldots, k$. In particular,

(9) $$1 = b_{j_k} + \ldots + b_{j_1} + \sum(b_j\colon\ j \in J') + \sum(a_i\colon\ i \in I' - \{i_1, \ldots, i_k\}).$$

Again the element $a_{i_2} \vee \ldots \vee a_{i_k}$ is compact, and there exists a finite subset $\{j_{k+1}, \ldots; j_m\} \subseteq J'' - J_1$ such that

$$a_{i_2} \vee \ldots \vee a_{i_k} \leqslant \bigvee(b_j\colon\ j \in J' \cup J_2), \quad \text{where } J_2 = J_1 \cup \{j_{k+1}, \ldots, j_m\}.$$

Now we apply Lemma 2 to the two θ-decompositions (2) and (9), and to the elements $b_{j_{k+1}}, \ldots, b_{j_m}$. As before, we get the existence of distinct elements $i_{k+1}, \ldots, i_m \in I' - \{i_1, \ldots, i_k\}$ such that (6) holds for each $n = k+1, \ldots, m$. By continuing this process, we obtain two subsets $I_0 = \{i_1, \ldots, i_n, \ldots\}$ and $J_0 = \{j_1, \ldots, j_n, \ldots\}$ such that (6) and (7) hold. This ends the proof of Lemma 3. ∎

3. Upper continuous lattices. A complete lattice L is called *upper continuous* iff, for every $a \in L$ and for every chain $C \subseteq L$, $a \wedge \bigvee C = \bigvee(a \wedge c\colon\ c \in C)$.

We need the following lemma.

LEMMA 4. *Let L be an upper continuous lattice and let T be a subset of $K(L)$. Let $\theta \in \Theta(L)$, and suppose that θ has the following property*:

(*) *for every subset A of L, if $a \equiv 0\ (\theta)$ for each $a \in A$, then $\bigvee A \equiv 0\ (\theta)$.*

If all finite subsets of T are θ-independent, then T is θ-independent.

Proof. First we shall prove that the set T is irredundant. Suppose on the contrary that for some $t_0 \in T$ we have $t_0 \leqslant \bigvee(T - t_0)$. But t_0 is compact and hence $t_0 \leqslant \bigvee(T' - t_0)$, where T' is a finite subset of T containing t_0. Thus T' is redundant, contrary to the θ-independence of T'.

Let t be an arbitrary element of T. Now, we will prove that

(10) $$t \wedge \bigvee(T - t) \equiv 0\ (\theta).$$

P. Crawley ([2], 2.4) has shown that if a is an element of an upper continuous lattice L, A is a subset of L and $\mathfrak{A}$ is the set of all finite subsets of A, then

$$a \wedge \bigvee A = \bigvee(a \wedge \bigvee A'\colon\ A' \in \mathfrak{A}).$$

Therefore,

$$t \wedge \bigvee(T-t) = \bigvee(t \wedge \bigvee X\colon X \in \mathfrak{X}), \tag{11}$$

where $\mathfrak{X}$ is the set of all finite subsets of $T-\{t\}$. Since every finite subset of T is θ-independent, the set $X \cup \{t\}$, where $X \in \mathfrak{X}$ is θ-independent. Hence, for every $X \in \mathfrak{X}$, $t \wedge \bigvee X \equiv 0\ (\theta)$. Then, by property $(*)$, we conclude that

$$\bigvee(t \wedge \bigvee X\colon X \in \mathfrak{X}) \equiv 0\ (\theta).$$

From this and (11) we obtain (10). Therefore, T is θ-independent. ∎

We are now ready to prove the following theorem.

THEOREM 1. *Let L be an upper continuous lattice, and let θ be a congruence relation on L having the property $(*)$. Let the unit element of L have two θ-decompositions* (1) *and* (2) *into compact elements. If the θ-decomposition* (2) *has property* ($\bar{\text{R}}$) *with respect to* (1), *then there is a one-to-one mapping f of I onto J such that, for each $i \in I$,*

$$1 = a_i + \sum(b_j\colon j \neq f(i)). \tag{12}$$

Proof. The idea of the proof comes from [1]. Let $\mathfrak{M}$ be the set of all ordered triples $\langle M, \leqslant^M, f_M\rangle$ where $M \subseteq I$, $\leqslant^M$ is a well-ordering of M, f_M is a one-to-one mapping of M to J and the following conditions are satisfied:

For each $m \in M$,

$$\begin{aligned} 1 &= a_m + \sum(b_j\colon j \neq f_M(m)) \\ &= \sum(b_{f_M(i)}\colon i \in (m]) + \sum(a_i\colon i \in I-(m]), \quad \text{where } (m] = \{i \in M\colon i \leqslant^M m\}, \end{aligned} \tag{13}$$

$$\bigvee(a_i\colon i \in M) \leqslant \bigvee(b_{f_M(i)}\colon i \in M). \tag{14}$$

$\mathfrak{M}$ is nonempty since it contains the triple consisting of the empty set, the empty relation, and the empty mapping (here, we are considering relations and functions as sets of ordered pairs). Define a partial order $\leqslant_{\mathfrak{M}}$ in $\mathfrak{M}$ by $\langle M, \leqslant^M, f_M\rangle \leqslant_{\mathfrak{M}} \langle M', \leqslant^{M'}, f_{M'}\rangle$ if either $M = M'$ or $M = \{i \in M'\colon i <^{M'} m\}$ for some $m \in M'$, $\leqslant^{M'}$ restricted to M coincides with $\leqslant^M$, and the restriction of $f_{M'}$ to M coincides with f_M.

Let $\langle M_\lambda, \leqslant^\lambda, f_\lambda\rangle$ $(\lambda \in \Lambda)$ be a chain of elements in $\mathfrak{M}$. Set

$$N = \bigcup(M_\lambda\colon \lambda \in \Lambda), \qquad \leqslant^N = \bigcup(\leqslant^\lambda\colon \lambda \in \Lambda), \qquad f_N = \bigcup(f_\lambda\colon \lambda \in \Lambda).$$

It is obvious that $\langle N, \leqslant^N, f_N\rangle \in \mathfrak{M}$ and that $\langle N, \leqslant^N, f_N\rangle$ is an upper bound of the chain $\langle M_\lambda, \leqslant^\lambda, f_\lambda\rangle$ $(\lambda \in \Lambda)$. Therefore, by Zorn's lemma, $\mathfrak{M}$ contains a maximal element $\langle P, \leqslant^P, f_P\rangle$.

We consider the set

$$A = \{b_{f_P(i)}\colon i \in P\} \cup \{a_i\colon i \in I-P\}.$$

By (14) we have

$$\bigvee(a_i\colon i \in P) \leqslant \bigvee(b_{f_P(i)}\colon i \in P).$$

Hence,

$$1 = \bigvee (a_i\colon\ i \in P) \vee \bigvee (a_i\colon\ i \in I-P) \leqslant \bigvee (b_{f_P(i)}\colon\ i \in P) \vee \bigvee (a_i\colon\ i \in I-P).$$

Then $1 = \bigvee A$. By (13), all finite subsets of A are θ-independent. From Lemma 4 we conclude that A is θ-independent. Therefore, $1 = \sum A$, and hence, if we set $I' = I-P$ and $J' = f_P(P)$, then we obtain the θ-decomposition (5).

Now we will prove that $P = I$. Suppose on the contrary that $P \neq I$, that is $I' \neq \emptyset$. Consequently, $J' \neq J$. By assumption, the θ-decomposition (2) has property ($\bar{\mathrm{R}}$) with respect to (1), and, therefore, it has also this property with respect to (5). Then applying Lemma 3 to the θ-decompositions (2) and (5) we get two subsets $I_0 = \{i_1, \ldots, i_n, \ldots\} \subseteq I'$ and $J_0 = \{j_1, \ldots, j_n, \ldots\} \subseteq J-J'$ such that (6) and (7) hold.

Set $Q = P \cup I_0$. Define the well-ordering $\leqslant^Q$ of Q by the following rules: if $i, i' \in P$, then $i \leqslant^Q i'$ iff $i \leqslant^P i'$, and for every $i \in P$

$$i <^Q i_1 <^Q i_2 <^Q \ldots <^Q i_n <^Q \ldots$$

Define the mapping f_Q by $f_Q(i) = f_P(i)$ for every $i \in P$, and

$$f_Q(i_n) = j_n \qquad \text{for } n = 1, 2, \ldots$$

By (6) and (7) we obtain that the triple $\langle Q, \leqslant^Q, f_Q\rangle$ belongs to $\mathfrak{M}$. It is obvious that $\langle Q, \leqslant^Q, f_Q\rangle$ is greater than $\langle P, \leqslant^P, f_P\rangle$. This contradiction forces the equality $P = I$. Then $I' = \emptyset$ and from (5) we have $J' = J$. Therefore, $f = f_P$ is one-to-one mapping of I onto J such that, for each $i \in I$, θ-decomposition (12) holds, and the proof of Theorem 1 is complete. ■

4. Modular lattices. Preliminary lemmas. Throughout this section L will denote a complete modular lattice. Let θ be a congruence relation on L. For an element $a \in L$ we denote by $F(a)$ the set of all functions φ of L such that $a\varphi = a$ and from $x \leqslant a$, $x\varphi \equiv 0(\theta)$ follows $x \equiv 0(\theta)$. For two elements $x, y \in L$ we define $y/x = \{u \in L\colon\ x \leqslant u \leqslant y\}$.

Let a be a θ-summand of the element $c \in L$. An element b is called a *θ-complement* of a in the sublattice $c/0$ iff

$$c = a+b. \tag{15}$$

If an element $a \in L$ has a θ-decomposition

$$a = \sum (a_m\colon\ m \in M) \tag{16}$$

we define

$$\overline{a_{i,j,\ldots,n}} = \bigvee (a_m\colon\ m \in M - \{i, j, \ldots, n\})$$

for each subset $\{i, j, \ldots, n\}$ of M. Denote by α_m the function of L defined by the formula $x\alpha_m = a_m \wedge (x \vee \bar{a}_m)$. The maps α_m, $m \in M$ are called the *θ-decomposition functions* related to θ-decomposition (16); any α_m is called the *θ-decomposition function* with respect to the θ-summand a_m of θ-decomposition (16).

Let c be an element of L, and let a be a θ-summand of c. Define the set $F(c, a)$ of maps of L by the rule: $\alpha \in F(c, a)$ iff there exists a θ-complement b of a in $c/0$ such that $x\alpha = a \wedge (x \vee b)$ for every $x \in L$ (i.e., α is the θ-decomposition function with respect to θ-summand a of θ-decomposition (15)).

DEFINITION 3. Let a be a θ-summand of $c \in L$. We say that a satisfies the *B-condition in* $c/0$ if for every $\alpha \in F(c, a)$ and for every θ-decomposition of c with two summands

$$c = c_1 + c_2, \tag{17}$$

$\alpha\gamma_1\alpha \in F(a)$ or $\alpha\gamma_2\alpha \in F(a)$, where γ_1, γ_2 are the θ-decomposition functions related to θ-decomposition (17).

DEFINITION 4. If a belongs to the set $S(L)$, we shall say that a satisfies the *$\bar{B}$-condition*, if for every $c \in L$ such that a is a θ-summand of c, a satisfies the B-condition in $c/0$.

Now, we shall prove the following lemma.

LEMMA 5. *Let the unit element of the lattice L have two θ-decompositions*:

$$1 = a + b, \tag{18}$$

$$1 = d + e. \tag{19}$$

If the elements b and e are comparable, d is θ-indecomposable, and a satisfies the B-condition in L, then

$$1 = d + b = a + e.$$

Proof. Let α, β and δ, ε be the pairs of θ-decomposition functions related to θ-decompositions (18) and (19), respectively. Suppose $b \leqslant e$. If $e \leqslant b$, then the proof of this lemma is similar. Observe that $\alpha\varepsilon\alpha \notin F(a)$.

Indeed, suppose on the contrary that $a = a\alpha\varepsilon\alpha$. Then

$$a = a \wedge (a\alpha\varepsilon \vee b) \leqslant a \wedge e \quad \text{(since } b \leqslant e\text{)}.$$

Hence $a \leqslant e$, a contradiction. Since a satisfies the B-condition and $\alpha\varepsilon\alpha \notin F(a)$, so $\alpha\delta\alpha \in F(a)$. Therefore, by Lemma 2 of [3] we obtain $1 = d + b = a + e$. ■

We are now going to prove the fundamental lemma.

LEMMA 6. *Let the unit element of the lattice L have a θ-decomposition* (19) *such that the element d is θ-indecomposable and compact, and let θ-decomposition* (1) *of* 1 *be given. Let* a_i, $i \in I$, *and* δ, ε *be the θ-decomposition functions related to* (1) *and* (19), *respectively. If d satisfies the* $\bar{B}$*-condition, then there exists an* $i_0 \in I$ *such that* $\delta\alpha_{i_0}\delta \in F(d)$.

Proof. We shall consider two cases.

Case 1. Let $d \leqslant a_{i_0}$ for some $i_0 \in I$. Then

$$d\delta\alpha_{i_0}\delta = d\alpha_{i_0}\delta = [a_{i_0} \wedge (d \vee \overline{a_{i_0}})]\,\delta$$

(by modularity and $d \leqslant a_{i_0}$)

$$= [d \vee (a_{i_0} \wedge \overline{a_{i_0}})]\,\delta = d.$$

Suppose now that $x \leqslant d$ and $x\delta\alpha_{i_0}\delta \equiv 0\ (\theta)$. By Property II from [3], we have $x\delta\alpha_{i_0}\delta \equiv x\ (\theta)$. Consequently, $x \equiv 0\ (\theta)$. Therefore, $\delta\alpha_{i_0}\delta \in F(d)$.

Case 2. Let $d \nleqslant a_i$ for each $i \in I$. Since d is compact, there is a finite subset $I_1 \subseteq I$ such that $d \leqslant \bigvee(a_i\colon i \in I_1)$. Compute:

$$\bigvee(d\alpha_i\colon i \in I_1) = \bigvee(a_i \wedge (d \vee \bar{a}_i)\colon i \in I_1)$$

(observe $a_i \leqslant \bar{a}_{i'}$ for each $i \neq i'$

and apply modularity)

$$= \bigvee(a_i\colon i \in I_1) \wedge \bigwedge(d \vee \bar{a}_i\colon i \in I_1) \geqslant d.$$

If we set $c = \bigvee(d\alpha_i\colon i \in I_1)$, then we have $d \leqslant c$. Without any loss of generality we can assume that the set $\{d\alpha_i\colon i \in I_1\}$ is irredundant. Thus

$$c = \sum(d\alpha_i\colon i \in I_1). \tag{20}$$

Since $1 = d \vee e$, so $c = c \wedge (d \vee e)$ and by modularity we obtain

$$c = d \vee (c \wedge e).$$

Observe that the set $\{d, c \wedge e\}$ is irredundant. Indeed, $c \neq c \wedge e$ and also $c \neq d$, since otherwise d is not θ-indecomposable, a contradiction.

Moreover, $d \wedge (c \wedge e) \equiv 0\ (\theta)$. Therefore

$$c = d + (c \wedge e). \tag{21}$$

Let γ_i, $i \in I_1$, and δ', ε' be the θ-decomposition functions related to (20) and (21), respectively. Since d satisfies the $\bar{B}$-condition, by Definition 4 d satisfies the B-condition in the lattice $c/0$. Applying Lemma 4 of paper [3] to the two θ-decompositions (20) and (21) we conclude that there exists an $i_0 \in I_1$ such that $\delta'\gamma_{i_0}\delta' \in F(d)$. Then $d\delta'\gamma_{i_0}\delta' = d$, hence $d\gamma_{i_0}\delta' = d$. Therefore, $d \wedge [d\gamma_{i_0} \vee (c \wedge e)] = d$ and we obtain the inequality $d \leqslant d\gamma_{i_0} \vee (c \wedge e)$. From this, since $d\gamma_{i_0} \leqslant d\alpha_{i_0}$, we have $d \leqslant d\alpha_{i_0} \vee e$. Consequently, $1 = d \vee e \leqslant d\alpha_{i_0} \vee e$, and hence $1 = d\alpha_{i_0} \vee e = d\delta\alpha_{i_0} \vee e$. Then

$$d = d\delta\alpha_{i_0}\delta. \tag{22}$$

Suppose now that $x \leqslant d$ and $x\delta\alpha_{i_0}\delta \equiv 0\ (\theta)$. Hence, since $x\delta'\gamma_{i_0}\delta' \leqslant x\delta\alpha_{i_0}\delta$ we obtain that $x\delta'\gamma_{i_0}\delta' \equiv 0\ (\theta)$. Therefore, since $x \leqslant d$ and $\delta'\gamma_{i_0}\delta' \in F(d)$, we get $x \equiv 0\ (\theta)$. From this and (22) we conclude that $\delta\alpha_{i_0}\delta \in F(d)$. Thus the proof of Lemma 6 is completed. ∎

LEMMA 7. *Let two θ-decompositions* (1) *and* (19) *of* 1 *be given. Suppose that each* a_i *and* d *is θ-indecomposable and that* d *is compact. If* d *satisfies the* $\bar{B}$*-condition, then there exists an* $i_0 \in I$ *such that*

$$1 = a_i + e = d + \sum (a_i\colon\ i \neq i_0).$$

Proof. Let α_i, $i \in I$, and δ, ε be the θ-decomposition functions related to (1) and (19), respectively. By Lemma 6 we conclude that there exists an $i_0 \in I$ such that $\delta\alpha_{i_0}\delta \in F(d)$. We consider two θ-decompositions

$$1 = d + e = a_{i_0} + \overline{a_{i_0}}.$$

From Lemma 2 of [3] we have $1 = a_{i_0} + e = d + \overline{a_{i_0}}$. ∎

Now we prove that the set $A = \{d\} \cup \{a_i\colon\ i \neq i_0\}$ is irredundant. Assume on the contrary that there exists an $i_1 \in I - \{i_0\}$ such that

$$1 = d + \overline{a_{i_0, i_1}}. \tag{23}$$

Applying Lemma 5 to the two θ-decompositions (23) and $1 = a_{i_0} + \overline{a_{i_0}}$ we conclude that

$$1 = a_{i_0} + \overline{a_{i_0, i_1}} = \overline{a_{i_1}}.$$

This means that the set $\{a_i\colon\ i \in I\}$ is not θ-independent, contrary to our assumptions. Therefore, the set A is irredundant and hence by Lemma 1 we obtain

$$1 = d + \sum (a_i\colon\ i \neq i_0).$$

This completes the proof of the lemma. ∎

Finally, we shall prove the following lemma.

LEMMA 8. *A θ-decomposition* (2) *of the unit element of the lattice L into θ-indecomposable compact elements satisfying the* $\bar{B}$*-condition has the property* (R) *with respect to any other θ-decomposition* (1) *of* 1 *into θ-indecomposable summands.*

Proof. Let j_0 be an arbitrary element of J. We consider two θ-decompositions of $1\colon 1 = b_{j_0} + \overline{b_{j_0}}$ and (1). By Lemma 7, there exists an $i_0 \in I$ such that

$$1 = a_{i_0} + \overline{b_{j_0}} = b_{j_0} + \sum (a_i\colon\ i \neq i_0). \tag{24}$$

Observe that the set $A = \{a_{i_0}\} \cup \{b_j\colon\ j \neq j_0\}$ is irredundant. Assume on the contrary that $1 = a_{i_0} \vee \overline{b_{j_0, j_1}}$, for some $j_1 \in J - \{j_0\}$. Then we have two θ-decompositions of 1:

$$1 = b_{j_0} + \overline{b_{j_0}} = a_{i_0} + \overline{b_{j_0, j_1}}.$$

From Lemma 5 we get

$$1 = b_{j_0} + \overline{b_{j_0,j_1}} = b_{j_1}.$$

This means that the set $\{b_j: j \in J\}$ is not θ-independent and we obtain a contradiction. Thus, the set A is irredundant. Now, by Lemma 1, from (24) we get

$$1 = a_{i_0} + \sum(b_j: j \neq j_0) = b_{j_0} + \sum(a_i: i \neq i_0).$$

Therefore, by Definition 1, the θ-decomposition (2) has property (R) with respect to (1).

5. Upper continuous modular lattices. Main results. The following theorem holds.

THEOREM 2. *Let L be an upper continuous modular lattice and let θ be an element of $\Theta(L)$ having the property* $(*)$. *If the unit element of L has a θ-decomposition* (2) *into θ-indecomposable compact elements satisfying the $\bar{B}$-condition, then for any other θ-decomposition* (1) *into θ-indecomposable compact summands we have that there exists a one-to-one mapping f on I onto J such that, for each $i \in I$, θ-decomposition* (12) *holds.*

Proof. By Lemma 8 we conclude that the θ-decomposition (2) has property (R) with respect to every θ-decomposition of 1 of the form

$$1 = \sum(a_i: i \in I') + \sum(b_j: j \in J'),$$

where I' and J' are subsets of I and J, respectively. Then, by Definition 2, (2) has the property ($\bar{\text{R}}$) with respect to θ-decomposition (1). The statement now follows from Theorem 1. ■

We need the following lemma.

LEMMA 9. *Let L be a modular lattice and let $\theta \in \Theta(L)$. If a θ-indecomposable element $a \in S(L)$ is of finite length*[5], *then a satisfies the $\bar{B}$-condition.*

Proof. Let c be an element of L such that a is a θ-summand of c. The element a is of finite length in L, and, therefore, in $c/0$. Then, by Lemma 7 of [3], a satisfies the B-condition in the lattice $c/0$ (in the proof of Lemma 7, [3], it was not used that L is of finite length only that so is a). Hence, by Definition 4 we obtain the statement. ■

The main result of our paper is expressed in the following theorem.

THEOREM 3. *Let L be an upper continuous modular lattice and let θ be a congruence relation on L having the property* $(*)$. *If*

$$1 = \sum(a_i: i \in I) = \sum(b_j: j \in J)$$

---

[5] a is of finite length iff the lattice $a/0$ is of finite length.

are two θ-decompositions of the unit element of L such that each a_i and each b_j is of finite length and θ-indecomposable, then there exists a one-to-one mapping f of I onto J such that, for each $i \in I$,

$$1 = a_i + \sum (b_j\colon j \neq f(i)).$$

Proof. P. Crawley shows in [1] (Lemma 3) that every element of finite length (in an upper continuous lattice) is compact. Therefore, from Lemma 9 and Theorem 2 our theorem follows. ∎

Remark. The case $\theta = \omega$ yields the theorem of P. Crawley (cf. Theorem 3 in [1]).

It is obvious that every lattice of finite length is upper continuous. Therefore, from Theorem 3 we get at once

COROLLARY (cf. [3], p. 358). *Let L be a modular lattice of finite length and let $\theta \in \Theta(L)$. If the unit element of L has two θ-decompositions*

$$1 = a_1 + a_2 + \ldots + a_n, \qquad 1 = b_1 + b_2 + \ldots + b_m$$

into θ-indecomposable elements, then $n = m$ and for every a_i there is a b_j such that

$$1 = b_1 + \ldots + b_{j-1} + a_i + b_{j+1} + \ldots + b_n.$$

Now, we consider the case when $\theta = \iota$. The following lemma holds.

LEMMA 10. *Every join-irreducible ι-summand of the unit element of a modular lattice L satisfies the $\bar{B}$-condition.*

Proof. Let a be a join-irreducible ι-summand of 1 and let c be an arbitrary element of L such that a is a ι-summand of c. By the proof of Corollary 2 ([3]), it is obvious that a satisfies the B-condition in the sublattice $c/0$, and hence, by Definition 4, we conclude that a satisfies the $\bar{B}$-condition. ∎

From Theorem 2 we obtain now

THEOREM 4. *Let L be an upper continuous modular lattice. If the unit element of L has two irredundant decompositions*

$$1 = \tilde{\bigvee}(a_i\colon i \in I) = \tilde{\bigvee}(b_j\colon j \in J) \tag{25}$$

into completely join-irreducible elements, then there is a one-to-one mapping f of I onto J such that, for each $i \in I$,

$$1 = a_i \,\tilde{\vee}\, \tilde{\bigvee}(b_j\colon j \neq f(i)).$$

Proof. In the proof of Lemma 3 ([1]) it was shown that every completely join-irreducible element of an upper continuous lattice is compact. Therefore, all ι-summands of ι-decompositions (25) are compact. Moreover, since every completely join-irreducible element of a complete lattice is join-irreducible, by Lemma 10 we infer that each a_i and each b_j satisfies the $\bar{B}$-condition. Now, from Theorem 2 our theorem follows. ■

References

[1] P. Crawley, *Direct decompositions with finite dimensional factors*, Pacific J. Math. 12 (1962), 457–468.
[2] P. Crawley, R. P. Dilworth, *Algebraic Theory of Lattices*, Prentice-Hall, Englewood Cliffs, N.J., 1973.
[3] A. Walendziak, *Solution of Grätzer's problem*, Comm. Math. 26 (1986), 349–359.

(INSTYTUT MATEMATYKI, WYŻSZA SZKOŁA PEDAGOGICZNA, OPOLE)
INSTITUTE OF MATHEMATICS, PEDAGOGICAL COLLEGE,
OPOLE, POLAND