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Generalizations of Natanson's lemma

Abstract. Some integral inequalities of Natanson's type were proved in papers [1], [3], [6]. The aim of this paper is to generalize these inequalities to the case of Lebesgue–Bochner integral.

1. Introduction. I. P. Natanson has obtained an important integral inequality ([4], p. 243). Some generalizations of that inequality have been given by P. G. Mamedov ([3]), R. Taberski ([6]) and A. D. Gadzhijev ([1]), where also applications to the theory of singular integrals are presented.

In this paper, I am going to generalize these estimates to the case of Lebesgue–Bochner integral. I shall also prove that it is impossible to improve the generalized estimates (in the sense which will be specified in Theorems 3 and 4).

2. Notation. Throughout this paper we use the Lebesgue–Bochner integral from [5] (see [5], Chapter IV)(1).

We shall make use of some definitions.

Let $(H, \|\ \|)$ be a finite-dimensional and real Banach space.

The norm $\|\ \|$ is called a *sum-norm* if there is a basis $\{e_1, \ldots, e_n\}$ of H such that

$$(1) \qquad \Big\|\sum_{k=1}^{n} x_k e_k\Big\| = \sum_{k=1}^{n} |x_k| \quad \text{for all real } x_1, \ldots, x_n.$$

The basis will be called a *sum-basis*.

If $\psi\colon A \to \boldsymbol{R}$, where $\emptyset \neq A \subset \boldsymbol{R}$, then the variation of ψ in A will be denoted by $\operatorname{var}_{t\in A} \psi(t)$ or $\operatorname{var}_A \psi(t)$.

Let $\|\ \|$ be a sum-norm of H. For $\varphi\colon A \to H$ we define the variation in A by

$$(2) \qquad V(A; \varphi) = V\Big(A; \sum_{j=1}^{n} \varphi_j e_j\Big) = \sum_{j=1}^{n} \operatorname*{var}_{A} \varphi_j(s),$$

(1) The symbol $\rightarrow$ complicates the notation. Therefore we omit it in the sequel. So, $\int_{a+}^{b}$, $\int_{a}^{b-}$ will be written instead of $\int_{a\leftarrow}^{b}$, $\int_{a}^{\rightarrow b}$, respectively.

where $\varphi = \sum_{j=1}^{n} \varphi_j e_j$ is the expansion of φ with respect to the sum-basis $\{e_1, \ldots, e_n\}$.

Let E, G, F be three Banach spaces which may be identical or distinct.

The norm of an arbitrary bounded bilinear operator $B\colon E \times G \to F$ will be denoted by $\|B\|$ (see [5], p. 117); $\mathscr{L}(\langle a, b\rangle; E)$ will denote the linear space of all functions $f\colon \langle a, b\rangle \to E$ integrable in the Lebesgue–Bochner([2]) sense on the interval $\langle a, b\rangle$ (see [5], p. 504 with $p = 1$).

3. Formulation of the results. *Throughout all this and next sections, we shall denote by E, H, G real Banach spaces, where H is finite-dimensional and the norm in H is a sum-norm. Moreover, we shall denote by B a bounded bilinear operator with domain $E \times H$, whose range lies in G.*

We shall assume that $g\colon \langle 0, b-a\rangle \to \mathbf{R}$ is absolutely continuous and nondecreasing on $\langle 0, b-a\rangle$, positive on $(0, b-a)$, $g(0) = 0$.

We shall make use of the following conditions:

(3) $\varphi\colon \langle a, b\rangle \to H$ has bounded variation $V(\langle a+\eta, b\rangle;\ \varphi)$ for all $\eta \in (0, b-a)$;

(4) $\psi\colon \langle a, b\rangle \to H$ has bounded variation $V(\langle a, b-\eta\rangle;\ \psi)$ for all $\eta \in (0, b-a)$.

Under these notations we formulate the following theorems:

THEOREM 1. *Assume that the functions g, φ satisfy the conditions listed above and, in addition,*

$$\int_a^b V(\langle s, b\rangle;\ \varphi)\, g'(s-a)\, ds < \infty. \tag{5}$$

If $f \in \mathscr{L}(\langle a, b\rangle; E)$ and

$$M = \sup_{0 < h \leqslant b-a} \left\| \frac{1}{g(h)} \int_a^{a+h} f(t)\, dt \right\| < \infty, \tag{6}$$

then the improper Lebesgue–Bochner integral

$$I = \int_{a+}^{b} B\big(f(t), \varphi(t)\big)\, dt,$$

exists, and

$$\|I\| \leqslant \|B\|\, M \int_a^b \{V(\langle s, b\rangle;\ \varphi) + \|\varphi(b)\|\}\, g'(s-a)\, ds. \tag{7}$$

([2]) The name "Bochner" is omitted in [5].

COROLLARY 1. *Let φ be a function satisfying* (3) *together with*

$$\int_a^b V(\langle s, b\rangle; \varphi)\, ds < \infty.$$

Assume that $f \in \mathscr{L}(\langle a, b\rangle; E)$ and that f satisfies

$$M = \sup_{0<h\leqslant b-a} \left\| \frac{1}{h} \int_a^{a+h} f(t)\, dt \right\| < \infty.$$

Then the improper Lebesgue–Bochner integral

$$I = \int_{a+}^{b} B\big(f(t), \varphi(t)\big)\, dt$$

exists and the following inequality holds

$$\|I\| \leqslant \|B\|\, M \int_a^b \{V(\langle s, b\rangle; \varphi) + \|\varphi(b)\|\}\, ds.$$

THEOREM 2. *Suppose that the functions g, ψ satisfy the conditions listed before Theorem* 1 *and*

$$-\int_a^b V(\langle a, s\rangle; \psi) \cdot \{g(b-s)\}'_s\, ds < \infty.$$

If $f \in \mathscr{L}(\langle a, b\rangle; E)$ and

$$N = \sup_{0<h\leqslant b-a} \left\| \frac{1}{g(h)} \int_{b-h}^{b} f(t)\, dt \right\| < \infty,$$

then the improper integral

$$J = \int_a^{b-} B\big(f(t), \psi(t)\big)\, dt$$

exists, and

$$\|J\| \leqslant -\|B\|\, N \int_a^b \{V(\langle a, s\rangle; \psi) + \|\psi(a)\|\} \cdot \{g(b-s)\}'_s\, ds.$$

COROLLARY 2. *Let ψ be a function satisfying* (4) *and*

$$\int_a^b V(\langle a, s\rangle; \psi)\, ds < \infty.$$

If f is a function belonging to $\mathscr{L}(\langle a, b\rangle; R)$ and satisfying

$$N = \sup_{0<h\leqslant b-a} \left\| \frac{1}{h} \int_a^{b-h} f(t)\, dt \right\| < \infty,$$

then the improper Lebesgue–Bochner integral

$$J = \int_a^{b-} B\big(f(t), \psi(t)\big)\,dt$$

exists and

$$\|J\| \leqslant \|B\|\, N \int_a^b \{V(\langle a, s\rangle; \psi) + \|\psi(a)\|\}\, ds.$$

In the next two theorems we shall assume that $H = \boldsymbol{R}$, $G = E$ and $B\colon E\times \boldsymbol{R} \to \boldsymbol{R}$ is defined by

$$B(x, \alpha) = \alpha\cdot x \quad \text{for all } x\in E,\ \alpha\in \boldsymbol{R}\,(^3).$$

THEOREM 3. *Suppose that the functions φ, ψ satisfy the assumptions of Theorem 1 and Theorem 2, respectively. Assume also that φ is nonincreasing on $(a, b\rangle$, ψ is nondecreasing on $\langle a, b)$, and they are nonnegative on (a, b). Then there are functions $f_1, f_2 \in \mathscr{L}(\langle a, b\rangle; E)$ for which the equalities in the assertions of Theorem 1, Theorem 2, respectively, are fulfilled, and whose integrals are not equal to zero on $\langle a, b)$.*

THEOREM 4. *Suppose that the functions φ, ψ satisfy all assumptions mentioned in the previous theorem and f, h satisfy the assumptions of Theorem 1, Theorem 2, respectively. Then there exist two sequences of functions $\{\varphi_n\}$, $\{\psi_n\}$, such that*

(8) $\quad$ *φ_k, ψ_k satisfy the same conditions as φ, ψ, respectively ($k\in \boldsymbol{N}$);*

(9) $$\int_a^b \varphi_k(t)\{g(t-a)\}_t'\,dt = \int_a^b \psi_k(t)\cdot\{-g(b-t)\}_t'\,dt = 1 \qquad (k\in \boldsymbol{N}),$$

(10) $$\begin{aligned} M &= \lim_{k\to\infty} \Big\| \int_{a+}^{b} \varphi_k(t) f(t)\,dt \Big\| \\ &= \lim_{k\to\infty} M \int_{a+}^{b} \{V(\langle s, b\rangle; \varphi_k) + \|\varphi_k(b)\|\}\cdot\{g(s-a)\}_s'\,ds, \\ N &= \lim_{k\to\infty} \Big\| \int_{a}^{b-} \psi_k(t) h(t)\,dt \Big\| \\ &= -\lim_{k\to\infty} N \int_{a}^{b-} \{V(\langle a, s\rangle; \psi_k) + \|\psi_k(a)\|\}\cdot g(b-s)\}_s'\,ds, \end{aligned}$$

i.e., we have the equalities in Theorem 1 and Theorem 2 on letting $k\to\infty$.

4. Auxiliary results. Let w be an arbitrary function defined on $\langle a, b\rangle$, whose range lies in a finite-dimensional Banach space X with the sum-norm $\|\ \|$ and the sum-basis $e_1, \ldots, e_n$. We assume that the function has finite

(3) Let us observe that $\|B\| = 1$.

variation on every interval $\langle a+\eta, b\rangle$, where $\eta \in (0, b-a\rangle$. We shall denote by $\tilde{w}$ the function

$$w(s) = w(s-0) = \lim_{t \to s-} w(t), \qquad s \in (0, b-a\rangle.$$

LEMMA 1. *The function $\tilde{w}$ is left-hand continuous.*

Proof. Let $w_1, \ldots, w_n$ be the coordinates of w in the sum-basis. Fix k in $\{1, \ldots, n\}$. According to (2) and the Jordan decomposition, $w_k = u-v$ on $\langle a+\eta, b\rangle$ $(\eta \in (0, b-a\rangle)$ where u, v are nondecreasing and nonnegative.

The functions $\tilde{u}, \tilde{v}$ are left-hand continuous. Indeed, let $0 < s \leqslant b-a$ and let $\{t_k\}$ be an arbitrary sequence of points of the interval $(0, s)$, such that $\lim_{k\to\infty} t_k = s$. Then for sufficiently large k we have $0 < t_k - 1/k < t_k$ and $u(t_k - 1/k) \leqslant \tilde{u}(t_k) \leqslant \tilde{u}(s)$, so

$$\lim_{k\to\infty} \tilde{u}(t_k) = \tilde{u}(s)$$

because $\lim_{k\to\infty} u(t_k - 1/k) = \tilde{u}(s)$.

The same is true for $\tilde{v}$, whence $\tilde{u}, \tilde{v}$ are left-hand continuous and so $\tilde{w}_k$, because $\tilde{w}_k = \tilde{u} - \tilde{v}$. Our lemma is now evident. ■

LEMMA 2. *Let $f \in \mathscr{L}(\langle a, b\rangle; E)$ and φ be a function satisfying* (3). *Then either both integrals $\int_{a+}^{b} B(f, \varphi)\,dt$, $\int_{a+}^{b} B(f, \tilde{\varphi})\,dt$ exist and they are equal, or neither of them exists.*

Proof. Let $\varphi_1, \ldots, \varphi_n$ be the coordinates of φ in a sum-basis $e_1, \ldots, e_n$ of H. According to (3) and (2), the sets $\{s \in (a, b\rangle\colon \tilde{\varphi}_k(s) \neq \varphi_k(s)\}$, $k = 1, \ldots, n$, are at most countable, so the set $\{s \in (a, b\rangle\colon \tilde{\varphi}(s) \neq \varphi(s)\}$ is at most countable (see [2], p. 26, Th. 3.).

The rest of the proof is obvious. ■

LEMMA 3. *If the functions φ, g satisfy the assumptions of Theorem* 1, *then*

$$\int_a^b \{V(\langle s, b\rangle; \tilde{\varphi}) + \|\tilde{\varphi}(b)\|\} \cdot \{g(s-a)\}'_s\,ds$$
$$\leqslant \int_a^b \{V(\langle s, b\rangle; \varphi) + \|\varphi(b)\|\} \cdot \{g(s-a)\}'_s\,ds.$$

Proof. Let $\varphi_1, \ldots, \varphi_n$ be the coordinates of φ in a sum-basis $e_1, \ldots, e_n$ of H. By the Jordan decomposition ([2], p. 25) we have

$$\varphi_k = u_k - v_k \qquad \text{on } \langle a+\eta, b\rangle$$

for all $\eta \in (0, b-a\rangle$ and $k = 1, \ldots, n$ where u_k, v_k are nondecreasing and nonnegative. If for $k = 1, \ldots, n$ the functions u_k, v_k are continuous in the point $s \in (a, b)$, then we have

$$\begin{aligned}V(\langle s, b\rangle, \tilde{\varphi}_k) &\leqslant V(\langle s, b\rangle; \tilde{u}_k)+V(\langle s, b\rangle; \tilde{v}_k)\\ &= \tilde{u}_k(b)-\tilde{u}_k(s)+\tilde{v}_k(b)-\tilde{v}_k(s)\\ &= \tilde{u}_k(b)-u_k(b)+u_k(b)-u_k(s)+\tilde{v}_k(b)-v_k(b)+v_k(b)-v_k(s)\\ &= V(\langle s, b\rangle, u_k)+V(\langle s, b\rangle, v_k)+\tilde{u}_k(b)-u_k(b)+\tilde{v}_k(b)-v_k(b)\text{(4)}\\ &= V(\langle s, b\rangle, \varphi_k)+\tilde{u}_k(b)-u_k(b)+\tilde{v}_k(b)-v_k(b)\\ &= V(\langle s, b\rangle, \varphi_k)-|\varphi_k(b)-\tilde{\varphi}_k(b)|,\end{aligned}$$

because $u_k(b) \geqslant \tilde{u}_k(b)$ and $v_k(b) \geqslant \tilde{v}_k(b)$ $(k = 1, \ldots, n)$. Hence, according to (2), we get

$$V(\langle s, b\rangle, \tilde{\varphi}) \leqslant V(\langle s, b\rangle, \varphi)+\|\varphi(b)-\tilde{\varphi}(b)\|,$$

where $s \in (a, b)$ is such that φ is continuous in it. Our conclusion is now evident (see [2], p. 26, Th. 3.).

5. Proofs of the theorems

Proof of Theorem 1. Since the lemmas may be applied to the function φ, it is sufficient to prove Theorem 1 for $\tilde{\varphi}$ instead of φ. Thus, without loss of generality, *we may assume that φ is left-hand continuous.*

Let α be in (a, b) and define I_α by

$$I_\alpha = \int_\alpha^b B(f, \varphi)\,dt.$$

We shall apply Theorem 92, 2° from [5], pp. 635–636. to the following case:

$$M(t) = F(t) = \int_a^t f(u)\,du, \qquad N = \varphi,\ \mu = f\cdot m,\ v = p\cdot v_0,$$

where m is Lebesgue measure on the real line, v is a measure whose indefinite integral is φ, v_0 is a positive basis of the measure v.

Using the theorem we obtain

$$(11)\quad I_\alpha = \int_\alpha^b B(fdm, \varphi) = \int_\alpha^b B(du, \varphi) = B\big(F(t), \varphi(t)\big)\Big|_{t=\alpha}^{t=b} - \int_\alpha^b B(F, dv) \equiv A_\alpha - B_\alpha,$$

say.

Let us observe that

$$\left\|\frac{1}{g(t-a)}F(t)\right\| \leqslant M, \qquad t \in (a, b)$$

(4) Compare [2], p. 26, (4.2).

(see (6)) and

$$B_\alpha = \int_\alpha^b B(F, dv) = \int_\alpha^b B(F, p)\,dv_0 = \int_\alpha^b g(t-a)\,B\left(\frac{1}{g(t-a)}F(t),\, p(t)\right) dv_0(t),$$

whence we have

$$\|B_\alpha\| \leqslant \|B\|\, M \int_\alpha^b g(t-a)\cdot \|p(t)\|\, dv_0(t). \tag{12}$$

Since $\|p\|\, v_0$ is the least absolute majorant of the measure $v = pv_0$ ([5], p. 530, note*), then using Theorem 88, [5], p. 614, formula (IV, 9; 36), we get

$$\int_{\langle\alpha,x)} \|p(t)\|\, dv_0(t) = V(\langle\alpha, x\rangle, \varphi)$$

$$= V(\langle\alpha, b\rangle, \varphi) - V(\langle x, b\rangle, \varphi) \equiv c_\alpha - \vartheta(x), \quad \text{say.}$$

Thus $c_\alpha - \vartheta(x)$ and so $-\vartheta(x)$ are indefinite integrals of the measure $\|p\|\, v_0$.

Using Theorem 92 from [5] to the case: $\mu = m$, $M(t) = g(t-a)$, $N(t) = -\vartheta(t)$, B the usual multiplication, we obtain

$$\int_\alpha^b \{g(t-a)\cdot\{\|p(t)\|\, dv_0(t)\} = g(t-a)\cdot[-\vartheta(t)]|_\alpha^b - \int_\alpha^b [dg(t-a)]\cdot[-\vartheta(t)]$$

$$= g(\alpha-a)\,\vartheta(\alpha) + \int_\alpha^b \vartheta(t)\, dg(t-a)$$

$$\leqslant \int_a^\alpha \vartheta(t)\, dg(t-a) + \int_\alpha^b \vartheta(t)\, dg(t-a)$$

$$= \int_a^b \vartheta(t)\,\{g(t-a)\}'_t\, dt,$$

from which follows the inequality

$$\int_\alpha^b g(t-a)\, \|p(t)\|\, dv_0(t) \leqslant \int_a^b \vartheta(t)\,\{g(t-a)\}'_t\, dt.$$

Applying this inequality and (12), we find

$$\|B_\alpha\| \leqslant \|B\|\, M \int_a^b \vartheta(t)\,\{g(t-a)\}'_t\, dt.$$

In order to estimate $\|A_\alpha\|$ we write

$$\|A_\alpha\| = \left\|B\big(F(t), \varphi(t)\big)\big|_{t=\alpha}^{t=b}\right\| = \left\|B\big(F(b), \varphi(b)\big) - B\big(F(\alpha), \varphi(\alpha)\big)\right\|$$

$$\leqslant g(b-a)\, M\, \|B\|\, \|\varphi(b)\| + g(\alpha-a)\, M\, \|B\|\, \|\varphi(\alpha)\|$$

$$\leqslant M\,\|B\|\,\{\int_a^b \|\varphi(b)\|\,dg(t-a)+g(\alpha-a)[\|\varphi(\alpha)-\varphi(b)\|+\|\varphi(b)\|]\}$$

$$\leqslant M\,\|B\|\,\{\int_a^b \|\varphi(b)\|\,\{g(t-a)\}'_t\,dt+\int_a^\alpha \vartheta(t)dg(t-a)+g(\alpha-a)\cdot\|\varphi(b)\|\}.$$

In view of (5) we get

$$\lim_{\alpha\to a+}\,[\int_a^\alpha \vartheta(t)\{g(t-a)\}'_t\,dt+g(\alpha-a)\,\|\varphi(b)\|]=0.$$

Thus, using the triangle inequality in (11) and combining the estimates for $\|A_\alpha\|$, $\|B_\alpha\|$ we obtain the desired estimate (7) on letting $\alpha\to a+$ (5). The proof is complete. ∎

The proof of Theorem 2 is essentially similar to that of Theorem 1 and we omit it. The corollaries are obvious.

Proof of Theorem 3. Let e in E be such that $\|e\|=1$. Define the functions f_1, f_2 as follows:

$$f_1(t)=\{g(t-a)\}'_t\cdot e,\qquad f_2(t)=\{-g(b-t)\}'_t\cdot e\qquad \text{for } t \text{ in } \langle a, b\rangle.$$

Then we have

$$\left\|\frac{1}{g(h)}\int_a^{a+h} f_1(t)\,dt\right\|=\left\|\frac{1}{g(h)}\cdot g(h)\cdot e\right\|=1,$$

whence $M=1$, and, similarly, $N=1$. Moreover,

$$\left\|\int_a^b f_1(t)\,dt\right\|=g(b-a)>0,$$

which implies $\int_a^b f_1(t)\,dt\neq 0$. Similarly, $\int_a^b f_2(t)\,dt\neq 0$.

Let us consider the integral I from Theorem 1 (under the assumptions of Theorem 3)

$$\left\|\int_a^b \varphi f\,dt\right\|=\left\|\left(\int_a^b \varphi(t)\,g'(t-a)\,dt\right)\cdot e\right\|$$

$$=\int_a^b \varphi(t)\,g'(t-a)\,dt=\|B\|\,M\int_a^b\{V(\langle t, b\rangle;\,\varphi)+|\varphi(b)|\}\cdot g'(t-a)\,dt,$$

because $\|B\|=1$, $M=1$ and

$$V(\langle t, b\rangle,\,\varphi)+|\varphi(b)|=\varphi(t)-\varphi(b)+\varphi(b)=\varphi(t)$$

(since φ is nonincreasing and $\varphi\geqslant 0$).

(5) The existence of the integral $\int_{a+}^b B(f,\varphi)\,dt$ can be proven in the same way as in [4], p. 245.

Thus, for $f=f_1$, we have the equality in the assertion of Theorem 1. Similarly, for $f=f_2$, we have the equality in the assertion of Theorem 2 and the proof is finished. ■

Finally, we prove Theorem 4.

Proof of Theorem 4. By (6), there exists a sequence $\{h_n\} \subset (0, b-a\rangle$, such that

$$M = \lim_{n\to\infty} \left\| \frac{1}{g(h_n)} \int_a^{a+h_n} f(t)\,dt \right\| = \lim_{n\to\infty} \left\| \int_a^b \frac{1}{g(h_n)} \chi_{\langle a,a+h_n\rangle}(t) f(t)\,dt \right\|.$$

Whence it is sufficient to define

$$\varphi_n(t) = \frac{1}{g(h_n)} \chi_{\langle a,a+h_n\rangle}(t),$$

where $\chi_{\langle a,a+h_n\rangle}$ is the characteristic function of $\langle a, a+h_n\rangle$. The functions ψ_n may be defined similarly. It is easy to check that the assertions (8), (9), (10) are fulfilled. The proof is complete. ■

6. Final remarks. We remark that if $E = H = G = R$ and B is usual multiplication, then Corollary 1 yields the following results: Lemma 1, [1]; Theorem 2, [3]; 2.1., [6].

Similarly, Corollary 2 implies Lemma 2 from [1], Theorem 1 from [3] and Remark, [6], p. 175.

From Lemma 3 we infer that sometimes we can improve our inequalities if we take $\tilde{\varphi}$, $\tilde{\psi}$ instead of φ, ψ, respectively. The same is true for the results cited here.

In the papers [6], [1] some theorems of the Romanovski and Faddeev type are proved. We observe that the analogous theorems can be stated in the case of the Lebesgue–Bochner integral, because we have proved the suitable results in this case (see Section 3).

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