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Combinatorial operators *

Binomial sequences of polynomials are those which satisfy the identity

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y)$$

for all natural n and real x, y. Various examples of such sequences arise naturally in combinatorial analysis. A systematic theory of binomial sequences was presented by R. Mullin and G.-C. Rota ([2]). It is a theory of algebraic nature and the authors adopt the calculus of linear operators as the method of investigations, although only the class of so-called shift-invariant operators is actually used.

The notion of binomial sequences of polynomials was generalized by the present author in [1]. The author introduced the notion of combinaatorial sequence of polynomials and, using the method of exponential generting functions, derived the basic properties of such sequences. The binomial sequences form a proper subclass of the collection of combinatorial sequences.

The aim of this study is to extend the algebraic methods and the results of Mullin and Rota to the class of combinatorial sequences. To this end we introduce the class of combinatorial operators.

The basic notion is that of a reducing operator, which is a generalization of the operator of the derivative.

Let (b_n) be a sequence of real numbers ($n \in N_0 = N \cup \{0\}$) such that $b_0 = b_1 = 1$ and $b_n \neq 0$ for all n.

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* This is a version of a chapter of the author's doctoral dissertation presented to the University of Warsaw. The present version was prepared by his advisor, Wiktor Marek. Although various extensions of the theory outlined in this paper are possible we decided to leave the author's text as it was making only some minor linguistic corrections.

DEFINITION. A *reducing operator* is a linear operator defined by the formula

$$D(b_n x^n) = nb_{n-1} x^{n-1}$$

for $n \in N$ and $D(1) = 0$. (Here (b_n) is any sequence satisfying the preceding assumptions).

Any operator defined on x^n for all n may be unambiguously extended by linearity to the class of all polynomials, and even to the algebra of power series.**

We can see that such operators are uniquely determined by the sequences (b_n). The derivative is a special case of a reducing operator with $b_n = 1$ for all n. We shall call the sequence (b_n) the *basis of the reducing operator D.*

An important property of a reducing operator is the following:

LEMMA. *Let D be a reducing operator with the basis* (b_n) *and let*

$$B(x) = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n$$

be the exponential generating function of the basis. For $r \in R$ *we have*

$$DB(rx) = r \cdot B(rx).$$

Proof. We have

$$D \sum_{n=0}^{\infty} \frac{b_n}{n!} r^n x^n = \sum_{n=1}^{\infty} \frac{r^n}{n!} nb_{n-1} x^{n-1} = \sum_{n=0}^{\infty} \frac{r^{n+1}}{n!} b_n x^n = rB(rx). \blacksquare$$

Let us remark that this is a generalization of the property

$$(e^{rx})' = re^{rx}$$

for the derivative operator $'$ and the function $B(x) = e^x$.

As it was remarked in [1], a combinatorial sequence of polynomials $(V_n(x))$ is uniquely determined by two sequences (a_n), (b_n) $(n \in N)$ where $a_1 \neq 0$ and the sequence (b_n) satisfies the same assumptions as a basis of a reducing operator — we shall call them the generating sequence (a_n) and the basis (b_n) of a combinatorial sequence $V_n(x)$. In this way to any combinatorial sequence $(V_n(x))$ we assign the appropriate reducing operator D with the same basis (b_n).

THEOREM. *Let* $(V_n(x))$ *be a combinatorial sequence of polynomials with generating sequence* (a_n) *and basis* (b_n), *and D be the reducing operator with the*

** This fact (the extendability of operators from polynomials to formal power series) is, in fact, a topological property related to the topology considered by Rota *et al.* in the reference [2] of the first part of this paper ([1] in the references). It can be proved using the density of the set of polynomials in the space of formal power series.

same basis (b_n). For all $n \in N$ we have the following identity

$$DV_n(x) = \sum_{k=0}^{n-1} \binom{n}{k} a_{n-k} V_k(x).$$

Proof. We apply the operator D to the generating function of the combinatorial sequence $V_n(x)$

$$V(x, t) = \sum_{n=0}^{\infty} \frac{V_n(x)}{n!} t^n = B(xA(t))$$

and obtain by virtue of the lemma

$$\sum_{n=1}^{\infty} \frac{DV_n(x)}{n!} t^n = DV(x, t) = A(t) \cdot B(xA(t)) = \sum_{n=1}^{\infty} \frac{a_n}{n!} t^n \cdot \sum_{n=0}^{\infty} \frac{V_n(x)}{n!} t^n$$
$$= \sum_{n=1}^{\infty} \Big(\sum_{k=0}^{n-1} \binom{n}{k} a_{n-k} V_k(x)\Big) \frac{t^n}{n!}. *$$

The equality of power series implies equality of the coefficients, so we obtain the assertion. ■

This theorem apparently has not been known before even in the special case of binomial sequences.

Remark. We have used the formula

$$DV(x) = A(t) \cdot V(x)$$

where $t \in R$, $V(x) = V(x, t)$.

Let us pass to the main subject of this paper; i.e., the class of combinatorial operators. As we know, a linear operator Q is uniquely determined by the polynomials

$$Q_n(x) = Qx^n = \sum_{k=0}^{n} Q(k, n) x^k.$$

The following lemma gives a characterization of shift-invariant operators by the coefficients $Q(k, n)$. For $t \in R$, we denote by E^t the shift operator $E^t V(x) = V(x+t)$. An operator Q is shift-invariant if, and only if, $E^t Q = QE^t$ for every $t \in R$.

LEMMA. *The shift-invariant operators are precisely the operators Q determined by sequences (c_n) in the following fashion*

$$Q(k, n) = \binom{n}{k} c_{n-k}$$

In addition, the sequence (c_n) uniquely determines the operator.

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* This argument makes heavy use of the extendability property mentioned above, and, in fact, its extension to the power series in two variables. (Remark by the Editor.)

Proof. For any $t \in R$ and $n \in N_0$, we have

$$E^t x^n = \sum_{k=0}^{n} \binom{n}{k} t^{n-k} x^k,$$

$$QE^t x^n = \sum_{k=0}^{n} \binom{n}{k} t^{n-k} Q_k(x) = \sum_{k=0}^{n} \binom{n}{k} t^{n-k} \sum_{i=0}^{k} Q(i, k) x^i$$
$$= \sum_{i=0}^{n} \sum_{k=1}^{n} \binom{n}{k} t^{n-k} Q(i, k) x^i,$$

$$E^t Q x^n = Q_n(x+t) = \sum_{k=0}^{n} Q(k, n) \sum_{i=0}^{k} \binom{k}{i} t^{k-i} x^i = \sum_{i=0}^{n} \sum_{k=i}^{n} \binom{k}{i} t^{k-i} Q(k, n) x^i.$$

We infer that an operator Q is shift-invariant if and only if

$$\sum_{k=0}^{n-i} \binom{i+k}{i} Q(k+i, n) t^k = \sum_{k=0}^{n-i} \binom{n}{n-k} Q(i, n-k) t^k$$

for any $i \leqslant n$ and $t \in R$, i.e.

$$\binom{n}{k} Q(i, n-k) = \binom{i+k}{i} Q(i+k, n)$$

for any $k+i \leqslant n$. Any operator with coefficients of the given shape satisfies the equality. We have

$$\binom{i+k}{k} Q(i+k, n) = \binom{i+k}{n} \binom{n}{i+k} c_{n-k-i} = \binom{n}{k} \binom{n-k}{i} c_{n-k-i} = \binom{n}{k} Q(i, n-k).$$

and conversely, if the equality is satisfied, then denoting $c_n = Q(0, n)$ and putting $i = 0$, we have for any $k \leqslant n$

$$Q(k, n) = \binom{n}{k} Q(0, n-k) = \binom{n}{k} c_{n-k}.$$

Delta-operators are shift-invariant operators with $c_0 = 0$ and $c_1 \neq 0$. ■

DEFINITION. A *combinatorial operator* is an operator Q with coefficients $Q(k, n)$ of the shape

$$Q(k, n) = \frac{b_k}{b_n} \binom{n}{k} c_{n-k}$$

where (b_n) is a sequence satisfying $b_0 = b_1 = 1$ and $b_n \neq 0$ for all n, and (c_n) is an arbitrary sequence.

By virtue of the lemma, shift-invariant operators are combinatorial. Moreover, the assignment of a combinatorial operator Q to a pair of sequences (b_n), (c_n) is a bijection. We shall call (b_n) the basis and (c_n) the generating sequence of Q.

We may expand combinatorial operators in operator series in precisely the same fashion as shift-invariant operators were expressed in [2], except that the operator of derivative must be replaced by the appropriate reducing operator D.

THEOREM. *Let Q be a combinatorial operator with the basic (b_n) and the generating sequence (c_n). Let D be the reducing operator associated with basis (b_n). Then*

$$Q = \sum_{n=0}^{\infty} \frac{c_n}{n!} D^n .$$

where D is the reducing operator with basis (b_n).

Proof. For $0 \leqslant k \leqslant n$

$$D^k x^n = \frac{b_{n-1}}{b_n} n D^{k-1} x^{n-1} = \ldots = \frac{b_{n-k}}{b_n} \frac{n!}{(n-k)!} x^{n-k}.$$

For $n \in N_0$

$$\left(\sum_{k=0}^{\infty} \frac{c_k}{k!} D^k\right) x^n = \sum_{k=0}^{n} \binom{n}{k} c_k \frac{b_{n-k}}{b_n} x^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \frac{b_k}{b_n} c_{n-k} x^k = \sum_{k=0}^{n} Q(k, n) x^k = Qx^n$$

hence, the two operators are equal. ■

Let us consider the class of combinatorial operators with $c_0 = 0$ and $c_1 \neq 0$, which we call combinatorial delta-operators. Now after giving necessary definitions, we show that the main results of G.-C. Rota and R. Mullin remain true in the present generalization.

The first main result in [2] is establishing a one-to-one correspondence between all binomial sequences of polynomials $V_n(x)$ and delta-operators Q such that the identity

$$QV_n = nV_{n-1}$$

is satisfied for any binomial sequence V_n and the corresponding operator Q, for $n \in N$. We shall extend it to a one-to-one correspondence between all combinatorial sequences of polynomials and all combinatorial delta-operators and show that the identity $QV_n = nV_{n-1}$ remains true in the general case.

Remark. Let us remark that the equalities $a_1 c_1 = 1$ and

$$\sum_{\alpha | -n} B(\alpha) a_\alpha c_{|\alpha|} = 0$$

for $n \geqslant 1$ (where $B(\alpha)$ is the Bell function with the notation of [1]), establish a symmetric one-to-one correspondence in the family of sequences (a_n) $(n \in N)$ with $a_1 \neq 0$. This set of equalities means that the generating functions

$$A(t) = \sum_{n=1}^{\infty} \frac{a_n}{n!} t^n, \quad C(t) = \sum_{n=1}^{\infty} \frac{c_n}{n!} t^n$$

form an inverse pair; i.e., that

$$A(C(t)) = C(A(t)) = t.$$

We shall call (c_n) the inverse of (a_n). The inverse of (c_n) is (a_n) again.

THEOREM. *Let $V_n(x)$ be a combinatorial sequence of polynomials with generating sequence (a_n) and basis (b_n). Let Q be the combinatorial delta-operator with the same basis (b_n) and the generating sequence (c_n) inverse to (a_n) for all $n \in N$. Then*

$$QV_n = nV_{n-1}.$$

Proof. We show that the generating functions of sequences $QV_n(x)$ and $nV_{n-1}(x)$ are identical. Let $V(x)$ be the generating function of the sequence $V_n(x)$:

$$V(x) = \sum_{n=0}^{\infty} \frac{V_n(x)}{n!} t^n.$$

Using the remark, we have

$$\sum_{n=1}^{\infty} \frac{QV_n(x)}{n!} t^n = QV(x) = \sum_{n=1}^{\infty} \frac{c_n}{n!} D^n V(x) = \sum_{n=1}^{\infty} \frac{c_n}{n!} A(t)^n V(x)$$

$$= C(A(t)) V(x) = tV(x) = \sum_{n=0}^{\infty} \frac{V_n(x)}{n!} t^{n+1} = \sum_{n=1}^{\infty} \frac{nV_{n-1}(x)}{n!} t^n.$$

By virtue of preceding remarks and corollaries such assignment of the operator Q to a combinatorial sequence $V_n(x)$ is clearly a bijection on the family of all combinatorial delta-operators. ■

We conclude our study of combinatorial operators. We prove that the expansion theorem – another main result of [2] – remains true with no changes whatever for combinatorial operators.

THEOREM. *Let Q be a combinatorial delta-operator and $V_n(x)$ be the corresponding combinatorial sequence of polynomials. Let P be an arbitrary combinatorial operator with the same basis (b_n) as Q and $V_n(x)$. Then*

$$P = \sum_{n=0}^{\infty} \frac{d_n}{n!} Q^n$$

where for all n

$$d_n = (PV_n)(0).$$

Proof. The operator P is of the form

$$P = \sum_{n=0}^{\infty} \frac{p_n}{n!} D^n$$

where D is the reducing operator with basis (b_n) and similarly

$$Q = \sum_{n=1}^{\infty} \frac{c_n}{n!} D^n$$

whose inversion is

$$D = \sum_{n=1}^{\infty} Q^n$$

with notations as before. Hence, for $k \in N$

$$D^k = \left(\sum_{n=1}^{\infty} \frac{a_n}{n!} Q^n\right)^k = \sum_{n=k}^{\infty} \left(\sum_{i_1+\ldots+i_k=n} \frac{a_{i_1}}{i_1!} \cdots \frac{a_{i_k}}{i_k!}\right) Q^n = \sum_{n=k}^{\infty} \sum_{\alpha| -n:\, |\alpha|=k} \frac{k!}{\varkappa(\alpha)} \frac{a_\alpha}{\alpha!} Q^n$$

$$= \frac{k!}{b_k} \sum_{n=k}^{\infty} \Big(\sum_{\alpha| -n:\, |\alpha|=k} B(\alpha)\, a_\alpha\, b_k\Big) \frac{Q^n}{n!} = \frac{k!}{b_k} \sum_{n=k}^{\infty} V(k, n) \frac{Q^n}{n!}$$

(with notations and formulas of [1]), and

$$P = \sum_{k=0}^{\infty} \frac{p_k}{k!} D^k = \sum_{k=0}^{\infty} \frac{p_k}{b_k} \sum_{n=k}^{\infty} V(k, n) \frac{Q^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{p_k}{b_k} V(k, n)\right) \frac{Q^n}{n!}.$$

We have a formula for the coefficients d_n

$$d_n = \sum_{k=0}^{n} \frac{p_k}{b_k} V(k, n).$$

But for $0 \leqslant k \leqslant n$

$$D^k V_n(0) = D^k V(k, n) x^k = V(k, n) \frac{b_0}{b_k} k! = \frac{k!}{b_k} V(k, n)$$

and for every n

$$(PV_n) = \left(\sum_{k=0}^{n} \frac{p_k}{k!} D^k V_n\right)(0) = \sum_{k=0}^{n} \frac{p_k}{k!} \frac{k!}{b_k} V(k, n) = \sum_{k=0}^{n} \frac{p_k}{b_k} V(k, n) = d_n.$$

which completes the proof. ■

References

[1] T. Kreid, *Combinatorial sequences of polynomials*, Comment. Math. Prace Mat. 29 (1990).

[2] R. Mullin, G.-C. Rota, *On the foundations of combinatorial theory III — Theory of binomial enumeration*, Graph Theory Appl. (1970), 168–213.