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# Combinatorial sequences of polynomials*

The subject of this article is sequences of polynomials

$$V_n(x) = \sum_{k=1}^{n} V(k, n)\, x^k$$

for $n \in N$ and $V_0(x) = 1$, with coefficients $V(k, n)$ of the following form

$$V(k, n) = b_k \sum_{\alpha| - n:\, |\alpha| = k} B(\alpha)\, a_\alpha$$

where $(a_n)$, $(b_n)$ are two arbitrary infinite sequences of real numbers ($n \in N$). Presently we explain this formula. The symbol $\alpha| - n$ means: $\alpha$ is a partition of the number $n$, i.e. a nonincreasing sequence $(i_1, \ldots, i_k)$ of natural numbers such that

$$i_1 + \ldots + i_k = n.$$

$|\alpha|$ denotes the length of $\alpha$ $(= k)$, $a_\alpha$ denotes the product $a_{i_1} \ldots a_{i_k}$, we also write $\alpha!$ instead of $i_1! \ldots i_k!$, and $B(\alpha)$ is the following Bell function defined for all partitions $\alpha$

$$B(\alpha) = \frac{n!}{\alpha!\, \varkappa(\alpha)}$$

where

$$\alpha = (i_1, \underset{k_1}{\ldots}, i_1, \ldots, i_2, \underset{k_2}{\ldots}, i_2, \ldots, i_j, \underset{k_j}{\ldots}, i_j)$$

where $i_1 > \ldots > i_j$,

$$\varkappa(\alpha) = k_1!\, k_2! \ldots k_j!.$$

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* This paper is an expanded version (by the author) of a chapter of the author's doctoral dissertation presented to the University of Warsaw. The present version was prepared for printing by his advisor, Wiktor Marek. In spite of the fact that a different presentation (in the spirit of [3]) is possible, we left the paper almost as it was.

The number of arrangements (permutations) of a partition $\alpha$ is known to be equal to $|\alpha|!/\varkappa(\alpha)$ and we use this fact in calculations frequently.

The combinatorial interpretation of the Bell function is such that for $\alpha\dashv n$, $B(\alpha)$ is equal to the number of partitions $\{A_1, \ldots, A_k\}$ of an $n$-element set $X$ such that the sequence $(|A_1|, \ldots, |A_k|)$ put in nonincreasing order gives the partition $\alpha$ of the number $n$ ($|A|$ denotes the number of elements of a set $A$). Stirling's numbers of the second kind, $S(k, n)$, satisfy the identity, for $1 \leqslant k \leqslant n$,

$$S(k, n) = \sum_{\alpha\dashv n:\ |\alpha| = k} B(\alpha).$$

Let us notice that for $n \in N$

$$V(1, n) = a_n b_1, \qquad V(n, n) = a_1^n b_n.$$

We also denote $V(0, 0) = 1$ and $V(0, n) = 0$ for $n \in N$. With additional assumptions $a_1 \neq 0$ and $b_n \neq 0$ for all $n$, we obtain the property of the sequence $V_n(x)$ that the degree of each polynomial $V_n$ is equal to $n$, as $V(n, n) \neq 0$. Sequences of polynomials satisfying the above conditions will be called *combinatorial sequences.*

Let us observe that further assumption $b_1 = 1$ does not diminish the class of combinatorial sequences, as putting

$$a'_n = a_n b_1, \qquad b'_n = \frac{b_n}{b_1^n}$$

gives the same sequence $V_n(x)$, since

$$V(k, n) = b'_k \sum_{\alpha\dashv n:\ |\alpha| = k} B(\alpha) a'_\alpha = b'_k b_1^k \sum_{\alpha\dashv n:\ |\alpha| = k} B(\alpha) a_\alpha = b_k \sum_{\alpha\dashv n:\ |\alpha| = k} B(\alpha) a_\alpha$$

for any $(a_n)$, $(b_n)$, with $b_1 \neq 0$, and makes the equality $b'_1 = 1$ satisfied. Let us also notice that once this assumption is made, the assignment of the combinatorial sequence $V_n(x)$ to a pair of sequences $(a_n)$, $(b_n)$ becomes a bijection.

First, we derive the general formula for the two-variable generating function of a given combinatorial sequence $(V_n(x))$, $\sum_{n=0}^{\infty} V_n(x) t^n/n!$.

For this purpose we need the formula for superposition of two functions given in the form of power series. This formula seems to be known to Faa di Bruno ([1]), although he did not consider power series but gave the formula for the $n$th derivative of superposition of functions. Also Gian-Carlo Rota ([2]) gives the special case of this formula with null free component in the inner function and, in fact, we need only this case here. Nevertheless, we give a new combinatorial proof of the general formula

THEOREM. *Let $(a_n)$, $(b_n)$ be arbitrary sequences of real numbers,*

$$A(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n, \qquad B(x) = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n$$

*and $\varrho$ be the radius of convergence of the series $B(x)$. For $x \in R$ such that*

$$\sum_{n=0}^{\infty} \left|\frac{a_n}{n!} x^n\right| < \varrho$$

*we have*

$$B(A(x)) = \sum_{n=0}^{\infty} \frac{c_n}{n!} x^n$$

*where $c_0 = B(a_0)$ and for $n \in N$*

$$c_n = \sum_{\alpha \mid -n} B(\alpha) a_\alpha B^{(|\alpha|)}(a_0)$$

*(one function $B$ is defined on partitions and the other on real numbers, so we shall not confuse them; $B^{(l)}$ denotes the $l$th derivative of $B$).*

Proof. We shall confine ourselves to calculations, the convergence questions are set correct by the assumptions. We have

$$B(A(x)) = \sum_{k=0}^{\infty} \frac{b_k}{k!} A(x)^k = \sum_{k=0}^{\infty} \frac{b_k}{k!} \sum_{n=0}^{\infty} \sum_{i_1+\ldots+i_k=n} \frac{a_{i_1} \ldots a_{i_k}}{i_1! \ldots i_k!} x^n$$
$$= \sum_{k=0}^{\infty} \frac{b_k}{k!} \frac{a_0^k}{0!} + \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} \frac{b_k}{k!} \sum_{i_1+\ldots+i_k=n} \frac{a_{i_1} \ldots a_{i_k}}{i_1! \ldots i_k!} \right) x^n.$$

We denote by $l$ the number of nonzero entries in the sequence $(i_1, \ldots, i_k)$. For each value of $l = 1, 2. \ldots, m = \min\{k, n\}$ we obtain such a sequence by taking a partition $\alpha$ of $n$ into $l$ parts, permuting it arbitrarily, which may be done in $l!/\varkappa(\alpha)$ ways, and placing its $l$ entries on $k$ places, which may be done in $\binom{k}{l}$ ways. The other $k-l$ entries of the sequence are zeros. So the formula may be transformed into

$$B(A(x)) = B(a_0) + \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} \frac{b_k}{k!} \sum_{l=1}^{m} \sum_{\alpha \mid -n: |\alpha| = 1} \frac{l!}{\varkappa(\alpha)} \binom{k}{l} \frac{1}{\alpha!} a_\alpha a_0^{k-l} \right) x^n$$
$$= B(a_0) + \sum_{n=1}^{\infty} \left( \sum_{l=1}^{n} \sum_{k=1}^{\infty} \frac{b_k}{(k-l)!} a_0^{k-l} \sum_{\alpha \mid -n: |\alpha| = l} \frac{n!}{\varkappa(\alpha)\, \alpha!} a_\alpha \right) \frac{x^n}{n!}$$
$$= B(a_0) + \sum_{n=1}^{\infty} \left( \sum_{l=1}^{n} \sum_{\alpha \mid -n: |\alpha| = l} B(\alpha) a_\alpha \cdot B^{(l)}(a_0) \right) \frac{x^n}{n!}$$
$$= B(a_0) + \sum_{n=1}^{\infty} \left( \sum_{\alpha \mid -n} B(\alpha) a_\alpha B^{(|\alpha|)}(a_0) \right) \frac{x^n}{n!}.$$

In the special case where $a_0 = 0$ the formula reduces to

$$c_n = \sum_{\alpha \mid -n} B(\alpha) a_\alpha b_{|\alpha|}$$

for $n \in N$ and $c_0 = b_0$. Applying this formula to combinatorial sequences of polynomials, we obtain the required generating function immediately. ■

THEOREM. *Let* $(a_n)$, $(b_n)$ *be arbitrary sequences of numbers* ($n \in N$) *with the generating functions*

$$A(t) = \sum_{n=1}^{\infty} \frac{a_0}{n!} t^n, \qquad B(t) = \sum_{n=0}^{\infty} \frac{b_0}{n!} t^n$$

*where* $b_0 = 1$. *The polynomial sequence* $(V_n(x))$ *determined by* $(a_n)$ *and* $(b_n)$ *satisfies the identity*

$$\sum_{n=0}^{\infty} \frac{V_n(x)}{n!} t^n = B(xA(t))$$

*for all real* $x, t$ *such that*

$$|x| \cdot \sum_{n=1}^{\infty} \left| \frac{a_n}{n!} t^n \right| < \varrho^*.$$

Proof. We apply the previous theorem to the functions

$$xA(t) = \sum_{n=1}^{\infty} \frac{xa_n}{n!} t^n, \qquad B(t)$$

and obtain $c_0 = 1 = V_0(x)$ and for $n \in N$

$$c_n = \sum_{\alpha | -n} B(\alpha) a_\alpha x^{|\alpha|} b_{|\alpha|} = \sum_{k=1}^{n} \sum_{\alpha | -n : |\alpha| = k} B(\alpha)_\alpha b_k x^k = V_n(x)$$

(the sequence $a_n$ is replaced by $xa_n$, so $a_\alpha$ is replaced by $x^{|\alpha|} a_\alpha$).

As an example, let us take the special case of $a_n = 1$ for $n \in N$. We have

$$A(t) = e^t - 1, \qquad V(k, n) = b_k S(k, n).$$

The generating function takes the form

$$\sum_{n=0}^{\infty} \frac{V_n(x)}{n!} t^n = B\left(x(e^t - 1)\right).$$

And so for the sequence

$$S_n(x) = \sum_{k=1}^{n} kS(k, n) x^k$$

with $b_k = k$ we have $B(t) = 1 + e^t$ and

$$\sum_{n=1}^{\infty} \frac{S_n(x)}{n!} t^n = x(e^t - 1) e^{x(e^t - 1)}.$$

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* Actually this result is related to formal power series and the convergence assumption does not seem to be necessary. (Remark of the Editor.)

For the sequence

$$T_n(x) = \sum_{k=1}^{n} k!\,S(k, n)\,x^k$$

we have $B(t) = 1/(1-t)$ and so we obtain the identity

$$\sum_{n=0}^{\infty} \frac{T_n(x)}{n!} t^n = \frac{1}{1-x(e^t-1)}$$

whose special case for $x = 1$ is the formula of Gross ([2])

$$\frac{1}{2-e^t} = \sum_{n=0}^{\infty} \frac{T_n}{n!} t^n$$

where

$$T_n = \sum_{k=1}^{n} k!\,S(k, n)$$

is the number of labelled partitions $(A_1, \ldots, A_k)$ of an $n$-element set $X$, equal to the number of chains of subsets

$$\emptyset \neq B_1 \subsetneq B_2 \subsetneq \ldots \subsetneq B_k = X.$$

(The assignment of $(B_1, \ldots, B_k)$ to $(A_1, \ldots, A_k)$ is a bijection.) Another corollary to this formula is

$$\sum_{k=1}^{n} (-1)^{n-k}\,k!\,S(k, n) = 1,$$

$$\sum_{n=0}^{\infty} \frac{T_n(-1)^n}{n!} t^n = e^{-t} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^n$$

and $T_n(-1) = (-1)^n$. Similarly, with $b_n = (n-1)!$ we obtain

$$\sum_{k=1}^{n} (-1)^k (k-1)!\,S(k, n) = 0$$

for $n > 1$.

Let us drop the line of generating functions for awhile to prove the following identity concerning the coefficients $V(k, n)$:

THEOREM. *For arbitrary sequences $(a_n), (b_n)$ $(n \in N)$, for any numbers $i, j, n \in N \cup \{0\}$ such that $i+j \leqslant n$ we have the following identity*

$$b_{i+j} \sum_{i=1}^{n-j} \binom{n}{k} V(i, k)\, V(j, n-k) = b_i b_j \binom{i+j}{i} V(i+j, n).$$

Proof. We have

$$b_{i+j}\sum_{i=1}^{n-j}\binom{n}{k} V(i, k)\, V(j, n-k)$$

$$= b_{i+j} b_i b_j \sum_{i=1}^{n-j}\binom{n}{k} \sum_{\alpha| -k:|\alpha|=i} B(\alpha)\, a_\alpha \sum_{\beta| -n-k:|\beta|=j} B(\beta)\, a_\beta$$

$$= b_{i+j} b_i b_j \frac{n!}{i!\,j!} \sum_{\alpha,\beta:\ |\alpha|=i,|\beta|=j} \frac{i!}{\varkappa(\alpha)\,\alpha!}\,\frac{j!}{\varkappa(\beta)\,\beta!}\, a_\alpha\, a_\beta$$

$$= b_{i+j} b_i b_j \frac{n!}{i!\,j!} \sum_{t_1+\ldots+t_i+s_1+\ldots s_j=n} \frac{1}{\alpha!\,\beta!}\, a_\alpha\, a_\beta$$

$$= b_{i+j} b_i b_j \frac{(i+j)!}{i!\,j!} \sum_{\gamma| -n:|\gamma|=i+j} \frac{(i+j)!\,n!}{\varkappa(\gamma)\ \gamma!}\, a_\gamma = b_i b_j \binom{i+j}{i} V(i+j, n).$$

Let us notice that this theorem may be also proved with the help of generating functions. The proof which follows is more elegant and highlights the sense of the identity in a better way: it is also shorter.

LEMMA. *For $k \in N \cup \{0\}$, and with notations as above*

$$\sum_{i=k}^{\infty} \frac{k!}{n!} V(k, n)\, t^n = n_k\, A(t)^k.$$

Proof. In fact, the lemma follows from the previous theorem; we merely take the coefficients of $x^k$ on both sides of the identity

$$\sum_{n=0}^{\infty} \frac{V_n(x)}{n!} t^n = \sum_{n=0}^{\infty} \frac{n_n}{n!} A(t)^n x^n$$

and multiply them by $k!$.

Now we have for any $i, j$

$$b_{i+j}\sum_{n=i}^{\infty} \frac{i!}{n!} V(i, n)\, t^n \cdot \sum_{n=j}^{\infty} \frac{j!}{n!} V(j, n)\, t^n = b_i b_j \sum_{n=i+j}^{\infty} \frac{(i+j)!}{n!} V(i+j, n)\, t^n$$

since $A(t)^i A(t)^{i+j}$. The coefficients of $t^n$ on both sides of the equality must be the same and so we obtain after multiplying the series

$$b_{i+j}\sum_{k=i}^{n-j} \frac{i!}{k!} V(i, k) \frac{j!}{(n-k)!} V(j, n-k) = b_i b_j \frac{(i+j)!}{n!} V(i+j, n)$$

which is the required identity.

Let us consider the special case of combinatorial sequences of polynomials generated by the sequence $(n_n)$ constantly equal to 1; that is the polynomial sequence

$$V_n(x) = \sum_{\alpha| -n} B(\alpha)\, a_\alpha\, x^{|\alpha|}$$

where $(a_n)$ is an arbitrary sequence of real numbers with $a_1 \neq 0$. Of course, the sequences are different for different sequences $(a_n)$. Our main result here is the following theorem.

THEOREM. *Polynomial sequences of the form*

$$V_n(x) = \sum_{\alpha| - n} B(\alpha)\, a_\alpha\, x^{|\alpha|}$$

*where $(a_n)$ is any sequence with $a_1 \neq 0$, are exactly the binomial sequences of polynomials.*

Proof. It is enough to prove that all such sequences are binomial, for a binomial sequence is uniquely determined by the coefficients $V(1, n)$, and the sequence $V(1, n) = a_n$ is arbitrary. By virtue of the last theorem we have in this case

$$\sum_{k=i}^{n-j} \binom{n}{k} V(i, k)\, V(j, n-k) = \binom{i+j}{i} V(i+j, n)$$

for $i+j \leqslant n$, and this identity is equivalent to the property that the sequence $V_n(x)$ is binomial, as it was remarked in [3]. ■

There is an alternative way of proving the theorem by means of generating functions, as follows:

Remark. For the sequences $V_n(x)$ under consideration we have $B(t) = e^t$ and so the formula

$$\sum_{n=0}^{\infty} \frac{V_n(x)}{n!} t^n = e^{xA(t)}$$

holds as a special case of the general formula for combinatorial sequences.

So we have for $x, y \in R^*$

$$\sum_{n=0}^{\infty} \frac{V_n(x+y)}{n!} t^n = e^{(x+y)A(t)} = e^{xA(t)}\, e^{yA(t)}$$

$$= \sum_{n=0}^{\infty} \frac{V_n(x)}{n!} t^n \cdot \sum_{n=0}^{\infty} t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} V_k(x)\, V_{n-k}(y) \frac{t^n}{n!},$$

$$V_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} V_k(x)\, X_{n-k}(y)$$

which exactly means that we deal with a binomial sequence.

COROLLARY. *If $V_n(x)$ is a binomial sequence, then denoting $a_n = V(1, n)$ we*

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* The fact we deal here with real numbers is immaterial; the argument holds for any field of characteristic 0. (Remark of the Editor.)

*have for every* $1 \leqslant k \leqslant n$

$$V(k, n) = \sum_{\alpha \mid -n:\, |\alpha| = k} B(\alpha)\, a_\alpha .$$

For example, with

$$V_n(x) = \sum_{k=1}^{n} S(k, n)\, x^k$$

we have $a_n = 1$ and we have the well-known formula for Stirling's numbers of the second kind

$$S(k, n) = \frac{n!}{k!} \sum_{\alpha \mid -n:\, |\alpha| = k} \frac{k!}{\varkappa(\alpha)\, \alpha!} = \frac{n!}{k!} \sum_{i_1 + \ldots + i_k = n} \frac{1}{i_1! \ldots i_k!},$$

and with

$$V_n(x) = x(x-1)\ldots(x-n+1) = \sum_{k=1}^{n} s(k, n)\, x^k$$

we have $a_n = (-1)^{n-1}(n-1)!$ and we have a formula for Stirling's numbers of the first kind

$$s(k, n) = (-1)^{n-k} \frac{n!}{k!} \sum_{i_1 + \ldots + i_k = n} \frac{1}{i_1 \ldots i_k}.$$

Our last theorem concerns again all combinatorial sequences, even without the assumptions that $a_1 \neq 1$ or $b_n \neq 0$ for $n \in N$.

THEOREM. *Let* $(a_n)$, $(b_n)$ *be arbitrary sequences of numbers* ($n \in N$). *Let* $V_n(x)$ *be the combinatorial sequence of polynomials with coefficients*

$$V(k, n) = b_k \sum_{\alpha \mid -n:\, |\alpha| = k} B(\alpha)\, a_\alpha .$$

*Let* $v_n(x)$ *be the binomial sequence of polynomials with coefficients*

$$v(k, n) = \sum_{\alpha \mid -n:\, |\alpha| = k} B(\alpha)\, d_\alpha$$

*where*

$$d_n = \frac{1}{n+1} a_{n+1} .$$

*For every* $n \in N \cup \{0\}$ *the identity*

$$\sum_{k=0}^{\infty} \frac{n!}{(n+k)!} V(k, k+n)\, x^k = \sum_{k=0}^{n} v(k, n)\, x^k B^{(k)}(a_1 x)$$

*holds for* $x \in R$ *such that* $|a_1 x| < \varrho$, *where* $\varrho$ *is the radius of convergence of the series* *

$$B(x) = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n.$$

Proof. For $n = 0$ we have

$$\sum_{k=0}^{\infty} \frac{1}{k!} V(k, k) x^k = \sum_{k=0}^{\infty} \frac{1}{k!} b_k a_1^k x^k = B(a_1 x) = v(0, 0) x^0 B(a_1) x.$$

For $n \in N$

$$\sum_{l=0}^{n} v(l, n) x^l B^{(l)}(a_1 x) = \sum_{l=1}^{n} v(l, n) x^l \sum_{k=1}^{\infty} \frac{b_k}{(k-1)!} (a_1 x)^{k-l}$$

$$= \sum_{k=1}^{\infty} \left( \sum_{1 \leqslant l \leqslant k,n} \sum_{\alpha | -n: |\alpha| = l} B(\alpha) d_\alpha \frac{1}{(k-l)!} a_1^{k-l} \right) b_k x^k$$

$$= \sum_{k=1}^{\infty} \left( \sum_{\alpha | -n: |\alpha| \leqslant k} B(\alpha) d_\alpha a_1^{k-|\alpha|} \frac{1}{(k-|\alpha|)!} \right) b_k x^k.$$

For $k \in N$, to every partition $\alpha = (i_1, \ldots, i_l)$ of $n$ with $1 = |\alpha| \leqslant k$ we assign the partition

$$\beta = (i_1 + 1, \ldots, \underset{k}{i_l + 1, 1, \ldots, 1})$$

of $n+k$ into $k$ parts — this is a bijection. The following relations hold

$$\varkappa(\beta) = \varkappa(\alpha) \cdot (k-l)!,$$

$$\frac{1}{\alpha!} d_\alpha a_1^{k-l} = \frac{1}{i_1!} \cdots \frac{1}{i_l!} \cdot \left( \frac{1}{i_1 + 1} a_{i_1 + 1} \cdots \frac{1}{i_l + 1} a_{i_l + 1} \right) a_1^{k-l} = \frac{1}{\beta!} a_\beta.$$

So for every $k$

$$\left( \sum_{\alpha | -n: |\alpha| < k} \frac{n!}{\alpha! \varkappa(\alpha)} d_\alpha a_1^{k-|\alpha|} \frac{1}{(k-|\alpha|!} \right) b_k = \frac{n!}{(n+k)!} \sum_{\beta | -n+k: |\beta| = k} \frac{(n+k)!}{\beta! \varkappa(\beta)} a_\beta b_k$$

$$= \frac{n!}{(n+k)!} V(k, k+n).$$

This theorem may also be proved by means of generating functions, but this method is more difficult, and we will not attempt it here.

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* As above, let us notice that the author proves the result for formal power series. (Remark of the Editor.)

Remark. Especially interesting is the case of binomial sequences, when $B(x) = e^x$. We have $B^{(k)}(a_1 x) = e^{a_1 x}$ for every $k$, and so we have that the identity

$$\sum_{k=0}^{\infty} \frac{n!}{(n+k)!} V(k, k+n) x^k = v_n(x) \cdot e^{a_1 x}$$

is valid for all real $x$.

To show an application let us condier the example where $V(k, n) = d(k, n)$ is the number of $n$-permutations with no fixed points with $k$ cycles. In this case $a_1 = 0$ and $V(k, n) = 0$ for $2k > n$. So the generating functions under consideration are evidently polynomials, and we have

$$\sum_{k=0}^{\infty} \frac{n!}{(n+k)!} d(k, k+n) x^k = \sum_{k=0}^{n} \frac{n!}{(n+k)!} d(k, k+n) x^k.$$

Our theorem states that they are precisely the polynomials $v_n(x)$, and so they also form a binomial sequence.

We may add that in this case

$$d(1, n) = (n-1)!,$$

and by virtue of the corollary to the previous theorem the following formula for the computation of the numbers $d(k, n)$ holds

$$d(k, n) = \frac{n!}{k!} \sum_{j_1 + \ldots + j_k = n} \frac{1}{j_1 \cdots j_k}$$

where $j_1, \ldots, j_k > 1$ (when $2k > n$ the sum is empty and gives 0, which is correct).

## References


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