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On zeros and sign changes of certain Dirichlet polynomials

Dedicated to Professor Władysław Orlicz on the occasion of his 85-th birth anniversary.

1. In the analytic theory of numbers, the so-called *explicit formulae* play a very remarkable and interesting role. They express values of arithmetic functions in terms of zeros and poles of zeta-functions (see [3], [11]). A number of such formulae can be written in the form

$$F(x) = \sum_{|\gamma| \leqslant T} a_\varrho x^\varrho + R(x, T),$$

the summation being over all zeros or poles $\varrho = \beta + i\gamma$ of certain zeta-functions with $|\mathrm{Im}\,\varrho| = |\gamma| \leqslant T$; $R(x, T)$ denotes a "small" remainder. If we write

$$\Theta_T = \sup_{|\gamma| \leqslant T} \mathrm{Re}\,\varrho, \qquad x = e^t,$$

then the formula above can be written as

$$F^*(x) = e^{-\Theta_T t} F(x) = \sum_{|\gamma| \leqslant T, \beta = \Theta_T} a_\varrho e^{i\gamma_t} + R^*(x, T).$$

Hence, to some extent, the discussion on the properties of F^* (oscillations, changes of sign, and so on) reduces to the investigation of the Dirichlet polynomial $\sum_{|\gamma| \leqslant T, \beta = \Theta_T} a_\varrho e^{i\gamma_t}$ (see [8], [9], [10]).

The aim of this note is to examine some properties of polynomials of this type.

2. Let $\gamma_1, \ldots, \gamma_N$ be given different real numbers and $a_1, \ldots, a_N, b_1, \ldots, b_N$ any real numbers. We shall study functions of the form

$$f(t) = \sum_{j=1}^{N} a_j \cos(2\pi\gamma_j t + b_j) \tag{2.1}$$

which are the real parts of the Dirichlet polynomials $\sum_{j=1}^{N} \alpha_j \exp(2\pi i \gamma_j t)$, where $\alpha_j = a_j \exp(ib_j)$.

We denote the field, and also the topological additive group of real numbers, by the same letter R, and its discrete subgroup consisting of rational integers by Z.

Let $M \subset R$ denote the Z-module generated by exponents $\gamma_1, \ldots, \gamma_N$. M is a free module since it is finitely generated and torsionfree. Hence, there exist numbers $\mu_1, \ldots, \mu_n$ such that

$$M = \mu_1 T \oplus \ldots \oplus \mu_n Z.$$

We can suppose that μ_i are all positive. Moreover, $n \leqslant N$ and

$$\gamma_j = \sum_{m=1}^{n} R_{jm} \mu_m$$

with some integers R_{jm}. Let us denote n-dimensional torus by

$$\Omega = T^1 \times \ldots \times T^1,$$

where T^1 denotes the interval $[0, 1)$ supplied with quotient group R/Z topology. Obviously, the group operation in T^1 is addition mod 1.

Besides, Ω has the natural structure of an n-dimensional differentiable manifold.

We define the functions F_k: $\Omega \to R$, $k = 1, 2, \ldots$, by the formulae

$$F_k(t_1, \ldots, t_n) = \sum_{l=1}^{k} \big(\operatorname{Re} \sum_{j=1}^{N} \alpha_j (2\pi i \gamma_j)^{l-1} \exp(2\pi i \sum_{m=1}^{n} R_{jm} t_m)\big)^2.$$

Denoting by λ the continuous homomorphism λ: $R \to \Omega$,

$$\lambda(t) = (\|\mu_1 t\|, \|\mu_2 t\|, \ldots, \|\mu_n t\|),$$

where $\|\alpha\|$ is the fractional part of a real α, we have

$$F_k(\lambda(t)) = \sum_{l=0}^{k-1} \left(\frac{d^l}{dt^l} f(t)\right)^2.$$

Let us write

$$M_k = \{\mathrm{t} = (t_1, \ldots, t_n)\colon\ F_k(\mathrm{t}) = 0\}$$

and let Ω_0 be a subgroup of Ω defined by

$$\Omega_0 = \{\mathrm{t} = (t_1, \ldots, t_n)\colon\ t_1 = 0\}.$$

Let π: $\Omega \to \Omega_0$ be the projection

$$\pi(t_1, \ldots, t_n) = \left(\left\| t_j - \frac{t_1}{\mu_1} \mu_j \right\|\right)_{j=1}^{n}.$$

Furthermore, let ν_k: $\Omega_0 \to N \cup \{0\}$ be a function defined by

$$\nu_k(\mathrm{t}) = \#\,(\pi^{-1}(\{\mathrm{t}\}) \cap M_k)$$

($\#\, A$ denotes the cardinality of A). Later we shall prove that the definition of v_k is correct, i.e. the set $\pi^{-1}(\{\mathrm{t}\}) \cap M_k$ is finite for each k and every $\mathrm{t} \in \Omega_0$.

Let $N_k(T)$ denote the number of zeros of order k of the function f in the interval $[0, T]$, $T > 0$, where each such zero is counted only once.

Now, we are able to state the first result of the present note.

THEOREM 1. *If* $n \geqslant 2$, *then*

$$\lim_{T\to\infty} \frac{N_k(T)}{T} = \varkappa_k, \tag{2.2}$$

where

$$\varkappa_k = \mu_1 \int_{\Omega_0} \big(v_k(\mathrm{t}) - v_{k+1}(\mathrm{t})\big)\, d\mu(\mathrm{t}) \tag{2.3}$$

and $d\mu$ *denotes the normalized Haar measure on the group* Ω_0.

The condition $n \geqslant 2$ is natural since for $n = 1$ the function is periodic and the investigation of the number of its zeros of a given order is trivial in this case.

Since sign changes appear at zeros of odd order, we have the following

COROLLARY.

$$\lim_{T\to\infty} \frac{V(T)}{T} = \varkappa,$$

where

$$\varkappa = \mu_1 \int_{\Omega_0} \sum_{2\nmid k} \big(v_k(\mathrm{t}) - v_{k+1}(\mathrm{t})\big)\, d\mu(\mathrm{t}).$$

It can easily be seen that the set M_k is empty for sufficiently large k. Thus, the sum $\sum_{2\nmid k}\big(v_k(\mathrm{t}) - v_{k+1}(\mathrm{t})\big)$ is correctly defined.

The problem of the number of zeros of Dirichlet polynomials was the subject of interest of several authors. The existence of the limit (2.2) can be deduced from the results of a joint paper of B. Jessen and H. Tornehave (see [6]) but without (2.3), M. Kac proved in [7] that if all exponents γ_j are linearly independent over Q and $b_j = 0, j = 1, \ldots, N$, then for the number of zeros of the polynomial (2.1) in the interval $[-T, T]$, we have

$$\frac{N_T}{T} - \frac{1}{2\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Big(\prod_{j=1}^{N} I_0(|a_j|\,\xi) - \prod_{j=1}^{N} I_0\big(|a_j| \sqrt{\xi^2 + \gamma_j^2 \eta^2}\big)\Big) \frac{d\xi\, d\eta}{\eta^2},$$

where I_0 denote a Bessel function of index zero. P. Turán gives in [13] an upper estimate for the number of zeros of (2.1) in the interval $[a, a+d]$, depending on a, d, N and $\max|\gamma_j|$, $1 \leqslant j \leqslant N$ (if $a_j \geqslant 0, j = 1, \ldots, N$).

Before stating the second theorem, let us introduce some definitions. Suppose that $t_1 < t_2 < t_3$ are consecutive zeros of the Dirichlet polynomial f. We call a zero t_2 *δ-small* if

$$\max_{t_1 \leqslant t \leqslant t_2} |f(t)| < \delta \quad \text{or} \quad \max_{t_2 \leqslant t < t_3} |f(t)| < \delta.$$

Denote by $N(T, \delta)$ the number of δ-small zeros of f in the interval $[0, T]$, where each zero is counted according to its multiplicity.

We shall prove that almost all zeros of f are not δ-small.

THEOREM 2. *For every $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$\limsup_{T \to \infty} \frac{N(T, \delta)}{T} \leqslant \varepsilon.$$

If t_1 and t_2 are some consecutive zeros of the polynomial f and $t_0 \in (t_1, t_2)$ is such that

$$|f(t_0)| = \max_{t_1 \leqslant t \leqslant t_2} |f(t)| > \delta,$$

then

$$\delta < \int_{t_1}^{t_0} |f'(t)|\, dt \ll t_2 - t_1.$$

We shall call a zero t_1 of f *δ-isolated* if the interval $(t_1 - \delta, t_1 + \delta)$ is free of zeros different from t_1. Denoting by $N_1(T, \delta)$ the number of zeros of f in the interval $[0, T]$ which are not δ-isolated, we get from Theorem 2 the following corollary.

COROLLARY. *For every $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$\limsup_{T \to \infty} \frac{N_1(T, \delta)}{T} \leqslant \varepsilon.$$

It means that the zeros of f which are situated "close to each other" are rather exceptional.

3. We say that a mapping is *smooth* if it belongs to the class C^∞. Now let us prove that the mapping π is smooth on the set $\Omega \setminus \Omega_0$.

Let $p_1\colon \Omega \to T^1$ be a function defined by

$$p_1(\mathrm{t}) = p_1(t_1, \ldots, t_n) = t_1,$$

and let $r\colon T^1 \to R$ be a function defined by $r(t) = t$. Obviously, p_1 is a smooth mapping and r is also smooth except for the point $t = 0$ where it is not even continuous.

For every $\mathrm{t} = (t_1, \ldots, t_n)$, let us determine the value of

$$a(\mathrm{t}) = \left(\mathrm{id}_\Omega - \lambda\left(\frac{1}{\mu_1} r p_1\right)\right)(\mathrm{t}),$$

where "$-$" denotes subtraction in the group Ω_0. We get

$$a(\mathrm{t}) = \mathrm{t}-\lambda\left(\frac{1}{\mu_1} rp_1(\mathrm{t})\right) = \mathrm{t}-\lambda\left(\frac{t_1}{\mu_1}\right) = \left(\left\|t_j - \frac{t_1}{\mu_1}\mu_j\right\|\right)_{j=1}^{n} = \pi(\mathrm{t}).$$

Thus we have $\pi = \mathrm{if}_\Omega - \lambda(rp_1/\mu_1)$. This means that π is a composition of a number of mappings. They are smooth, except for the mapping r at the point $t_1 = 0$. Therefore, π is smooth on $\Omega \setminus \Omega_0$.

4. Next, let us examine in detail the set M_k for a given k. Since the function F_k is analytic on Ω, therefore the set M_k is analytic and compact in Ω. Hence

$$M_k = M_k^0 \cup M_k^1, \qquad M_k^0 \cap M_k^1 = \emptyset,$$

where

(1) M_k^0 is an open subset of M_k (in the topology induced on M_k);
(2) M_k^0 is an $(n-1)$-dimensional submanifold of Ω;
(3) M_k^1 is closed in M_k and its dimension is not greater than $n-2$.

The existence of such a decomposition of a complex analytic set is a well-known fact from the theory of analytic functions of several variables (see [5]). In the real case, a similar result can be proved as follows. Suppose that U is an open subset of R^n and suppose further that $\tilde{U}$ is an open subset of C^n such that $\tilde{U} \cap R^n = U$. We can assume that $\tilde{U}$ is symmetrical, i.e. together with the point $(z_1, \ldots, z_n)$ it contains $(\bar{z}_1, \ldots, \bar{z}_n)$. Let f be a real-valued function defined on U, which can be continued as an analytic function over $\tilde{U}$. This analytic continuation will also be denoted by f. Thus we have

$$f(\bar{z}_1, \ldots, \bar{z}_n) = \overline{f(z_1, \ldots, z_n)}. \tag{4.1}$$

Let us write

$$M = \{(z_1, \ldots, z_n) \in \tilde{U}\colon f(z_1, \ldots, z_n) = 0\}.$$

Since M is a complex analytic set, there exists a decomposition $M = M^0 \cup M^1$ satisfying conditions (1), (2), and (3). Obviously, the dimensions of the sets mentioned in the formulation of those conditions are understood as the dimensions over the field of complex numbers and, therefore, they are equal to twice the real dimensions.

We shall prove that $(M^0 \cap R^n) \cup (M^1 \cap R^n)$ is the needed decomposition of the real analytic set $M \cap R^n$. Condition (1) is obviously satisfied. To prove condition (2), consider a point $p_0 \in M^0 \cap R^n$. Let V be a symmetric neighbourhood of p_0 in C^n, such that (after appropriate renumbering of coordinates, if necessary)

$$V \cap M^0 = \{(\phi(z_2, \ldots, z_n), z_2, \ldots, z_n)\colon (z_2, \ldots, z_n) \in V_1\}$$

for an open set $V_1 \subset c^{n-1}$ and holomorphic ϕ. Since, owing to (4.1), the set

$V \cap M^0$ is symmetric, we have

$$(\overline{\phi(z_2, \ldots, z_n)}, \bar{z}_2, \ldots, \bar{z}_n) \in V \cap M^0$$

for any $(z_2, \ldots, z_n) \in V_1$. Hence, $\overline{\phi(z_2, \ldots, z_n)} = \phi(\bar{z}_2, \ldots, \bar{z}_n)$ and so ϕ is real for real $(z_2, \ldots, z_n)$. Therefore,

$$V \cap (M^0 \cap R^n) = \{(\phi(x_2, \ldots, x_n), x_2, \ldots, x_n): (x_2, \ldots, x_n) \in V_1 \cap R^n\}$$

and this proves that $M^0 \cap R^n$ is a submanifold of U, having the dimension equal to $n-1$.

We shall now prove (3). Owing to the Cardan theorem (see [5]), the set M^1 is analytic and its dimension is not greater than $n-2$. Hence, if $p \in M^1 \cap R^n$ and the set M^1 is determined in the neighbourhood of the point p, by the formulae

$$\begin{aligned} z_1 &= \phi_1(z_{k+1}, \ldots, z_n), \\ &\cdots\cdots\cdots\cdots \qquad (z_{k+1}, \ldots, z_n) \in V_1 \subset C^{n-k}, \\ z_k &= \phi_k(z_{k+1}, \ldots, z_n), \end{aligned}$$

where ϕ_i, $i = 1, 2, \ldots, k$, are holomorphic and $k \geqslant 2$, then the set $M^1 \cap R^n$ is contained in the submanifold which has dimension equal to $n-k$, and is determined by the equations

$$\begin{aligned} x_1 &= \operatorname{Re} \phi_1(x_{k+1}, \ldots, x_n), \\ &\cdots\cdots\cdots\cdots \\ x_k &= \operatorname{Re} \phi_k(x_{k+1}, \ldots, x_n), \end{aligned}$$

and $(x_{k+1}, \ldots, x_n) \in V_1 \cap R^{n-k}$. Hence, the set $M^1 \cap R^n$ is at most $(n-2)$-dimensional, and the existence of the decomposition of the set M_k is completely proved.

Since the mapping $\pi|_{\Omega \setminus \Omega_0}$ is smooth and M_k^0 is a submanifold of Ω, the mapping $\pi|_{M_k^0 \setminus \Omega_0}$ is also smooth. Let K be the set of critical points of this mapping. From the well-known theorem of Sard (see [4]) we have that $\mu(\pi(K)) = 0$. Let us consider

$$S_0 = \pi(K) \cup (M_k \cap \Omega_0) \cup \pi(M_k^1).$$

Obviously,

$$\mu(S_0) \leqslant \mu(\pi(K)) + \mu(M_k \cap \Omega_0) + \mu(\pi(M_k^1)),$$

and since the sets $M_k \cap \Omega_0$, $\pi(M_k^1)$ have dimensions not greater than $n-2 = \dim \Omega_0 - 1$, we have $\mu(S_0) = 0$.

If $t \in \pi(M_k) \setminus S_0$ then t is a regular value of the mapping $\pi|_{\Omega \setminus \Omega_0}$. Hence, there exists a neighbourhood $U \subset \pi(M_k) \setminus S_0$ of t which is properly covered by π. It follows in particular that the set $\pi(M_k) \setminus S_0$ is open in Ω_0.

5. We shall now prove that the functions ν_k are bounded. Let $\mathrm{t} \in \pi(M_k)$. There exists t_0, $0 < t_0 < 1/\mu_1$, such that

$$E_k(\mathrm{t}+\lambda(t_0)) = \alpha > 0.$$

Then the function of a real variable

$$\left(0, \frac{1}{\mu_1}\right) \ni u \mapsto F_k(\mathrm{t}+\lambda(u))$$

cannot be identically equal to zero in any nonempty interval, because it is the restriction to the interval $(0, 1/\mu_1)$ of a certain function $\psi_{k,\mathrm{t}}$ which is holomorphic in the domain $s = \sigma + i\tau$, $|\sigma| < 2/\mu_1$, $|\tau| < 1$. Therefore, t_0 always exists.

There exists also a neighbourhood U of t such that for every $\mathrm{t}' \in U$ we have

$$F_k(\mathrm{t}'+\lambda(t_0)) \geqslant \tfrac{1}{2}\alpha. \tag{5.1}$$

Since the functions $\psi_{k,\mathrm{t}'}$, $\mathrm{t}' \in U$, are uniformly bounded in the domain $s = \sigma + i\tau$, $|\sigma| < 2/\mu_1$, $|\tau| < 1$, hence owing to (5.1), also the numbers of zeros of the functions $\varphi_{k,\mathrm{t}'}$ are uniformly bounded (from Jensen's formula, see [12]). It follows that the numbers $\nu_k(\mathrm{t}')$, $\mathrm{t}' \in U$, are bounded by a constant $C(U)$ depending on U but independent of the choice of t'. In other words, the function ν_k is locally bounded on $\pi(M_k)$. Since $\nu_k = 0$ outside $\pi(M_k)$, thus it is locally bounded on Ω_0. But Ω_0 is compact, so ν_k is globally bounded.

From the above, we can easily deduce that the set $\pi(M_k)\setminus S_0$ can be written in the form

$$\pi(M_k)\setminus S_0 = U_1 \cup U_2 \cup \ldots \cup U_L,$$

where

$$U_j = \{\mathrm{t} \in \pi(M_k)\setminus S_0 \colon \nu_k(t) = j\}.$$

The function ν_k is constant in a neighbourhood of every $\mathrm{t} \in \pi(M_k)\setminus S_0$ and so U_j are open subsets of $\pi(M_k)$. This follows from the fact that each point $\mathrm{t} \in \pi(M_k)\setminus S_0$ is properly covered by π. Hence the sets $\pi(M_k)\setminus(U_1 \cup \ldots \cup U_{j-1} \cup U_{j+1} \cup \ldots \cup U_L)$ are closed and therefore

$$\bar{U}_j \subset \pi(M_k)\setminus(U_1 \cup \ldots \cup U_{j-1} \cup U_{j+1} \cup \ldots \cup U_L)$$

for every j, and so, for the boundary $\mathrm{bd}\, U_j$ of the set U_j, we have

$$\mathrm{bd}\, U_j = \bar{U}_j \setminus U_j \subset \pi(M_k)\setminus(U_1 \cup \ldots \cup U_L) = S_0.$$

Hence,

$$\mu(\mathrm{bd}\, U_j) \leqslant \mu(S_0) = 0.$$

This gives an essential information about the sets U_j since by the classic

Kronecker–Weyl theorem (see [1], [2]) on Diophantine approximation we have for every j, $1 \leqslant j \leqslant L$,

$$\frac{1}{T}\sum_{k\in D_j T} 1 \to \mu(U_j), \qquad T\to\infty,$$

where

$$D_j(T) = \left\{k\colon\ k \leqslant T, \left(\left\|k\frac{\mu_l}{\mu_1}\right\|\right)_{l=1}^{n} \in U_j\right\}. \tag{5.2}$$

6. Let $N^*(T)$ be the number of zeros of order not lower than k of the function f in the interval $[0, T]$.

If t is such that

$$\frac{k}{\mu_1} \leqslant t < \frac{k+1}{\mu_1},$$

then

$$\pi(\lambda(t)) = \left(\left\|\mu_1 t - \frac{\|\mu_1 t\|\,\mu_l}{\mu_1}\right\|\right)_{l=1}^{n} = \left(\left\|\frac{k\mu_l}{\mu_1}\right\|\right)_{l=1}^{n}.$$

Consequently,

$$N^*(T) = \sum_{m\leqslant[\mu_1 T]} \nu_k\left(\left(\left\|\frac{m\mu_l}{\mu_1}\right\|\right)_{l=1}^{n}\right) + O(1) = \sum_{j=1}^{L} j \sum_{m\in D_j([\mu_1 T])} 1 + \sum_{m\in D_0} 1 + O(1),$$

where the sets D_j, $j = 1, 2, \ldots, L$, are defined by (5.2) and

$$D_0 = D_0(T) = \left\{r\colon\ r \leqslant [\mu_1 T], \left(\left\|\frac{r\mu_l}{\mu_1}\right\|\right)_{l=1}^{n} \in S_0\right\}.$$

Hence,

$$N^*(T) = \sum_{j=1}^{L} j\mu_1 T\mu(U_j) + o(T) = \mu_1 T \int_{\Omega_0} \nu_k(\mathrm{t})\, d\mu(\mathrm{t}) + o(T).$$

But

$$N_k(T) = N_k^*(T) - N_{k+1}^*(T) \sim \mu_1 T \int_{\Omega_0} \big(\nu_k(\mathrm{t}) - \nu_{k+1}(\mathrm{t})\big)\, d\mu(\mathrm{t})$$

and the proof of Theorem 1 is completed.

7. We shall now prove Theorem 2. All the notation introduced previously are still valid. Since we are now interested in zeros of any multiplicity, we shall only deal with the set M_1. For instance, the sets U_j, $j = 1, 2, \ldots, L$, and S_0 are now subsets of $\pi(M_1)$.

Let $\mathrm{t} \in U_1 \cup \ldots \cup U_L$. There exists an open neighbourhood V of t such that $V \subset U_1 \cup \ldots \cup U_L$, which is properly covered by the mapping $\pi|_{M_1}$.

We can also suppose that the closures of the connected components of $\pi^{-1}(V)\cap M_1$ are disjoint. Hence, there exists a positive constant $c = c(V)$, depending on V, such that for every $\mathrm{t}'\in V$ and every

$$\mathrm{t}'+\lambda(t_1), \quad \mathrm{t}'+\lambda(t_2)\in\pi^{-1}(\{\mathrm{t}'\})\cap M_1,$$

$0 \leqslant t_1 < t_2 \leqslant 1/\mu_1$ we have

$$\max_{t_1\leqslant t\leqslant t_2} F_1(\mathrm{t}'+\lambda(t)) \geqslant c, \tag{7.1}$$

$$\max_{t_2\leqslant t\leqslant 1/\mu_1} F_1(\mathrm{t}'+\lambda(t)) \geqslant c, \tag{7.1}$$

$$\max_{0\leqslant t\leqslant t_1} F_1(\mathrm{t}'+\lambda(t)) \geqslant c, \tag{7.1}$$

Thus, if $F\subset U_1\cup\ldots\cup U_L$ is a closed subset, then there exists a constant $C = C(F) > 0$ depending on F and satisfying (7.1), (7.2), (7.3) for every

$$\mathrm{t}'+\lambda(t_1), \quad \mathrm{t}'+\lambda(t_2)\in\pi^{-1}(\{\mathrm{t}'\})\cap M_1,$$

$0 \leqslant t_1 < t_2 \leqslant 1/\mu_1$, $\mathrm{t}'\in F$. This obviously follows from the fact that F is compact.

It follows also from the above that for every open neighbourhood V of S_0 there exists a positive number $\delta = \delta(V)$ such that, if t_0 is a δ-small zero of the polynomial f then $\pi(\lambda(t_0))\in V$. Since $\mu(S_0) = 0$, therefore for every $\varepsilon > 0$ one can choose a neighbourhood V such that

$$\mu(V) < \frac{\varepsilon}{2\mu_1 m_1 k}, \qquad \mu(\mathrm{bd}\, V) = 0,$$

where

$$m_1 = \max_{\mathrm{t}\in\Omega_0} \nu_1(\mathrm{t})$$

and k denotes the maximal multiplicity of zeros of f.

Thus we have

$$N(T,\delta) \leqslant k \sum_{m\in D(T)} \nu_1\left(\left(\left\|\frac{m\mu_l}{\mu_1}\right\|_{l=1}^{n}\right)\right) + O(1),$$

where

$$D(T) = \left\{r\colon\ 1\leqslant r\leqslant [\mu_1 T],\ \left(\left\|\frac{r\mu_l}{\mu_1}\right\|\right)_{l=1}^{n}\in V\right\}.$$

Finally,

$$N(T,\delta) \leqslant k m_1\,\mu(V)\,\mu_1 T + o(T) \leqslant \varepsilon T,$$

for $T \geqslant T_0(\varepsilon)$ and the proof of Theorem 2 is completed.

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