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On the Baire category of some collections of sets in measure spaces

Abstract. The main results of this paper, Theorems 1 and 2, assert that under some quite simple assumptions, a collection $\mathscr{C}$ of sets in a measure space (S, Σ, λ) either coincides with Σ or is of first Baire category (in the associated semimetric space). These results are illustrated with several examples of such collections $\mathscr{C}$ arising in a natural way in the theory of vector measures. In particular, a recent result of R. Anantharaman, which was an inspiration for this paper, is shown to hold in a more general setting.

In general, our terminology and notation concerning measures are as in [3] and [4]. Throughout, we let (S, Σ, λ) be a finite positive measure space; however, in some of our examples below we make a specific choice of λ. We equip Σ with the Fréchet–Nikodym semimetric $(A, B) \to \lambda(A \Delta B)$ associated to λ. It is well known that the semimetric space Σ_λ thus obtained is complete (see e.g. [4; III.7.1]). Hence, by the Baire category theorem, if a set $\mathscr{C} \subset \Sigma$ is of first category in Σ_λ, then $\mathscr{C} \neq \Sigma$.

The starting point for this paper was the following recent result of R. Anantharaman [1; Theorem 1]: If (S, Σ, λ) is the usual Lebesgue measure space on $S = [0, 1]$ and $F\colon \Sigma \to l_2$ is the vector measure defined by $F(E) = 2(\int_E r_n\, d\lambda)$, where (r_n) is the Rademacher sequence, then $F^{-1}(l_p)$ for $1 \leqslant p < 2$ (and hence also $F^{-1}(\bigcup_{p<2} l_p)$ is of first category in Σ_λ.

It turned out very quickly that this result can be quite easily extended to general vector measures. After a short time it also became clear that our generalization of Anantharaman's result can be put in a more "abstract" form dealing with the Baire category of some subcollections $\mathscr{C}$ in Σ_λ. Accordingly, we prove two such "abstract" results in this paper, Theorem 1 and 2. Roughly speaking they say that, under some (quite simple) assumptions on $\mathscr{C}$, either $\mathscr{C} = \Sigma$ or $\mathscr{C}$ is of first category in Σ_λ.

As will be seen in Example 1, Anantharaman's theorem — and our extension of it — follow immediately from Theorem 1. The only fact about Anantharaman's measure F that we need here is that, for $1 \leqslant p < 2$, $F(\Sigma) \not\subset l_p$; in [1] also a "local" version of that property of F was required.

Theorem 2 is a stronger variant of Theorem 1. In fact, Theorem 1 is a direct consequence of part (i) of Theorem 2, but we preferred to state and prove these two results separately.

We illustrate our results, particularly Theorem 1, with several examples of collections $\mathscr{C} \subset \Sigma$ arising quite naturally in the theory of vector measures. Typically, we show that if $\mathscr{C} \neq \Sigma$, then $\mathscr{C}$ is of first category (sometimes even closed and nowhere dense) in Σ_λ.

Finally, let's point out that all our arguments are very simple and entirely standard, similar to those used in the classical proofs of the Vitali–Hahn–Saks or Nikodym theorems (see [4]); the reader may also consult the paper [6] of Labuda.

THEOREM 1. *Let $\mathscr{C}$ be a subset of Σ such that*
(a) *every λ-atom is in $\mathscr{C}$;*
(b) *if $E, F \in \mathscr{C}$ and $E \cap F = \emptyset$, then $E \cup F \in \mathscr{C}$;*
(c) *if $E, F \in \mathscr{C}$ and $E \subset F$, then $F \setminus E \in \mathscr{C}$.*
Then:
(i) *If $\mathscr{C}$ has a nonempty interior in Σ_λ, then $\mathscr{C} = \Sigma$.*
(ii) *If $\mathscr{C}$ is an F_σ in Σ_λ, then either $\mathscr{C} = \Sigma$ or $\mathscr{C}$ is of first category in Σ_λ.*

Proof. (i) Suppose $\operatorname{Int}\mathscr{C} \neq \emptyset$; then there are $E_0 \in \mathscr{C}$ and $\varepsilon > 0$ such that

$$\mathscr{B}(E_0, \varepsilon) := \{F \in \Sigma\colon \dot{\lambda}(E_0 \Delta F) < \varepsilon\} \subset \mathscr{C}.$$

Take any $E \in \Sigma$ with $\lambda(E) < \varepsilon$. Then both $E_0 \cup E$ and $E_0 \setminus E$ are in $\mathscr{B}(E_0, \varepsilon)$, hence in $\mathscr{C}$. Since $E = (E_0 \cup E) \setminus (E_0 \setminus E)$, applying (c) we see that $E \in \mathscr{C}$. We have thus verified that

$$(*) \qquad \text{if } E \in \Sigma \text{ and } \lambda(E) < \varepsilon, \text{ then } E \in \mathscr{C}.$$

Now, by Saks' decomosition [4]; IV.9.7, there is a finite Σ-partition $S = E_1 \cup \ldots \cup E_l$ such that, for each i, either $\lambda(E_i) < \varepsilon$ or E_i is an atom for λ.

Let $E \in \Sigma$, and fix any i, $1 \leqslant i \leqslant l$. If $\lambda(E \cap E_i) < \varepsilon$, then $E \cap E_i \in \mathscr{C}$ by (*); in the other case $E \cap E_i$ is a λ-atom and so $E \cap E_i \in \mathscr{C}$ by (a). From (b) it is now immediate that $E \in \mathscr{C}$.

(ii) follows directly from (i). ■

EXAMPLE 1. Let X be a locally convex space (or, more generally, a topological abelian group), and let $\mu\colon \Sigma \to X$ be a finitely additive set function. Let Y be a linear subspace (resp., a subgroup) of X. Then $\mathscr{C} := \mu^{-1}(Y)$ satisfies conditions (b) and (c). Suppose that μ is λ-continuous (hence countably additive), i.e., $\mu\colon \Sigma_\lambda \to X$ is a continuous mapping, and that Y is F_σ in X. Then, of course, $\mathscr{C}$ is F_σ in Σ_λ. Hence, if we known in addition that $\mu(A) \in Y$ for every λ-atom A, then also condition (a) is satisfied for our $\mathscr{C}$, and we conclude that either $\mathscr{C} = \Sigma$ or $\mathscr{C}$ is of first category in Σ_λ. Finally, let's note that if the subspace Y is equipped with a norm whose closed unit ball is closed in X, then Y is obviously an F_σ in X. And this is precisely the situation we encounter in Anantharaman's result, for if $1 \leqslant p < 2$, then $l_p \subset l_2$ and the closed unit ball of l_p is closed in l_2.

Remark 1. Let $S = [0, 1]$, Σ be the Borel σ-algebra on S, λ_1 be Lebesgue measure, δ_1 the Dirac measure concentrated at 1, and let $\lambda = \lambda_1 + \delta_1$. Define μ: $\Sigma \to \boldsymbol{R}^2 = X$ by $\mu(E) = (\lambda_1(E), \delta_1(E))$, and let $Y = \boldsymbol{R} \times \{0\}$. Then $\mu^{-1}(Y) =$ Borel subsets of $[0, 1)$ is a proper open subset of Σ_λ. Thus the assumption (a) in Theorem 1 is essential.

EXAMPLE 2 (General). Let η: $\Sigma_\lambda \to \bar{\boldsymbol{R}}_+$ be a lower semicontinuous function such that

(α) $\eta(A) < \infty$ if $A \in \Sigma$ is a λ-atom;

(β) $\eta(E \cup F) \leqslant \eta(E) + \eta(F)$ if E, $F \in \Sigma$ and $E \cap F = \emptyset$;

(γ) $\eta(F \setminus E) \leqslant \eta(E) + \eta(F)$ if E, $F \in \Sigma$ and $E \subset F$.

Then $\mathscr{C} = \{E \in \Sigma\colon \eta(E) < \infty\}$ satisfies conditions (a)–(c) and is an F_σ in Σ_λ. (Indeed, $\mathscr{C}$ is the union of the closed sets $\mathscr{C}_n = \{E \in \Sigma\colon \eta(E) \leqslant n\}$, $n \in \boldsymbol{N}$.) By Theorem 1 (ii), either $\mathscr{C}$ is of first category in Σ_λ or $\mathscr{C} = \Sigma$, i.e., η is finite valued. Actually, in the latter case η is bounded, see Example 2′ below.

Remark 2. It is useful to observe that if a collection $\mathscr{C} \subset \Sigma$ is hereditary (i.e., $A \in \Sigma$, $A \subset B \in \mathscr{C}$ imply $A \in \mathscr{C}$), then a set F is in the closure of $\mathscr{C}$ in Σ_λ if and only if for every $\varepsilon > 0$ there exists $E \in \mathscr{C}$ such that $E \subset F$ and $\lambda(F \setminus E) < \varepsilon$.

EXAMPLE 3. Let f: $S \to \boldsymbol{R}_+$ be a measurable function.

(A) Suppose f is not λ-essentially bounded and consider the set $\mathscr{C} = \{E \in \Sigma\colon f \text{ is } \lambda\text{-essentially bounded on } E\}$. Then $\mathscr{C}$ is an F_σ in Σ_λ and satisfies conditions (a)–(c). Hence, by Theorem 1 (ii), $\mathscr{C}$ is of first category in Σ_λ.

(B) Suppose f is not λ-integrable, and define η: $\Sigma \to \bar{\boldsymbol{R}}_+$ by $\eta(E) = \int_E f\,d\lambda$. Then η satisfies the assumptions of Example 2 above; in particular, its lower semicontinuity on Σ_λ can be easily checked using Fatou's lemma. (It also follows from the Lemma proved below.) Therefore, by Theorem 1 (ii), $\mathscr{C} = \{E \in \Sigma\colon \int_E f\,d\lambda < \infty\}$ is of first category in Σ_λ.

LEMMA. *Let η: $\Sigma \to \bar{\boldsymbol{R}}_+$ be a submeasure* (i.e., *subadditive and nondecreasing*) *such that $\eta(E_n) \to \eta(E)$ whenever $E_n \nearrow E$. Suppose that $\eta \ll \lambda$ on every set $E \in \Sigma$ such that $\eta(E) < \infty$ (that is, for each $\varepsilon > 0$ there is a $\delta > 0$ so that $\eta(A) < \varepsilon$ whenever $A \subset E$ and $\lambda(A) < \delta$), and that $\eta(E) = 0$ whenever $\lambda(E) = 0$. Then η is lower semicontinuous on Σ_λ.*

In consequence, by Example 2, if $\eta(S) = \infty$ and $\eta(A) < \infty$ for all λ-atoms A, then $\mathscr{C} = \{E \in \Sigma\colon \eta(E) < \infty\}$ is of first category in Σ_λ.

Proof. Let $0 < r < \infty$; we have to show that $\mathscr{C}_r = \{E \in \Sigma\colon \eta(E) \leqslant r\}$ is closed in Σ_λ. Let F be in the closure of $\mathscr{C}_r$ in Σ_λ. Then, as noted in Remark 2 above, for every $\varepsilon > 0$ we can find $E \in \mathscr{C}_r$ such that $E \subset F$ and $\lambda(F \setminus E) < \varepsilon$.

Fix an $E \subset F$ with $\eta(E) < \infty$, and let $\varepsilon > 0$ and $\gamma > 0$ be arbitrary.

Since $\eta \ll \lambda$ on E, there is $0 < \delta < \varepsilon$ such that if $A \subset E$ and $\lambda(A) < \delta$, then $\eta(A) < \gamma$. Since $F \in \mathscr{C}_r$, there is $E' \subset F$ such that $\eta(E') \leqslant r$ and $\lambda(F \setminus E') < \delta$. Then $\lambda(E \setminus E') < \delta$, hence $\eta(E \setminus E') < \gamma$. Therefore,

$$\eta(E) \leqslant \eta(E \cap E') + \eta(E \setminus E') < r + \gamma,$$

$$\eta(E \cup E') \leqslant \eta(E') + \eta(E \setminus E') < r + \gamma.$$

Since $\gamma > 0$ was arbitrary, it follows that whenever $E \subset F$ and $\eta(E) < \infty$, then $\eta(E) \leqslant r$. In particular, also $\eta(E \cup E') \leqslant r$.

We have thus shown that whenever we have a set $E \subset F$ with $\eta(E) < \infty$, then for every $\varepsilon > 0$ we can find a set E'' such that $E \subset E'' \subset F$, $\eta(E'') \leqslant r$, and $\lambda(F \setminus E'') < \varepsilon$. It follows that there exists an increasing sequence of sets $E_n \subset F$ such that $\lambda(F \setminus E_n) \to 0$ and $\eta(E_n) \leqslant r$ for all n. Let $E_0 = \bigcup_{n=1}^{\infty} E_n$. Then $\eta(E_0) = \lim_n \eta(E_n) \leqslant r$. Moreover, as $\lambda(F \setminus E_0) = 0$, we also have $\eta(F \setminus E_0) = 0$, and we conclude that $\eta(F) \leqslant r$. ■

EXAMPLE 4. Let X be a Banach space and $\mu: \Sigma \to X$ be a countably additive vector measure. Then, by the Bartle–Dunford–Schwartz theorem [3]; p. 14, there exists a finite positive measure λ on Σ such that μ is λ-continuous.

(A) Let $\eta = |\mu|: \Sigma \to \bar{\boldsymbol{R}}_+$ be the variation of μ. It is clear that η satisfies the assumption of the above Lemma. Therefore, if μ is not of bounded variation, then $\mathscr{C} = \{E \in \Sigma: |\mu|(E) < \infty\}$ is of first category in Σ_λ.

(B) Let $\mathscr{C} = \{E \in \Sigma: \mu$ is of σ-finite variation on $E\}$. Then, using Remark 2, it is easy to see that $\mathscr{C}$ is closed in Σ_λ. Moreover, it is obvious that $\mathscr{C}$ satisfies conditions (a)–(c) of Theorem 1. Consequently, if μ is not of σ-finite variation on S, then $\mathscr{C}$ is nowhere dense in Σ_λ. Let's remark here that the vector measures μ such that $|\mu|(E) = \infty$ for every non-μ-null set E exist in abundance (see [5] and [2]; Theorem 2.4).

(C) Let now $\mathscr{C}$ be the collection of those $E \in \Sigma$ for which the range of μ over E, i.e., $\{\mu(F): F \subset E\}$, is a relatively (norm) compact subset of X. We easily verify that $\mathscr{C}$ is closed in Σ_λ. Hence, by Theorem 1 (i), if $\mu(\Sigma)$ is not relatively compact, then $\mathscr{C}$ is nowhere dense in Σ_λ. (We recall that $\mu(\Sigma)$ is always relatively weakly compact [3]; p. 14.) Let us also observe here that we can arrive at the same class $\mathscr{C}$ as follows: For every $E \in \Sigma$ define $\eta(E)$ to be the infimum of those $\varepsilon > 0$ for which the range of μ over E can be covered by a finite number of balls of radius ε. Then it is easily seen that $\eta: \Sigma \to \boldsymbol{R}_+$ is a submeasure, and that $\eta \ll \lambda$ (i.e., η is a continuous function on Σ_λ). Moreover, it is obvious that $\mathscr{C} = \{E \in \Sigma: \eta(E) = 0\}$.

(D) Finally, let us consider the collection $\mathscr{C}$ of those sets E in Σ over which μ is representable as the indefinite Bochner integral with respect to λ of some function $f_E: E \to X$ (i.e., $\mu(B) = \int_B f_E \, d\lambda$ for all measurable sets $B \subset E$). It is evident that $\mathscr{C}$ satisfies conditions (a)–(c). Moreover, with some help from Remark 2 and the Lemma, it is not very hard to verify that, for every $r > 0$, $\mathscr{C}_r = \{E \in \mathscr{C}: |\mu|(E) \leqslant r\}$ is closed in Σ_λ. Hence, by Theorem 1 (ii), either $\mathscr{C} = \Sigma$ or $\mathscr{C}$ is of first category in Σ_λ.

THEOREM 2. *Let* $(\mathscr{C}_n)$ *be an increasing sequence of subsets of* Σ. *Assume that*

(a′) *every* λ*-atom is in* $\mathscr{C}_n$ *for some* n,

and that for each n *there is an* m *such that*

(b′) *if* $E, F \in \mathscr{C}_n$ *and* $E \cap F = \emptyset$, *then* $E \cup F \in \mathscr{C}_m$;

(c′) *if* $E, F \in \mathscr{C}_n$ *and* $E \subset F$, *then* $F \setminus E \in \mathscr{C}_m$.

Then:

(i) *If one of the sets* $\mathscr{C}_n$ *has a nonempty interior in* Σ_λ, *then* $\mathscr{C}_k = \Sigma$ *for some* k.

(ii) *If all the sets* $\mathscr{C}_n$ *are closed in* Σ_λ, *then either* $\mathscr{C}_k = \Sigma$ *for some* k *or the union of these sets is of first category in* Σ_λ.

Proof. Let $\phi\colon N \to N$ be an increasing function such that, for each n, conditions (b′) and (c′) are satisfied with $m = \phi(n)$. Let n_0 be such that $\operatorname{Int}\mathscr{C}_{n_0} \neq \emptyset$, and choose $E_0 \in \Sigma$ and $\varepsilon > 0$ so that $\mathscr{B}(E_0, \varepsilon) \subset \mathscr{C}_{n_0}$. Let $m_0 = \phi(n_0)$. As in the proof of Theorem 1 we easily verify that

$$(*') \qquad \text{if } E \in \Sigma \text{ and } \lambda(E) < \varepsilon, \text{ then } E \in \mathscr{C}_{m_0}.$$

Let $S = E_1 \cup \ldots \cup E_l$ be a Saks decomposition corresponding to ε. We are now going to verify the following claim:

(+) For every $1 \leqslant i \leqslant l$ there is an m_i such that $E \cap E_i \in \mathscr{C}_{m_i}$ for all $E \in \Sigma$.

Fix $1 \leqslant i \leqslant l$. If $\lambda(E_i) < \varepsilon$, then $m_i = m_0$ is as required, by $(*')$. In the other case, E_i must be a λ-atom and therefore, by (a′), there is a k_i such that $E_i \in \mathscr{C}_{k_i}$; let $l_i = \max(m_0, k_i)$ and $m_i = \phi(l_i)$. Take any $E \in \Sigma$. If $\lambda(E \cap E_i) = 0$, then $E \cap E_i \in \mathscr{C}_{m_0} \subset \mathscr{C}_{m_i}$ by $(*')$. In the opposite case, let $F = E_i \setminus (E \cap E_i)$. Then $\lambda(F) = 0$ so that $F \in \mathscr{C}_{m_0} \subset \mathscr{C}_{l_i}$ by $(*')$. Moreover, since $E_i \in \mathscr{C}_{k_i} \subset \mathscr{C}_{l_i}$, $F \subset E_i$, and $E \cap E_i = E_i \setminus F$, we can apply (c′) to see that $E \cap E_i \in \mathscr{C}_{m_i}$. Thus m_i is again as required.

Applying (+) and (b′), it is now easy to conclude the proof of (i). Assertion (ii) is an obvious consequence of (i). ■

Remark 3. If the measure λ is atomless, then the above proof simplifies considerably. Also, if the sets $\mathscr{C}_n$ are closed, then $E \in \mathscr{C}_n$ and $\lambda(E \Delta F) = 0$ imply $F \in \mathscr{C}_n$, and again the above proof takes a bit simpler form.

EXAMPLE 1′. As at the end of Example 1, let X be a locally convex space, let Y be its subspace equipped with a norm whose closed unit ball B_Y is closed in X, and let $\mu\colon \Sigma \to X$ be a λ-continuous measure. For each n, set $\mathscr{C}_n = \mu^{-1}(nB_Y)$. Conditions (b′) and (c′) are then fulfilled with $m = 2n$. Applying Theorem 2(ii) we see that if $\mu(\Sigma) \subset Y$, then $\mu\colon \Sigma \to Y$ is a bounded finitely additive measure. More generally, the same conclusion can be obtained when $\mu(\Sigma) \subset Y$ and Y, instead of a norm, is equipped with a locally convex topology having a base at zero consisting of sets that are closed in X.

EXAMPLE 2′. Let $\eta\colon \Sigma \to \bar{\boldsymbol{R}}_+$ and $\mathscr{C}_n$ be as in Example 2. Then conditions (b′) and (c′) are satisfied with $m = 2n$. Now suppose $\eta(E) < \infty$ for all $E \in \Sigma$. From Theorem 2(ii) it then follows that η is bounded on Σ. This can be applied to get Example 1′ by letting $\eta(E) = \|\mu(E)\|_Y$. An application to Nikodym's uniform boundedness theorem is indicated below.

EXAMPLE 5. Let (μ_i) be a sequence of scalar valued countably additive measures on Σ, and define a finite positive measure λ on Σ by

$$\lambda = \sum_{i=1}^{\infty} 2^{-i} |\mu_i| \left(1 + |\mu_i|(S)\right)^{-1}.$$

Assume that

$$\eta(E) := \sup_i |\mu_i(E)| < \infty \qquad \text{for all } E \in \Sigma.$$

Since $\mu_i \ll \lambda$ for every i, η is lower semicontinuous on Σ_λ. Appealing to Example 2′ we easily see that η is bounded on Σ. That is, the family (μ_i) is uniformly bounded on Σ. From this the general form of Nikodym's boundedness theorem [4]; III.9.8, follows readily.

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