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On linear functionals in Hardy–Orlicz spaces in the half-plane. I

Abstract. In this paper we give the representation of linear functionals norm continuous on the space of finite elements $H^{0\psi}$ in the Hardy–Orlicz space $H^{*\psi}$.

The linear functionals in Hardy–Orlicz spaces in the unit disc were considered by Leśniewicz ([4] and [5]).

Strömberg in [8] considers dual spaces of the Hardy spaces making use of bounded mean oscillation with Orlicz norms.

This paper can be regarded as a continuation of papers [9]–[13], which contain the study of Hardy–Orlicz spaces of analytic functions in the half-plane. Some results of papers [9]–[13] and other papers will be needed here. We collect them in the first section.

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1. Let $\psi\colon \langle 0, \infty) \to \langle 0, \infty)$ be an N-function, i.e., ψ is increasing and convex and satisfies the following conditions:

$$(0_1)\qquad \lim_{u\to 0^+} \frac{\psi(u)}{u} = 0 \quad \text{and} \quad (\infty_1)\qquad \lim_{u\to\infty} \frac{\psi(u)}{u} = \infty.$$

ψ satisfies the condition (Δ_2) if for some constant $d > 1$ the inequality $\psi(2u) \leqslant d\psi(u)$ for $u \geqslant 0$ holds. For any N-function the function

$$\psi^*(v) = \sup\{uv - \psi(u)\colon u \geqslant 0\} \qquad \text{for } v \geqslant 0$$

is also an N-function. Moreover, $(\psi^*)^* = \psi$ ([3], Chapter I).

1.1. If S denotes the space of all complex-valued functions, defined and measurable on the interval $(-\infty, \infty)$, then the functional

$$\varrho_\psi(f) = \int_{-\infty}^{\infty} \psi(|f(t)|)\,dt$$

is a convex modular in the sense of Musielak and Orlicz. We define an Orlicz space $L^{*\psi}$ as

$$L^{*\psi} = \{f \in S\colon \varrho_\psi(kf) < \infty \text{ for some } k > 0\},$$

whereas a space of finite elements $L^{0\psi}$ of $L^{*\psi}$ as

$$L^{0\psi} = \{f \in S\colon \varrho_\psi(kf) < \infty \text{ for every } k > 0\}.$$

Two equivalent norms can be defined in the space $L^{*\psi}$

$$\|f\|_\psi = \inf\{\varepsilon > 0\colon \varrho_\psi(f/\varepsilon) \leqslant 1\} \qquad \text{for } f \in L^{*\psi},$$

and

$$\|f\|_{(\psi)} = \sup\{|\int_{-\infty}^{\infty} f(t)g(t)dt|\colon \varrho_{\psi^*}(g) \leqslant 1,\, g \in L^{\psi^*}\} \qquad \text{for } f \in L^{*\psi}$$

([3], [6], [7]).

1.2. If $A(\Omega)$ denotes the space of analytic functions F in the half-plane $\Omega = \{w \in \boldsymbol{C}\colon \operatorname{Re} w > 0\}$ (in the sense of [9]), then the functional

$$\varrho_\psi(F) = \sup\{\int_{-\infty}^{\infty} \psi(|F(x+iy)|)dy\colon x > 0\}$$

defined on the space $A(\Omega)$ is a convex modular.

We define the following classes of functions:

$$H^\psi = \{F \in A(\Omega)\colon \varrho_\psi(F) < \infty\},$$
$$H^{*\psi} = \{F \in A(\Omega)\colon kF \in H^\psi \text{ for some } k > 0\},$$
$$H^{0\psi} = \{F \in A(\Omega)\colon kF \in H^\psi \text{ for every } k > 0\}.$$

The class H^ψ is an absolutely convex set in $A(\Omega)$ and the classes $H^{*\psi}$ and $H^{0\psi}$ are linear subspaces of $A(\Omega)$. Obviously,

$$H^{0\psi} \subset H^\psi \subset H^{*\psi} \subset A(\Omega).$$

The class H^ψ we call *Hardy–Orlicz class in Ω*, $H^{*\psi}$ *Hardy–Orlicz space in Ω* and $H^{0\psi}$ the *space of finite elements in $H^{*\psi}$*. In the space $H^{*\psi}$ we introduce the norm by the formula

$$\|F\|_\psi = \inf\{\varepsilon > 0\colon \varrho_\psi(F/\varepsilon) \leqslant 1\} \qquad \text{for } F \in H^{*\psi}.$$

In $H^{*\psi}$ we can introduce another norm by the formula

$$\|F\|_{(\psi)} = \sup\{|\int_{-\infty}^{\infty} F(x+iy)g(y)dy|\colon \varrho_{\psi^*}(g) \leqslant 1,\, g \in L^{\psi^*},\ x > 0\}$$

for $F \in H^{*\psi}$. The norms $\|\cdot\|_\psi$ and $\|\cdot\|_{(\psi)}$ are equivalent on $H^{*\psi}$; namely

$$\|F\|_\psi \leqslant \|F\|_{(\psi)} \leqslant 2\|F\|_\psi \qquad \text{for every } F \in H^{*\psi}.$$

If $F \in H^{*\psi}$, then F has non-tangential limits in almost every point of the imaginary axis and the boundary function $F(i\cdot)$ belongs to the space $L^{*\psi}$ ([9]).

For $F \in H^{*\psi}$ we have

$$\|F\|_{\psi} = \|F(i\cdot)\|_{\psi} \quad \text{and} \quad \|F\|_{(\psi)} = \|F(i\cdot)\|_{(\psi)} \quad ([10]).$$

In $H^{*\psi}$, similarly as in $L^{*\psi}$, we can define two convergences: a norm convergence and a modular convergence. A sequence $\{F_n\} \subset H^{*\psi}$ is convergent in norm to $F \in H^{*\psi}$ if $\|F_n - F\|_{\psi} \to 0$ as $n \to \infty$; this holds iff $\varrho_{\psi}(\lambda(F_n - F)) \to 0$ as $n \to \infty$ for any $\lambda > 0$. Moreover, a sequence $\{F_n\} \subset H^{*\psi}$ is convergent in modular to $F \in H^{*\psi}$ if $\varrho_{\psi}(\lambda(F_n - F)) \to 0$ as $n \to \infty$ for some $\lambda > 0$ (in general, dependent on $\{F_n - F\}$). In the case where ψ satisfies the condition (Δ_2), $H^{*\psi} = H^{0\psi}$ and norm and modular convergences are equivalent. Otherwise, we have $H^{0\psi} \subset H^{*\psi}$ only and the norm convergence implies the modular convergence ([11]).

1.3. By H^1 in the disc $D = \{z \in \mathbf{C}\colon |z| < 1\}$ we denote the class of all functions G analytic in D for which

$$\|G\|_1 = \sup\Big\{\int_0^{2\pi} |G(re^{i\theta})|\,d\theta\colon\ 0 \leqslant r < 1\Big\} < \infty \qquad \text{holds.}$$

By H^1 in the half-plane Ω we denote the class of all functions F analytic in Ω for which the integrals

$$\int_{-\infty}^{\infty} |F(x+it)|\,dt$$

are uniformly bounded for $x > 0$ ([1], [2]).

1.3.1. *If the function F belongs to H^1 in Ω, then*

$$\int_{-\infty}^{\infty} F(it)\,dt = 0.$$

Proof. Let F belong to H^1 in Ω. Then by (Theorem 1.9, [12]), the function

$$G(z) = \frac{2}{(1-z)^2} F\left(\frac{1+z}{1-z}\right) \qquad (z \in D),$$

belongs to H^1 in D. We denote

$$h(z) = \left(1 - \left(\frac{1+z}{1-z}\right)^2\right) F\left(\frac{1+z}{1-z}\right) \qquad \text{for } z \in D.$$

Since $|z| \leqslant 1$, so $|h(z)| \leqslant 2|G(z)|$ for $z \in D$. Hence we deduce that h belongs to H^1 in D. From above, in virtue of the fact that $z = (w-1)/(w+1)$ is the homographic transformation of the half-plane Ω into the disc D, in view of the Poisson's integral formula for H^1 in D, we get

$$\frac{1}{\pi}\int_{-\infty}^{\infty} F(it)dt = \frac{1}{\pi}\int_{-\infty}^{\infty} (1-(it)^2)F(it)\cdot\frac{dt}{1+t^2}$$

$$= \frac{1}{2\pi}\int_{0}^{2\pi}\left(1-\left(\frac{1+e^{i\theta}}{1-e^{i\theta}}\right)^2\right)F\left(\frac{1+e^{i\theta}}{1-e^{i\theta}}\right)d\theta$$

$$= \frac{1}{2\pi}\int_{0}^{2\pi} h(e^{i\theta})d\theta = h(0) = 0.$$

2. Since every function $F \in H^{*\psi}$ has the boundary function defined off a set of measure zero, so if each $F \in H^{*\psi}$ is identified with its boundary function, $H^{*\psi}$ can be regarded as a subspace of $L^{*\psi}$. Besides we have $\|F\|_\psi = \|F(i\cdot)\|_\psi$ and $\|F\|_{(\psi)} = \|F(i\cdot)\|_{(\psi)}$. Similarly, $H^{0\psi}$ is a subspace of $L^{0\psi}$.

2.1. *For every function $g \in L^{*\psi^*}$ the formula*

$$(*)\qquad \xi(F) = \int_{-\infty}^{\infty} F(it)g(t)dt \quad \textit{for } F \in H^{0\psi}$$

defines a linear functional, norm continuous on $H^{0\psi}$; besides,

$$\|\xi\|_\psi = \sup\{|\xi(F)|:\ F \in H^{0\psi},\ \|F\|_\psi \leqslant 1\} \leqslant \|g\|_{(\psi^*)}$$

and

$$\|\xi\|_{(\psi)} = \sup\{|\xi(F)|:\ F \in H^{0\psi},\ \|F\|_{(\psi)} \leqslant 1\} \leqslant \|g\|_{\psi^*}.$$

Proof. This is obvious if $g = 0$. Let us suppose that the function $g \in L^{*\psi^*}$ is different from zero. By the Hölder's inequality (see [3], Chapter II), we have

$$\int_{-\infty}^{\infty} |F(it)g(t)|dt \leqslant \|F(i\cdot)\|_\psi \|g\|_{(\psi^*)} = \|F\|_\psi \|g\|_{(\psi^*)}$$

and

$$\int_{-\infty}^{\infty} |F(it)g(t)|dt \leqslant \|F(i\cdot)\|_{(\psi)} \|g\|_{\psi^*} = \|F\|_{(\psi)} \|g\|_{\psi^*}$$

for any function $F \in H^{0\psi}$, and we get

$$|\xi(F)| \leqslant \|F\|_\psi \|g\|_{(\psi^*)} \quad \text{and} \quad |\xi(F)| \leqslant \|F\|_{(\psi)} \|g\|_{\psi^*}.$$

Hence we deduce that the linear functional ξ is norm continuous on $H^{0\psi}$; moreover,

$$\|\xi\|_\psi \leqslant \|g\|_{(\psi^*)} \quad \text{and} \quad \|\xi\|_{(\psi)} \leqslant \|g\|_{\psi^*}.$$

2.2. *For every linear functional ξ, norm continuous on $H^{0\psi}$, there exists a function $g \in L^{*\psi^*}$ such that*

$$(*) \qquad \xi(F) = \int_{-\infty}^{\infty} F(it)g(t)dt \quad \text{for } F \in H^{0\psi};$$

moreover,

$$\|\xi\|_\psi = \|g\|_{(\psi^*)} \quad \text{and} \quad \|\xi\|_{(\psi)} = \|g\|_{\psi^*}.$$

Proof. Since $H^{0\psi}$ can be embedded isometrically in $L^{0\psi}$, then by the Hahn–Banach theorem there exists a linear functional l, norm continuous on $L^{0\psi}$, such that $\xi(F) = l(F(i\,\cdot))$ for $F \in H^{0\psi}$ and $\|l\|_\psi = \|\xi\|_\psi$, $\|l\|_{(\psi)} = \|\xi\|_{(\psi)}$. We know ([3], Chapter II) that for a functional l there exists a function $g \in L^{*\psi^*}$ such that

$$l(f) = \int_{-\infty}^{\infty} f(t)g(t)dt \quad \text{for } f \in L^{0\psi}$$

and

$$\|g\|_{(\psi^*)} = \|l\|_\psi, \quad \|g\|_{\psi^*} = \|l\|_{(\psi)}.$$

There holds formula (*) and equalities for norms.

2.3. Let $L_0^{*\psi^*}$ denote the class of all functions $g \in L^{*\psi^*}$ such that

$$\int_{-\infty}^{\infty} F(it)g(t)dt = 0$$

for any function $F \in H^{0\psi}$.

2.4. $L_0^{*\psi^*} = H^{*\psi^*}$ *(isomorphically).*

Proof. Let $G \in H^{*\psi^*}$. Then $G(i\cdot) \in L^{*\psi^*}$. We take an arbitrary function $F \in H^{0\psi}$. By the Hölder inequality we state that $FG \in H^1(\Omega)$. Hence and from 1.3.1 we deduce that

$$\int_{-\infty}^{\infty} F(it)G(it)dt = 0.$$

This fact proves the inclusion $H^{*\psi^*} \subset L_0^{*\psi^*}$.

Now, we take an arbitrary function $g \in L_0^{*\psi^*}$. Then we have

$$\int_{-\infty}^{\infty} U_n(it)g(t)dt = 0,$$

where U_n $(n = 1, 2, \ldots)$, denote functions from Lemma 1.4 ([13]). We consider the homographic transformation $z = (w-1)/(w+1)$ of the half-plane Ω into the disc D. Since this transformation has on the boundary the form $e^{i\theta} = (it-1)/(it+1)$ therefore $t = \cot\frac{1}{2}\theta$ $(0 < \theta < 2\pi)$ and hence $\theta = 2\cdot\operatorname{arccot} t$.

If $t \to \infty$, then $\theta \to 0$; whereas if $t \to -\infty$, then $\theta \to 2\pi$. We have $(-2dt)/(1+t^2) = d\theta$, also. From above, we obtain

$$0 = \int_{-\infty}^{\infty} U_n(it)g(t)dt = \int_{-\infty}^{\infty} \frac{(it-1)^{n-1}}{(it+1)^{n+1}} g(t)dt$$

$$= \tfrac{1}{2} \int_{-\infty}^{\infty} \frac{(it-1)^n}{(it+1)^{n+1}} \cdot g(t) \cdot \frac{-2dt}{1+t^2} = -\tfrac{1}{2} \int_{0}^{2\pi} e^{in\theta} \tilde{g}(\theta) d\theta,$$

where $\tilde{g}(\theta) = g(\cot\frac{1}{2}\theta)$. The function $\tilde{g}$ is integrable on the interval $[0, 2\pi)$, because for $u \geqslant 1$ we have $u\psi^*(1) \leqslant \psi^*(u)$; and

$$\int_{0}^{2\pi} |\tilde{g}(\theta)| d\theta \leqslant 2\pi + \frac{1}{\psi^*(1)} \int_{0}^{2\pi} \psi^*(|\tilde{g}(\theta)|) d\theta$$

$$\leqslant 2\pi + \frac{2}{\psi^*(1)} \int_{-\infty}^{\infty} \psi^*(|\tilde{g}(t)|) \frac{dt}{1+t^2} \leqslant 2\pi + \frac{2}{\psi^*(1)} \int_{-\infty}^{\infty} \psi^*(|\tilde{g}(t)|) dt.$$

From the above we deduce ([2], Chapter III) that $\tilde{g}$ is a boundary function of the some analytic function $\tilde{G}$ in D and that $\tilde{G}$ can be represented by the Poisson integral of $\tilde{g}$, i.e.,

$$\tilde{G}(z) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1-|z|^2}{|z-e^{i\theta}|^2} \tilde{g}(\theta) d\theta.$$

Since the reciprocal map $w = (1+z)/(1-z)$ of disc D into the half-plane Ω is conformal and

$$\frac{1-|z|^2}{|z-e^{i\theta}|^2} = \frac{x(1+t^2)}{x^2+(y-t)^2},$$

therefore the function G, corresponding to function $\tilde{G}$ in D, is analytic in Ω and

$$G(x+iy) = G(w) = \tilde{G}\left(\frac{w-1}{w+1}\right) = \frac{1}{\pi} \int_{-\infty}^{\infty} g(t) \frac{x}{x^2+(y-t)^2} dt,$$

because $g(t) = \tilde{g}(2 \cdot \operatorname{arc\,cot} t)$. Hence we get

$$\psi^*(|G(x+iy)|) \leqslant \frac{1}{\pi} \int_{-\infty}^{\infty} \psi^*(|g(t)|) \frac{x}{x^2+(y-t)^2} dt.$$

Integrating both sides of this inequality with respect to y on the interval $(-\infty, \infty)$ and changing the order of integration at the right-hand side we obtain for $x > 0$

$$\int_{-\infty}^{\infty} \psi^*(|G(x+iy)|)dy \leqslant \int_{-\infty}^{\infty} \psi^*(|g(t)|)dt.$$

Hence and from the fact that $g \in L^{*\psi^*}$, we deduce that $\varrho_{\psi^*}(G) < \infty$. This means, in virtue of the analyticity of G, that $G \in H^{*\psi^*}$. Therefore the inclusion $L_0^{*\psi^*} \subset H^{*\psi^*}$ holds.

2.5. We denote by $(H^{0\psi})^{\#}$ the space of all norm continuous linear functionals on $H^{0\psi}$ equipped with the norms

$$\|\xi\|_{\psi} = \sup\{|\xi(F)|: \ F \in H^{0\psi}, \ \|F\|_{\psi} \leqslant 1\}, \qquad \xi \in (H^{0\psi})^{\#}$$

or

$$\|\xi\|_{(\psi)} = \sup\{|\xi(F)|: \ F \in H^{0\psi}, \ \|F\|_{(\psi)} \leqslant 1\}, \qquad \xi \in (H^{0\psi})^{\#}.$$

Let us note that norms $\|\cdot\|_{\psi}$ and $\|\cdot\|_{(\psi)}$ satisfy the inequalities

$$\tfrac{1}{2}\|\xi\|_{\psi} \leqslant \|\xi\|_{(\psi)} \leqslant \|\xi\|_{\psi} \qquad \text{for } \xi \in (H^{0\psi})^{\#}.$$

2.6. In the quotient space $L^{*\psi^*}/H^{*\psi^*}$ we identify functions in $H^{*\psi^*}$ with their boundary functions. Let $\tilde{g}$ denote the element of $L^{*\psi^*}/H^{*\psi^*}$ determined by the element $g \in L^{*\psi^*}$, i.e.,

$$\tilde{g} = g + H^{*\psi^*} = \{g + f: \ f \in H^{*\psi^*}\}, \qquad g \in L^{*\psi^*}.$$

Now the quotient space $L^{*\psi^*}/H^{*\psi^*}$ is equipped with two norms

$$\|\tilde{g}\|_{\psi^*} = \inf\{\|g+f\|_{\psi^*}: \ f \in H^{*\psi^*}\}, \qquad g \in L^{*\psi^*},$$

or

$$\|\tilde{g}\|_{(\psi^*)} = \inf\{\|g+f\|_{(\psi^*)}: \ f \in H^{*\psi^*}\}, \qquad g \in L^{*\psi^*}.$$

These norms are equivalent; namely,

$$\|\tilde{g}\|_{\psi^*} \leqslant \|\tilde{g}\|_{(\psi^*)} \leqslant 2\|\tilde{g}\|_{\psi^*} \qquad \text{for } \tilde{g} \in L^{*\psi^*}/H^{*\psi^*}.$$

2.7. *The space* $(H^{0\psi})^{\#}$ *is isometrically isomorphic to the quotient space* $L^{*\psi^*}/H^{*\psi^*}$ (the space $L^{*\psi^*}/H^{*\psi^*}$ is equipped with the norm $\|\cdot\|_{(\psi^*)}$ $[\|\cdot\|_{\psi^*}]$ if the space $(H^{0\psi})^{\#}$ is equipped with the norm $\|\cdot\|_{\psi}$ $[\|\cdot\|_{(\psi)}]$).

This isomorphism establishes formula (*) from 2.1.

Proof. Two functions $g_1, g_2 \in L^{*\psi^*}$ represent (according to formula (*)) the same functional $\xi \in (H^{0\psi})^{\#}$, i.e.,

$$\xi(F) = \int_{-\infty}^{\infty} F(it)g_1(t)dt = \int_{-\infty}^{\infty} F(it)g_2(t)dt \quad \text{for } F \in H^{0\psi}$$

iff the difference $g_1 - g_2 \in L^{*\psi^*}$ satisfies the condition

$$\int_{-\infty}^{\infty} F(it)\big(g_1(t) - g_2(t)\big)dt = 0 \quad \text{for } F \in H^{0\psi};$$

which by 2.4 is equivalent to the property $g_1 - g_2 \in H^{*\psi^*}$. In this case formula (*) establishes the isomorphism of the space $L^{*\psi^*}/H^{*\psi^*}$ with the space $(H^{0\psi})^{\#}$. By 2.1 and 2.2, this isomorphism is an isometry.

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