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On the complex convexity of Orlicz–Musielak sequence spaces(*)

Abstract. Complex extreme point, complex strict convexity and complex uniform convexity of complex Banach space are natural generalizations of extreme point, strict convexity and uniform convexity. In this paper, we give criteria of the complex convexities of Orlicz–Musielak sequence spaces.

Definition 1 [1]. Let C be a convex set in a complex Banach space X. A point u of C is said to be a *complex extreme point of* C if $u+\lambda v \in C$ whenever $|\lambda| \leqslant 1$ and v in X, then $v = 0$.

Definition 2 [1]. Let X be a complex Banach space. We say that X is a *complex strictly convex space* if any point x, $\|x\| = 1$, is a complex extreme point of the closed unit ball $U(X)$ of X.

Definition 3 [2]. A complex Banach space is called *complex uniformly convex* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that x, y in X, $\|x+\lambda y\| \leqslant 1$ ($|\lambda| \leqslant 1$) and $\|y\| \geqslant \varepsilon$ imply $\|x\| \leqslant 1-\delta$.

Let X be a complex Banach space, N be the set of all natural numbers. Let $\Phi = (\varphi_n)$: $X \times N \mapsto [0, +\infty]$ be a sequence of Young functions, i.e., φ_n is convex, $\varphi_n(e^{it}x) = \varphi_n(x)$, $t \in (-\infty, +\infty)$, and $\varphi_n(0) = 0$ for every n in N. Furthermore, for each n in N, the following conditions are assumed:

(a) there exists nonzero $x \in X$ such that $\varphi_n(x) < \infty$;

(b) for each x in X, $\varphi_n(tx)$: $(0, +\infty) \mapsto [0, +\infty]$ is a left-continuous function of t.

For a sequence $x = (x_n)$ of X, define $I_\varphi(x) = \sum_{n=1}^{\infty} \varphi_n(x_n)$ and

$$l_\varphi = \{x = (x_n) \in X:\ I_\varphi(kx) < \infty \text{ for some } k > 0\},$$

$$\|x\|_\varphi = \inf\{k > 0:\ I_\varphi(x/k) \leqslant 1\} \qquad (x \text{ in } l_\varphi),$$

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then $(l_\varphi, \|\cdot\|_\varphi)$, so-called *Orlicz–Musielak sequence space*, is a Banach space.

LEMMA 1. *For every n in N and x, y in X, $y \neq 0$, if*

$$\varphi_n(x+\lambda y) \leqslant M < \infty \qquad (|\lambda| \leqslant 1),$$

then $\varphi_n(x+\lambda y)$ is a continuous function of λ on $\{\lambda: |\lambda| < 1\}$.

Proof. Taking n in N and x, y in X, $y \neq 0$, we suppose that there is $M > 0$ such that for $|\lambda| \leqslant 1$

$$\varphi_n(x+\lambda y) \leqslant M < \infty .$$

(If $M = 0$, the assertion of the lemma is obvious.) For all $|\lambda_0| < 1$ and $0 < \varepsilon < M$, set $\delta = (1-|\lambda_0|)\varepsilon/M$. If $|\lambda-\lambda_0| < \delta$, we have

$$\begin{aligned}
\varphi_n(x+\lambda y) &= \varphi_n(x+\lambda_0 y+\lambda y-\lambda_0 y) \\
&= \varphi_n\left[x+\lambda_0 y+\frac{|\lambda-\lambda_0|}{1-|\lambda_0|}\cdot\frac{1-|\lambda_0|}{|\lambda-\lambda_0|}(\lambda-\lambda_0)\, y\right] \\
&\leqslant \left(1-\frac{|\lambda-\lambda_0|}{1-|\lambda_0|}\right)\varphi_n(x+\lambda_0 y) \\
&\qquad\qquad +\frac{|\lambda-\lambda_0|}{1-|\lambda_0|}\varphi_n\left[x+\lambda_0 y+\frac{1-|\lambda_0|}{|\lambda-\lambda_0|}(\lambda-\lambda_0)\, y\right] \\
&\leqslant \varphi_n(x+\lambda_0 y)+\varepsilon .
\end{aligned}$$

From the convexity of φ_n,

$$\begin{aligned}
2\varphi_n(x+\lambda_0 y) &\leqslant \varphi_n[x+\lambda_0 y+(\lambda-\lambda_0) y]+\varphi_n[x+\lambda_0 y-(\lambda-\lambda_0) y] \\
&\leqslant \varphi_n(x+\lambda y)+\varphi_n(x+\lambda_0 y)+\varepsilon ,
\end{aligned}$$

since

$$\varphi_n(x+\lambda_0 y) \leqslant \varphi_n(x+\lambda y)+\varepsilon$$

and so

$$|\varphi_n(x+\lambda y)-\varphi_n(x+\lambda_0 y)| \leqslant \varepsilon .$$

The lemma is proved.

LEMMA 2. *For all n in N, if there are x_n, y_n in X, $y_n \neq 0$, such that for each $|\lambda| \leqslant 1$*

$$(*) \qquad 2\varphi_n(x_n) = \varphi_n(x_n+\lambda y_n)+\varphi_n(x_n-\lambda y_n) < M < \infty ,$$

then there is a t' in $[0, \pi)$ such that

$$\varphi_n(x_n) = \varphi_n(x_n+\tfrac{1}{2}e^{t'i}\, y_n);$$

If there is still a t'' in $[0, \pi)$, $t'' \neq t'$, satisfying

$$\varphi_n(x_n) = \varphi_n(x_n + \tfrac{1}{2} e^{t''i} y_n),$$

then $\varphi_n(x) = \varphi_n(x_n)$ *for* x *in* $\{x_n + \lambda y_n \colon |\lambda| \leqslant \frac{1}{2}\}$.

Proof. If $\varphi_n(x_n) = \varphi_n(x_n + \frac{1}{2} y_n)$ then we may take $t' = 0$. Now we suppose that

$$\varphi_n(x_n) < \varphi_n(x_n + \tfrac{1}{2} y_n).$$

From (*),

$$\varphi_n(x_n) > \varphi_n(x_n - \tfrac{1}{2} y_n).$$

Using Lemma 1, we get that $\varphi_n(x_n + \frac{1}{2} e^{it} y_n)$ is a continuous function of t on $[0, \pi]$. There is, therefore, a t' in $(0, \pi)$ such that

$$\varphi_n(x_n) = \varphi_n(x_n + \tfrac{1}{2} e^{it'} y_n).$$

If there is also t'' in $[0, \pi)$, $t'' \neq t'$, satisfying

$$\varphi_n(x_n) = \varphi_n(x_n + \tfrac{1}{2} e^{it''} y_n),$$

then take λ such that $x_n + \lambda y_n$ is on the line segment connecting $x_n + \frac{1}{2} e^{it'} y_n$ and $x_n + \frac{1}{2} e^{it''} y_n$ and $x_n - \lambda y_n$ is on that connecting $x_n - \frac{1}{2} e^{it'} y_n$ and $x_n - \frac{1}{2} e^{it''} y_n$. From the convexity of φ_n and (*), we have

$$\varphi_n(x_n + \lambda y_n) = \varphi_n(x_n - \lambda y_n) = \varphi_n(x_n).$$

Analogously, we can get that $\varphi_n(x) = \varphi_n(x_n)$ on the line segments connecting any two points among $x_n + \frac{1}{2} e^{it'} y_n$, $x_n + \frac{1}{2} e^{it''} y_n$, $x_n - \frac{1}{2} e^{it'} y_n$ and $x_n - \frac{1}{2} e^{it''} y_n$, which implies that $\varphi_n(x) = \varphi_n(x_n)$ on the quadrilateral B with vertices at the four points. For all $|\lambda| = \frac{1}{2}$, we may suppose without loss of the generality that

$$\varphi_n(x_n) < \varphi_n(x_n + \lambda y_n), \qquad \varphi_n(x_n) > \varphi_n(x_n - \lambda y_n)$$

if $\varphi_n(x_n) \neq \varphi_n(x_n + \lambda y_n)$. Consider the convex function $g(\alpha)$ and the linear function $f(\alpha)$ on $[-1, 1]$,

$$g(\alpha) = \varphi_n(x_n + \alpha \lambda y_n), \quad f(\alpha) = \tfrac{1}{2} \alpha [\varphi_n(x_n + \lambda y_n) - \varphi_n(x_n - \lambda y_n)] + \varphi_n(x_n).$$

From (*) and the convexity of g, we have that $f(1) = g(1)$, $f(-1) = g(-1)$ and $f(\alpha) \geqslant g(\alpha)$, α in $[-1, 1]$. Obviously, we can find $\alpha_0 < 0$ such that $x_n + \alpha_0 \lambda y_n$ is in B, i.e.,

$$g(\alpha_0) = \varphi_n(x_n + \alpha_0 \lambda y_n) = \varphi_n(x_n) > f(\alpha_0).$$

This contradiction implies that for $|\lambda| = \frac{1}{2}$

$$\varphi_n(x_n + \lambda y_n) = \varphi_n(x_n - \lambda y_n) = \varphi_n(x_n).$$

Using the convexity of φ_n and (*) again, we can obtain that

$$\varphi_n(x) = \varphi_n(x_n) \quad \text{for all } x \text{ in } \{x_n+\lambda y_n\colon |\lambda| \leqslant \tfrac{1}{2}\}.$$

THEOREM 1. *An element $x \in U(l_\varphi)$ is a complex extreme point of $U(l_\varphi)$ if and only if*

(i) *$I_\varphi(x) = 1$ or x_n is a complex extreme point of the set $\{w\colon \varphi_n(w) \leqslant 1\}$ $(n = 1, 2, \ldots)$;*

(ii) *for all n in N, $y_n \neq 0$, φ_n is not constant on $\{x_n+\lambda y_n\colon |\lambda| \leqslant 1\}$;*

(iii) *there is at most one number n in N such that x_n is not a complex strictly convex point of φ_n, i.e., if there exists a $v \in X$, $v \neq 0$, such that for $|\lambda| \leqslant 1$*

$$2\varphi_n(x_n) = \varphi_n(x_n+\lambda v)+\varphi_n(x_n-\lambda v).$$

Proof. *Sufficiency*. Let $I_\varphi(x) = 1$. For y in X, if $\|x+\lambda y\|_\varphi \leqslant 1$, $|\lambda| \leqslant 1$, then

$$2 = 2I_\varphi(x) \leqslant I_\varphi(x+\lambda y)+I_\varphi(x-\lambda y)$$
$$= \sum_{n=1}^{\infty} \varphi_n(x_n+\lambda y_n)+\sum_{n=1}^{\infty} \varphi_n(x_n-\lambda y_n) \leqslant 2.$$

From the convexity of φ_n,

$$2\varphi_n(x_n) \leqslant \varphi_n(x_n+\lambda y_n)+\varphi_n(x_n-\lambda y_n)$$

and thus

$$2\varphi_n(x_n) = \varphi_n(x_n+\lambda y_n)+\varphi_n(x_n-\lambda y_n).$$

From (iii), there is at most one m in N such that $y_m \neq 0$. From (ii), there exists λ_m such that $|\lambda_m| \leqslant 1$ and

$$\varphi_m(x_m+\lambda_m y_m) > \varphi_m(x_m),$$

which implies that

$$\sum_{n=1}^{\infty} \varphi_n(x_n+\lambda_m y_n) > \sum_{n=1}^{\infty} \varphi_n(x_n) = 1.$$

Hence

$$\|x+\lambda_m y\|_\varphi > 1,$$

which obviously contradicts our supposition. Therefore, $y = 0$, and x is a complex extreme point of $U(l_\varphi)$.

If $I_\varphi(x) \neq 1$, then from (i) x_n is a complex extreme point of $\{w\colon \varphi_n(w) \leqslant 1\}$ (n in N), i.e., for all v in X, if for $|\lambda| \leqslant 1$, $\varphi_n(x_n+\lambda v) \leqslant 1$, then

$v = 0$. Let $y \in l_\varphi$, $\|x+\lambda y\|_\varphi \leqslant 1$, $|\lambda| \leqslant 1$. Then

$$\sum_{n=1}^{\infty} \varphi_n(x_n+\lambda y_n) = I_\varphi(x+\lambda y) \leqslant \|x+\lambda y\|_\varphi \leqslant 1,$$

and thus for all n in N we must have $y_n = 0$, i.e., $y = 0$, and x is a complex extreme point of $U(l_\varphi)$.

Necessity. If (i) is not true, then $I_\varphi(x) < 1$ and there exist m in N and y'_m in X, $y'_m \neq 0$, such that for $|\lambda| \leqslant 1$

$$\varphi_m(x_m+\lambda y'_m) \leqslant 1.$$

From Lemma 1, there exists $0 < k < 1$ such that

$$\varphi_m(x_m+\lambda y'_m) \leqslant \varphi_m(x_m)+1-I_\varphi(x)$$

provided that $|\lambda| \leqslant k$. Set $y = \{y_n\}_{n=1}^{\infty}$, $y_n = 0$, $n \neq m$; $y_m = ky'_m$. Therefore for $|\lambda| \leqslant 1$ we have

$$I_\varphi(x+\lambda y) \leqslant I_\varphi(x)+1-I_\varphi(x) = 1,$$

and from [3], $\|x+\lambda y\|_\varphi \leqslant 1$, which implies that x is not a complex extreme point of $U(l_\varphi)$. This contradiction proves that (i) is true.

If (ii) is false then there exist m and $y_m \neq 0$ such that for $|\lambda| \leqslant 1$,

$$\varphi_m(x_m+\lambda y_m) = \varphi_m(x_m).$$

Take $y = \{y_n\}_{n=1}^{\infty}$, $y_n = 0$, $n \neq m$, then for $|\lambda| \leqslant 1$

$$I_\varphi(x+\lambda y) = I_\varphi(x) \leqslant 1.$$

We can get a contradiction as above.

Suppose that (iii) is false and there are m, k in N, $m \neq k$ and $y'_m \neq 0$, $y'_k \neq 0$, such that for $|\lambda| \leqslant 1$

$$2\varphi_m(x_m) = \varphi_m(x_m+\lambda y'_m)+\varphi_m(x_m-\lambda y'_m),$$

$$2\varphi_k(x_k) = \varphi_k(x_k+\lambda y'_k)+\varphi_k(x_k-\lambda y'_k).$$

Since (ii) is true, from Lemma 2, there exist $t_m, t_k \in [0, \pi)$ such that

$$\varphi_m(x_m+\tfrac{1}{2}e^{it_m}y'_m) = \varphi_m(x_m), \qquad \varphi_k(x_k+\tfrac{1}{2}e^{it_k}y'_k) = \varphi_k(x_k).$$

Set $y''_m = \frac{1}{2}e^{it_m}y'_m$, $y''_k = \frac{1}{2}e^{it_k}y'_k$, then

(a) $$\varphi_m(x_m+y''_m) = \varphi_m(x_m),$$

(b) $$\varphi_k(x_k+y''_k) = \varphi_k(x_k)$$

and for $|\lambda| \leqslant 1$

$$2\varphi_m(x_m) = \varphi_m(x_m+2\lambda y''_m)+\varphi_m(x_m-2\lambda y''_m),$$

$$2\varphi_k(x_k) = \varphi_k(x_k+2\lambda y''_k)+\varphi_k(x_k-2\lambda y''_k).$$

Consider the function $\varphi_m(x_m+e^{it}y''_m)$ of t on $[0, \pi)$, if there are $t', t''\in[0, \pi)$, such that

$$\varphi_m(x_m+e^{it'}y''_m) > \varphi_m(x_m), \qquad \varphi_m(x_m+e^{it''}y''_m) < \varphi_m(x_m).$$

From Lemma 1, there is a $t_0\in(t', t'')$ satisfying

$$\varphi_m(x_m+e^{it_0}y''_m) = \varphi_m(x_m).$$

Thus we get $t_0 = 0$ according to (ii) and Lemma 2. This contradiction gives that the function $\varphi_m(x_m+e^{it}y''_m)-\varphi_m(x_m)$ does not change the sign for $t\in(0, \pi)$. In this manner we can obtain that the function $\varphi_k(x_k+e^{it}y''_k)-\varphi_k(x_k)$ does not change the sign on $(0, \pi)$. Without loss of generality we suppose that for $t\in[0, \pi]$

$$\varphi_m(x_m+e^{it}y''_m) \geqslant \varphi_m(x_m), \qquad \varphi_k(x_k+e^{it}y''_k) \geqslant \varphi_k(x_k)$$

(otherwise we replace y''_m by y'''_m, $y'''_m = -y''_m$, or y''_k by y'''_k, $y'''_k = -y''_k$). Then clearly

$$\varphi_m(x_m-e^{it}y''_m) \leqslant \varphi_m(x_m), \qquad \varphi_k(x_k-e^{it}y''_k) \leqslant \varphi_k(x_k).$$

We also suppose that

$$\varphi_m(x_m+e^{\pi i/2}y''_m)+\varphi_k(x_k-e^{\pi i/2}y''_k) \geqslant \varphi_m(x_m-e^{\pi i/2}y''_m)+\varphi_k(x_k+e^{\pi i/2}y''_k).$$

From Lemma 1, there exists $0<\alpha<1$ such that

$$\varphi_m(x_m+\alpha iy''_m)+\varphi_k(x_k-iy''_k) = \varphi_m(x_m-\alpha iy''_m)+\varphi_k(x_k+iy''_k)$$

and thus

$$\begin{aligned}\varphi_m(x_m+\alpha iy''_m)&+\varphi_k(x_k-iy''_k)\\ &= \varphi_m(x_m)+\tfrac{1}{2}[\varphi_m(x_m+\alpha iy''_m)-\varphi_m(x_m-\alpha iy''_m)]+\varphi_k(x_k-iy''_k)\\ &= \varphi_m(x_m)+\tfrac{1}{2}[\varphi_k(x_k+iy''_k)-\varphi_k(x_k-iy''_k)]+\varphi_k(x-iy''_k)\\ &= \varphi_m(x_m)+\varphi_k(x_k) = \varphi_m(x_m-\alpha iy''_m)+\varphi_k(x_k+iy''_k).\end{aligned}$$

Set $y = \{y_n\}_{n=1}^{\infty}$, $y_n = 0$, $n\neq m, k$; $y_m = \alpha y''_m$, $y_k = -y''_k$, then

$$I_\varphi(x+y) = I_\varphi(x-y) = I_\varphi(x+iy) = I_\varphi(x-iy) = I_\varphi(x) \leqslant 1,$$

by (a), (b) and the convexity of φ_m. Thus from [3]

$$\|x\pm y\|_\varphi \leqslant 1, \qquad \|x\pm iy\|_\varphi \leqslant 1$$

and x is not a complex extreme point of $U(l_\varphi)$. The contradiction proves that (iii) is true.

DEFINITION 4 [3]. We say that $\Phi = (\varphi_n)$ *satisfies condition* Δ if there exist $\lambda > 1$, $a > 0$, $K > 1$ and a nonnegative convergent series $\sum_{n=1}^{\infty} c_n$ such that for all large n, we have $\varphi_n(\lambda u) \leqslant K\varphi_n(u) + c_n$ for all u in X satisfying $\varphi_n(u) \leqslant a$.

THEOREM 2. *l_φ is a complex strictly convex space if and only if*

(1) $\sup\{k:\ \varphi_n(ku) < \infty\} > 1$ *for all nonzero u in X with $\varphi_n(u) < 1$ and all n in N,*

(2) *Φ satisfies condition Δ,*

(3) *$\varphi_n(u)$ is not constant on $\{v + \lambda w:\ |\lambda| \leqslant 1\}$ for any v, w in X with $\varphi_n(v) \leqslant 1$, $w \neq 0$ and for all n in N,*

(4) *for any m, k in N, $m \neq k$, and each (u, v) in*

$$\{(x, y):\ \varphi_m(x) + \varphi_k(y) \leqslant 1,\ x, y \in X\},$$

u is a complex strictly convex point of φ_m or v is a complex strictly convex point of φ_k.

Proof. *Necessity.* If (3) is false then there are m in N and x_m, y_m in X with $\varphi_m(x_m) \leqslant 1$, $y_m \neq 0$, such that for $|\lambda| \leqslant 1$

$$\varphi_m(x_m + \lambda y_m) = \varphi_m(x_m).$$

Take k in N, $k \neq m$. Since there is x_k' in X such that $\varphi_k(x_k') < \infty$, we set

$$K_0 = \sup\{K \geqslant 0:\ \varphi_k(Kx_k') + \varphi_m(x_m) \leqslant 1\}.$$

Let $x = \{x_n\}_{n=1}^{\infty}$, $x_n = 0$, $n \neq m, k$, $x_k = K_0 x_k'$; then it is not difficult to obtain $\|x\|_\varphi = 1$. From Theorem 1, x is not a complex extreme point of $U(l_\varphi)$. This contradicts the complex strict convexity of l_φ and thus (3) holds.

Suppose that (1) is not true; then there are m in N and x_m in X with $\varphi_m(x_m) < 1$ such that, for any $\alpha > 1$, $\varphi_m(\alpha x_m) = \infty$. Write $x = \{x_n\}_{n=1}^{\infty}$, $x_n = 0$, $n \neq m$. It is clear that $\|x\|_\varphi = 1$. Take k in N, $k \neq m$. Since (3) is true, there is y_k in X, $y_k \neq 0$, such that $\varphi_k(y_k) < 1$. Thus 0 is not a complex extreme point of $U(l_\varphi)$, which contradicts the complex strict convexity of l_φ, too.

If (2) is false then, from [3], there are $x = \{x_n\}_{n=1}^{\infty}$ and $M > 1$ such that $\sum_{n=1}^{\infty} \varphi_n(x_n) < 1$, and $\|(0, \ldots, 0, x_M, x_{M+1}, \ldots)\|_\varphi = 1$. Since $M > 1$, we can find an x in l_φ, $\|x\|_\varphi = 1$, which is not a complex extreme point of $U(l_\varphi)$. This is proved in the same way as (1). So we get a contradiction.

If (4) is not true then there are m, k in N, $m \neq k$, and x_m, x_k in X, such that

$$\varphi_m(x_m) + \varphi_k(x_k) \leqslant 1$$

and x_m, x_k are not the complex strictly convex points of φ_m, φ_k, respectively, i.e., there are nonzero y_m, y_k in X such that for $|\lambda| \leqslant 1$

$$2\varphi_m(x_m) = \varphi_m(x_m+\lambda y_m)+\varphi_m(x_m-\lambda y_m),$$

$$2\varphi_k(x_k) = \varphi_k(x_k+\lambda y_k)+\varphi_k(x_k-\lambda y_k).$$

Take j in N, $j \neq m$, k, and x_j' in X with $x_j' \neq 0$, $\varphi_j(x_j') < \infty$. Let

$$K_0 = \sup\{K > 0\colon \varphi_m(x_m)+\varphi_k(x_k)+\varphi_j(Kx_j') \leqslant 1\}.$$

Define $x = \{x_n\}_{n=1}^{\infty}$, $x_n = 0$, $n \neq m$, k, j; $x_j = K_0 x_j'$ and thus $\|x\|_\varphi = 1$. According to Theorem 1, x is not a complex extreme point of $U(l_\varphi)$. The contradiction implies that (4) is true.

Sufficiency. If $\|x\|_\varphi = 1$, then from (1), (2) and [3] we can get $I_\varphi(x) = 1$. Since $\varphi_n(x_n) \leqslant 1$ for n in N, from (3) φ_n is not constant on $\{x_n+\lambda y\colon |\lambda| \leqslant 1\}$ for any y in X with $y \neq 0$. Let x_m be not a complex strictly convex point of φ_m, then since for $j \neq m$

$$\varphi_m(x_m)+\varphi_j(x_j) \leqslant 1,$$

x_j must be a complex strictly convex point of φ_j from (4). Therefore, from Theorem 1, x is a complex extreme point of $U(l_\varphi)$ provided that $\|x\|_\varphi = 1$. Thus we obtain that l_φ is a complex strictly convex space.

THEOREM 3. *l_φ is a complex uniformly convex space if and only if*

(α) *for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x\|_\varphi > \varepsilon$ implies $I_\varphi(x) > \delta$;*

(β) *for every $\varepsilon > 0$, there exists $\delta > 0$ such that $I_\varphi(x) \leqslant 1-\varepsilon$ implies $\|x\|_\varphi \leqslant 1-\delta$;*

(γ) *for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any x, y in l_φ with*

$$1-\delta \leqslant I_\varphi(x+\lambda y) \leqslant 1 \quad (\lambda = 1,\ -1,\ i,\ -i)$$

and $I_\varphi(y) \geqslant \varepsilon$ there exists a subset N' of N which satisfies that $\Sigma_{N'}\varphi_n(y_n) \geqslant \delta$ and for all n in N'

$$4\varphi_n(x_n) \leqslant (1-\delta)\{\varphi_n(x_n+y_n)+\varphi_n(x_n-y_n)+\varphi_n(x_n+iy_n)+\varphi_n(x_n-iy_n)\}.$$

Proof. *Sufficiency.* For all $\varepsilon > 0$, if there are x, y in l_φ with $\|y\|_\varphi \geqslant \varepsilon$ and $\|x+\lambda y\|_\varphi \leqslant 1$ $(\lambda = 1, -1, i, -i)$, then $I_\varphi(y) \geqslant \varepsilon' > 0$ and $I_\varphi(x+\lambda y) \leqslant 1$ $(\lambda = 1, -1, i, -i)$ from (α) and [3]. For the ε' we can find $\delta > 0$ satisfying (γ). Supposing

$$1-\delta \leqslant I_\varphi(x+\lambda y) \leqslant 1 \quad (\lambda = 1,\ -1,\ i,\ -i),$$

from (γ) there exists $N' \subset N$ such that $\Sigma_{N'}\varphi_n(y_n) \geqslant \delta$ and

$$4\varphi_n(x_n) \leqslant (1-\delta)\{\varphi_n(x_n+y_n)+\varphi_n(x_n-y_n)+\varphi_n(x_n+iy_n)+\varphi_n(x_n-iy_n)\}$$

for n in N'. According to the convexity of φ_n we have

$$1-I_\varphi(x)$$
$$\geqslant \Sigma_{N'}\{\tfrac{1}{4}[\varphi_n(x_n+y_n)+\varphi_n(x_n-y_n)+\varphi_n(x_n+iy_n)+\varphi_n(x_n-iy_n)]-\varphi_n(x_n)\}$$
$$\geqslant \tfrac{1}{4}\delta\Sigma_{N'}\{\varphi_n(x_n+y_n)+\varphi_n(x_n-y_n)+\varphi_n(x_n+iy_n)+\varphi_n(x_n-iy_n)\}$$
$$=\tfrac{1}{4}\delta\Sigma_{N'}\{\varphi_n(y_n+x_n)+\varphi_n[-(y_n-x_n)]+\varphi_n[i(y_n-ix_n)]+\varphi_n[-i(y_n+ix_n)]\}$$
$$=\tfrac{1}{4}\delta\Sigma_{N'}\{\varphi_n(y_n+x_n)+\varphi_n(y_n-x_n)+\varphi_n(y_n+ix_n)+\varphi_n(y_n-ix_n)\}$$
$$\geqslant \delta\Sigma_{N'}\varphi_n(y_n)\geqslant \delta^2,$$

which implies $I_\varphi(x)\leqslant 1-\delta^2$. If there exists $\lambda\in\{1,-1,i,-i\}$ such that $I_\varphi(x+\lambda y)\leqslant 1-\delta$, then

$$I_\varphi(x)\leqslant \tfrac{1}{4}\{I_\varphi(x+y)+I_\varphi(x-y)+I_\varphi(x+iy)+I_\varphi(x-iy)\}\leqslant 1-\tfrac{1}{4}\delta.$$

Take $\delta'=\min\{\delta^2,\tfrac{1}{4}\delta\}$, then $I_\varphi(x)\leqslant 1-\delta'$. From (β), there exists $\delta''>0$ such that $\|x\|_\varphi\leqslant 1-\delta''$ and thus from Definition 3 we obtain that l_φ is a complex uniformly convex space.

Necessity. We suppose that (α) is false and there exist $0<\varepsilon_0<1$ and $\{x^m\}_{m=1}^\infty\subset l_\varphi$ with $\|x^m\|_\varphi>\varepsilon_0$ and $I_\varphi(x^m)<1/m$ for all m in N. Since l_φ is complex strictly convex (see [2]), Φ satisfies condition Δ from Theorem 2. Therefore, there are $K>1$, $\lambda>1$, $a>0$, $M_0>1$, $p>1$ and a nonnegative convergent series $\sum_{n=1}^{\infty} c_n$ such that

$$\varphi_n(2u/\varepsilon_0)\leqslant\varphi_n(\lambda^p u)\leqslant K^p\varphi_n(u)+pc_n$$

for all $n>M_0$ and u in X when $\varphi_n(u)<a/\lambda^{p-1}$. Take M_1 in N, which satisfies that $\sum_{n=M_0}^{\infty} pc_n<\tfrac{1}{2}$, and define $M_2=\max\{M_0,M_1\}$ then

$$\sum_{n=M_2}^{\infty}\varphi_n(2x_n^m/\varepsilon_0)\leqslant K^p\sum_{n=M_2}^{\infty}\varphi_n(x_n^m)+\tfrac{1}{2}<1$$

when $m>\max\{\lambda^{p-1}/a,\,1/2K^p\}$. Set $[x^m]_{M_2}=(x_1^m,x_2^m,\ldots,x_{M_2-1}^m,0,\ldots)$, then $\|x^m-[x^m]_{M_2}\|_\varphi\leqslant\tfrac{1}{2}\varepsilon_0$, which implies that

$$\|[x^m]_{M_2}\|_\varphi=\|x^m-(x^m-[x^m]_{M_2})\|_\varphi\geqslant\|x^m\|_\varphi-\|x^m-[x^m]_{M_2}\|_\varphi$$
$$>\varepsilon_0-\tfrac{1}{2}\varepsilon_0=\tfrac{1}{2}\varepsilon_0.$$

From the properties of φ_{M_2} there exists z_{M_2} in X, $z_{M_2}\neq 0$, such that $\varphi_{M_2}(z_{M_2})<\infty$. For every m in N, take

$$K^m=\sup\{K>0\colon \sum_{n=1}^{M_2-1}\varphi_n(x_n^m)+\varphi_{M_2}(K^p z_{M_2})\leqslant 1\}.$$

Set $z^m = \{z_n^m\}_{n=1}^\infty \in l_\varphi$, $z_n^m = 0$, $n \neq M_2$, $z_{M_2}^m = K^m z_{M_2}$; $y^m = \{y_n^m\}_{n=1}^\infty$, $[y^m]_{M_2} = [x^m]_{M_2}$, $y_n^m = 0$, $n > M_2$. Obviously, $\|y^m\|_\varphi \geqslant \frac{1}{2}\varepsilon_0$ and $\|z^m + \lambda y^m\|_\varphi = 1$ ($\lambda = 1, -1, i, -i$). Since l_φ is complex strictly convex, from Theorem 2 we have that $I_\varphi(z^m + \lambda y^m) = 1$ for m in N and $\lambda = 1, -1, i, -i$. Since $I_\varphi(y^m) \to 0$, we get that $I_\varphi(z^m) \to 1$ and, further, $\|z^m\|_\varphi \to 1$, which contradicts the complex uniform convexity of l_φ.

If (β) is not true then there are $\varepsilon_0 > 0$ and $\{x^m\}_{m=1}^\infty \subset l_\varphi$ with $I_\varphi(x^m) < 1 - \varepsilon_0$ and $\|x^m\|_\varphi \geqslant 1 - 1/m$ for m in N. Since (α) is true, for every m in N there exists n_m in N such that $\varphi_{n_m}(mx_{n_m}) < 1$, that is to say, if we set $x' = \{x_n'\}_{n=1}^\infty$, $x_n' = 0$, $n \neq n_m$, $x_{n_m}' = x_{n_m}$, then $\|x'\|_\varphi \leqslant 1/m$. We define $z^m = \{z_n^m\}_{n=1}^\infty \in l_\varphi$, $z_n^m = x_n^m$, $n \neq n_m$, $z_n^m = 0$, $n = n_m$ and take $y_{n_m}^{m'}$ in X such that $y_{n_m}^{m'} \neq 0$ and $\varphi_n(y_{n_m}^{m'}) < \infty$. Let

$$K^{n_m} = \sup\{K \geqslant 0:\ I_\varphi(z^m) + \varphi_{n_m}(Ky_{n_m}^{m'}) \leqslant 1\}$$

and $y^m = \{y_n^m\}_{n=1}^\infty \in l_\varphi$, $y_n^m = 0$, $n \neq n_m$, $y_{n_m}^m = K^{n_m} y_{n_m}^{m'}$, then $\|y^m + z^m\|_\varphi = 1$. According to Theorem 2, we have

$$I_\varphi(y^m + z^m) = I_\varphi(y^m) + I_\varphi(z^m) = 1$$

and

$$\|y^m\|_\varphi \geqslant I_\varphi(y^m) = 1 - I_\varphi(z^m) \geqslant \varepsilon_0, \qquad \|z^m + y^m\|_\varphi = 1 \qquad (|\lambda| \leqslant 1),$$

for all m in N. But

$$\|z^m\|_\varphi \geqslant \|x^m\|_\varphi - 1/m \geqslant 1 - 2/m \to 1 \qquad (m \to \infty),$$

l_φ is not a complex uniformly convex space, by the definition. The contradiction gives that (β) is true.

Now we suppose that (γ) is false and there are $\varepsilon_0 > 0$ and $\{x_n^m\}_{n=1}^\infty$, $\{y_n^m\}_{n=1}^\infty \subset l_\varphi$ with $I_\varphi(y^m) \geqslant \varepsilon_0$, $1 - 1/m \leqslant I_\varphi(x^m + \lambda y^m) \leqslant 1$ ($\lambda = 1, -1, i, -i$) for all m in N such that for any set N' of natural numbers $\Sigma_{N'}\varphi_n(y_n^m) > 1/m$, or there exists n in N' such that

$$4\varphi_n(x_n^m) \geqslant (1 - 1/m)\{\varphi_n(x_n^m + y_n^m) + \varphi_n(x_n^m - y_n^m) + \varphi_n(x_n^m + iy_n^m) + \varphi_n(x_n^m - iy_n^m)\}.$$

Take M_0 in N, $1/M_0 > \varepsilon_0$. Then for any $m > M_0$ we have $I_\varphi(y^m) > 1/m$ and thus there exists n_{m_1} in N which satisfies the following inequality

$$4\varphi_{n_{m_1}}(x_{n_{m_1}}^m) \geqslant (1 - 1/m)\{\varphi_{n_{m_1}}(x_{n_{m_1}}^m + y_{n_{m_1}}^m) + \varphi_{n_{m_1}}(x_{n_{m_1}}^m - y_{n_{m_1}}^m) + \varphi_{n_{m_1}}(x_{n_{m_1}}^m + iy_{n_{m_1}}^m) + \varphi_{n_{m_1}}(x_{n_{m_1}}^m - iy_{n_{m_1}}^m)\}.$$

Considering $\sum_{n \neq n_{m_1}} \varphi_n(y_n^m)$, if $\sum_{n \neq n_{m_1}} \varphi_n(y_n^m) \leqslant 1/m$, we set $N_m' = N - \{n_{m_1}\}$

if $\sum_{n \neq n_m} \varphi_n(y_n^m) > 1/m$, then there exists $n_{m_2} \neq n_{m_1}$ such that

$$4\varphi_{n_{m_2}}(x_{n_{m_2}}^m) \geqslant (1-1/m)\{\varphi_{n_{m_2}}(x_{n_{m_2}}^m + y_{n_{m_2}}^m) + \varphi_{n_{m_2}}(x_{n_{m_2}}^m - y_{n_{m_2}}^m) + \varphi_{n_{m_2}}(x_{n_{m_2}}^m + iy_{n_{m_2}}^m) + \varphi_{n_{m_2}}(x_{n_{m_2}}^m - iy_{n_{m_2}}^m)\}$$

and we go on with this process. Since $I_\varphi(y^m) < \infty$, we can find a set N'_m of the natural numbers such that $\Sigma_{N'_m} \varphi_n(y_n) \leqslant 1/m$ and

$$4\varphi_n(x_n^m) \geqslant (1-1/m)\{\varphi_n(x_n^m + y_n^m) + \varphi_n(x_n^m - y_n^m) + \varphi_n(x_n^m + iy_n^m) + \varphi_n(x_n^m - iy_n^m)\}$$

for $n \notin N'_m$. From the complex uniform convexity of l_φ, there exists $\delta > 0$ such that for all m in N

$$I_\varphi(x^m) \leqslant \|x^m\|_\varphi \leqslant 1-\delta,$$

which implies that

$$\tfrac{1}{4}\{I_\varphi(x^m + y^m) + I_\varphi(x^m - y^m) + I_\varphi(x^m + iy^m) + I_\varphi(x^m - iy^m)\} - I_\varphi(x^m) \geqslant 1 - 1/m - (1-\delta) = \delta - 1/m.$$

Since

$$\sum_{n \notin N'_m} \{\tfrac{1}{4}[\varphi_n(x_n^m + y_n^m) + \varphi_n(x_n^m - y_n^m) + \varphi_n(x_n^m + iy_n^m) + \varphi_n(x_n^m - iy_n^m)] - \varphi_n(x_n^m)\}$$
$$\leqslant 1/4m \sum_{n \notin N'_m} [\varphi_n(x_n^m + y_n^m) + \varphi_n(x_n^m - y_n^m) + \varphi_n(x_n^m + iy_n^m) + \varphi_n(x_n^m - iy_n^m)] \leqslant 1/m,$$

we have

$$\sum_{n \in N'_m} \{\tfrac{1}{4}[\varphi_n(x_n^m + y_n^m) + \varphi_n(x_n^m - y_n^m) + \varphi_n(x_n^m + iy_n^m) + \varphi_n(x_n^m - iy_n^m)] - \varphi_n(x_n^m)\} \geqslant \delta - 1/m - 1/m = \delta - 2/m > \delta/2$$

when $m > 4/\delta$. Define $\{\bar{x}_n^m\}_{n=1}^\infty \in l_\varphi$, $\bar{x}_n^m = x_n^m$, $n \in N'_m$, $\bar{x}_n^m = 0$, $n \notin N'_m$, $m \geqslant \max\{M_0, 4/\delta\}$; $\{\bar{y}_n^m\}_{n=1}^\infty \in l_\varphi$, $\bar{y}_n^m = y_n^m$, $n \in N'_m$, $\bar{y}_n^m = 0$, $n \notin N'_m$. Then for $|\lambda| \leqslant 1$

$$\left\| \bar{x}^m + \frac{\delta}{\|\bar{y}^m\|_\varphi} \lambda \bar{y}^m \right\|_\varphi \leqslant \|\bar{x}^m\|_\varphi + \delta \leqslant 1 - \delta + \delta = 1$$

and thus

$$I_\varphi\left(\bar{x}^m + \frac{\delta}{\|\bar{y}^m\|_\varphi} \lambda \bar{y}^m\right) \leqslant 1.$$

Consider the convex function

$$g_m(\alpha) = I_\varphi(\bar{x}^m + \alpha\lambda\bar{y}^m)$$

and the linear function

$$f_m(\alpha) = [I_\varphi(\bar{x}^m+\bar{y}^m) - I_\varphi(\bar{x}^m)]\,\alpha - I_\varphi(x^m)$$

on $[0, \infty)$. Note $g_m(0) = f_m(0)$ and $g_m(1) = f_m(1)$. From the convexity of g_m we have that $g_m(\alpha) \geqslant f_m(\alpha)$ for $\alpha > 1$. In virtue of $\|\bar{y}^m\|_\varphi \to 0$, we have also $\delta/\|\bar{y}^m\|_\varphi > 1$ for all large m and thus

$$\begin{aligned} 1 &\geqslant \tfrac{1}{4}\Big\{I_\varphi\Big(\bar{x}^m+\frac{\delta\bar{y}^m}{\|\bar{y}^m\|_\varphi}\Big)+I_\varphi\Big(\bar{x}^m-\frac{\delta\bar{y}^m}{\|\bar{y}^m\|_\varphi}\Big)+I_\varphi\Big(\bar{x}^m+i\frac{\delta\bar{y}^m}{\|\bar{y}^m\|_\varphi}\Big) \\ &\qquad +I_\varphi\Big(\bar{x}^m-i\frac{\delta\bar{y}^m}{\|\bar{y}^m\|_\varphi}\Big)\Big\} \\ &\geqslant \tfrac{1}{4}\{I_\varphi(\bar{x}^m+\bar{y}^m)+I_\varphi(\bar{x}^m-\bar{y}^m)+I_\varphi(\bar{x}^m+i\bar{y}^m)+I_\varphi(\bar{x}^m-i\bar{y}^m) \\ &\qquad -4I_\varphi(\bar{x}^m)\}\frac{\delta}{\|\bar{y}^m\|_\varphi}+I_\varphi(\bar{x}^m) \\ &\geqslant \frac{\delta^2}{2\|\bar{y}^m\|_\varphi}+I_\varphi(\bar{x}^m). \end{aligned}$$

But $\delta^2/2\|y^m\|_\varphi \to \infty$, and this contradiction proves that (γ) is true.

J. E. Jamison, Irene Loomis and C. C. Rousseau discuss the complex strict convexity of Orlicz spaces in [4]. The criterion of rotundity of Orlicz–Musielak sequence spaces was given first by A. Kamińska in [5].

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