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On some semilinear integro-differential equation of parabolic type

1. Preliminaries. In applied mathematics there appear the so-called loaded equations of various types (see e.g. [9] and the references therein). Integro-differential equations constitute an important class of loaded equations. The following integro-differential equation

$$(1.1)\qquad \sum_{i=1}^{n} u_{x_i x_i}(x, t) = u_t(x, t) + K \int_D u_t(y, t)\,dy, \qquad x = (x_1, \ldots, x_n)$$

appears in thermoelasticity (see [3]–[7]), where D is a domain of the Euclidean space $\boldsymbol{R}^n$ and K is a real constant. Therefore, it is advisable to investigate equations involving (1.1).

In this paper, we consider the first Fourier problem for a semilinear parabolic integro-differential equation (involving (1.1)) in a Banach space. Using the results of papers [13], [14], we establish some existence and uniqueness theorems for the above problem. The employment of a Banach space instead of the Euclidean space $\boldsymbol{R}$ is justified, because it enables us to obtain various classes of equations (see Section 7).

In order to formulate the problem in question, we first introduce some notation. Let G be a bounded domain of the Euclidean space $\boldsymbol{R}^{n+1}$ of the variables $(x, t) = (x_1, \ldots, x_n, t)$ whose boundary consists of sets $E_0 \times \{0\}$ and $E_T \times \{T\}$ (E_0 and E_T being bounded domains of $\boldsymbol{R}^n$), and of a surface S included in the strip $\boldsymbol{R}^n \times [0, T]$, where T is a positive constant. We put

$$E_t = \{x\colon (x, t) \in G\}, \qquad 0 < t < T,\ \Gamma = S \cup (E_0 \times \{0\}),$$

$$S_t = \{x\colon (x, t) \in S\}, \qquad 0 \leqslant t \leqslant T.$$

Let B be a real Banach space with a norm $\|\cdot\|_B$. The limit, continuity and partial derivatives of functions of real variables with values in B are understood in the strong sense. Integrals of these functions are taken in the Bochner sense. We shall use the Banach spaces $C(\bar{G}, B)$ and $C^{(k+\alpha)}(\bar{G}, B)$ with the norms $\|\cdot\|_{B,\bar{G}}$ and $\|\cdot\|_{B,\bar{G}}^{(k+\alpha)}$, respectively, introduced in [13] (p. 441), where $k = 0, 1, 2$, $\alpha \in (0, 1)$ is a constant, and $\bar{G}$ denotes the closure of G. Moreover, we introduce the Banach spaces

$$C^{(1,0)}(\bar{G}, B) = \{u \in C(\bar{G}, B)\colon\ u_{x_i} \in C(\bar{G}, B),\ i = 1, \ldots, n\},$$
$$C^{(2,1)}(\bar{G}, B) = \{u \in C^{(1,0)}(\bar{G}, B)\colon\ u_{x_i x_i},\ u_t \in C(\bar{G}, B),\ i, j = 1, \ldots, n\}$$

with the norms

$$\|u\|_{B,\bar{G}}^{(1,0)} = \|u\|_{B,\bar{G}} + \sum_{i=1}^{n} \|u_{x_i}\|_{B,\bar{G}},$$

$$\|u\|_{B,\bar{G}}^{(2,1)} = \|u\|_{B,\bar{G}}^{(1,0)} + \sum_{i,j=1}^{n} \|u_{x_i x_j}\|_{B,\bar{G}} + \|u_t\|_{B,\bar{G}},$$

respectively. All the above functional Banach spaces and norms will be particularly used in the case $B = \boldsymbol{R}$. In this case we shall omit the symbol $\boldsymbol{R}$.

We shall consider the problem

$$(1.2)\qquad (Lu)(x, t) + K(L_0 u)(t) = f\big(x, t, u(x, t), u_x(x, t), (L_0 u)(t)\big), \quad (x, t) \in \bar{G} \setminus \Gamma,$$

$$(1.3)\qquad u(x, t) = \varphi(x, t), \quad (x, t) \in \Gamma,$$

where $u_x = (u_{x_1}, \ldots, u_{x_n})$, K is a real constant,

$$(1.4)\qquad (Lu)(x, t) = \sum_{i,j=1}^{n} a_{ij}(x, t) u_{x_i x_j}(x, t) - u_t(x, t)$$

and

$$(1.5)\qquad (L_0 u)(t) = \int_{E_t} g(y, t) u_t(y, t)\, dy.$$

In the above problem, the functions u, f and φ take values in B, whereas L is a parabolic operator with real-valued coefficients and g is a real-valued function as well. We shall investigate the existence and uniqueness of a solution of problem (1.2), (1.3) in the space $C^{(2,1)}(\bar{G}, B)$.

2. Elimination of the integral from equation (1.2). In this section we eliminate the function $L_0 u$ from equation (1.2). Consequently, we obtain a new form of equation (1.2) which is more convenient for our investigation.

We introduce the following assumptions.

(2.I) The surface S is of class $\bar{C}^{(2+\alpha)} \cap C^{(2-0)}$ (see [11], p. 838), where $\alpha \in (0, 1)$ is a constant.

(2.II) The coefficients a_{ij} $(a_{ij}=a_{ji})$ belong to $C^{(\alpha)}(\bar{G})$. Moreover, $a_{ij}\in C^{(1-0)}(S)$ (see [11], p. 838) and there exist derivatives $a_{ijx_j}(x,t)$ continuous in $\bar{G}$ and satisfying in $\bar{G}$ the uniform Hölder condition of exponent $\alpha/2$ in t.

(2.III) The operator L is uniformly parabolic in $\bar{G}$, i.e.

$$\sum_{i,j=1}^{n} a_{ij}(x,t)r_i r_j \geqslant A_0|r|^2, \qquad (x,t)\in\bar{G},\ r=(r_1,\dots,r_n)\in\boldsymbol{R}^n,$$

A_0 being a positive constant.

(2.IV) The function g belongs to $C^{(1,0)}(\bar{G})$ and $g(x,t)$, $g_{x_i}(x,t)$ $(i=1,\dots,n)$ satisfy in $\bar{G}$ the uniform Hölder condition of exponent $\alpha/2$ in t. Moreover, if $K\neq 0$, then

$$\int_{E_t} g(y,t)\,dy \neq K^{-1}, \qquad t\in[0,T]\ (^1).$$

(2.V) The function $f\colon G\times B^{n+2}\to B$ is continuous (in the strong sense) and

$$\|f(x,t,u,p,q_1)-f(x,t,u,p,q_2)\|_B \leqslant A_1\|q_1-q_2\|_B$$

for any $(x,t)\in\bar{G}$, $u, q_1, q_2\in B$, $p\in B^n$, where A_1 is a positive constant less than

$$A_2=\left[\sup\left\{|g_0(t)|\int_{E_t}|g(y,t)|\,dy\colon t\in[0,T]\right\}\right]^{-1}$$

and

$$g_0(t)=\left[1-K\int_{E_t} g(y,t)\,dy\right]^{-1}. \tag{2.1}$$

Now let us denote by $z(y,t)=\big(z_1(y,t),\dots,z_n(y,t)\big)$ the unit exterior normal vector at $y\in S_t$ and introduce the following notation:

$$(L_1u)(t)=g_0(t)\sum_{i,j=1}^{n}\Big[\int_{S_t} a_{ij}(y,t)u_{y_i}(y,t)z_j(y,t)g(y,t)\,dy - \int_{E_t} u_{y_i}(y,t)\frac{\partial}{\partial y_j}\big(a_{ij}(y,t)g(y,t)\big)\,dy\Big], \tag{2.2}$$

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(1) It follows from assumptions (2.IV) and (2.I) that $g\in C^{(\alpha)}(\bar{G})$.

$$(L_2(u, v))(t) = (L_1 u)(t) + \int_{E_t} g_1(y, t)(F_1(u, v))(y, t)\,dy, \tag{2.3}$$

$$(F_1(u, v))(x, t) = f(x, t, u(x, t), u_x(x, t), v(t)), \tag{2.4}$$

$$g_1(x, t) = -g_0(t) g(x, t). \tag{2.5}$$

We shall use the Banach space $C([0, T], B)$ consisting of all continuous functions v: $[0, T] \to B$ and provided with norm

$$\|v\|_{B,[0,T]} = \sup\{\|v(t)\|_B\colon\ t \in [0, T]\}.$$

Now let us consider the equation

$$v = L_2(u, v). \tag{2.6}$$

Assumptions (2.I), (2.II), (2.IV), (2.V), relations (2.1)–(2.5) and Lemma 6.1 (Section 6) imply that

$$L_2(u, \cdot)\colon\ C([0, T], B) \to C([0, T], B)$$

for any $u \in C^{(1,0)}(\bar{G}, B)$ and

$$\|L_2(u, v_1) - L_2(u, v_2)\|_{B,[0,T]} \leqslant A_3 \|v_1 - v_2\|_{B,[0,T]}$$

for any $u \in C^{(1,0)}(\bar{G}, B)$ and $v_1, v_2 \in C([0, T], B)$, where

$$A_3 = A_1 A_2^{-1} < 1. \tag{2.7}$$

Hence, by the Banach fixed point theorem, for any $u \in C^{(1,0)}(\bar{G}, B)$ there exists a unique solution $v \in C([0, T], B)$ of equation (2.6). This enables us to define an operator

$$L_3\colon\ C^{(1,0)}(\bar{G}, B) \to C([0, T], B)$$

setting $L_3 u = v$.

THEOREM 2.1. *Let assumptions* (2.I)–(2.V) *be satisfied. Then a function* $u \in C^{(2,1)}(\bar{G}, B)$ *is a solution of equation* (1.2) *if and only if it is a solution of the equation*

$$(Lu)(x, t) = (F_2 u)(x, t), \qquad (x, t) \in \bar{G} \setminus \Gamma, \tag{2.8}$$

where

$$(F_2 u)(x, t) = -K(L_3 u)(t) + (F_1(u, L_3 u))(x, t). \tag{2.9}$$

Proof. Let $u \in C^{(2,1)}(\bar{G}, B)$ be a solution of equation (1.2). Then, multiplying (1.2) by $g(x, t)$ and integrating with respect to x over E_t we obtain, by (1.4), (1.5), (2.1)–(2.5), the equality

$$L_0 u = L_2(u, L_0 u).$$

Hence it follows from the definition of L_3 that $L_0 u = L_3 u$. Consequently, (1.2), (2.4) and (2.9) imply that u satisfies (2.8).

Now let $u \in C^{(2,1)}(\bar{G}, B)$ be a solution of equation (2.8). Multiplying (2.8) by $g(x, t)$ and integrating with respect to x over E_t, we obtain by (1.4), (1.5), (2.2) and (2.9) the equality

$$[g_0(t)]^{-1}(L_1 u)(t) - (L_0 u)(t)$$
$$= -K(L_3 u)(t) \int_{E_t} g(x, t)\,dx + \int_{E_t} g(x, t)\big(F_1(u, L_3 u)\big)(x, t)\,dx.$$

Hence, using (2.3), (2.5) and the equality

$$K \int_{E_t} g(x, t)\,dx = 1 - [g_0(t)]^{-1},$$

we have

$$[g_0(t)]^{-1}\big(L_2(u, L_3 u)\big)(t) - (L_0 u)(t) = [g_0(t)]^{-1}(L_3 u)(t) - (L_3 u)(t).$$

In virtue of $L_3 u = L_2(u, L_3 u)$ the last equality yields $L_3 u = L_0 u$. Consequently, (2.8), (2.9) and (2.4) imply that u satisfies (1.2). This completes the proof.

3. Existence and uniqueness of a solution of problem (1.2), (1.3). We use the notation and assumptions of Sections 1 and 2. Moreover, we need the following assumptions.

(3.I) There are constants $A_4, A_5 > 0$ such that

$$\|f(x, t, u, p, q)\|_B \leqslant A_4 + A_5(\|u\|_B + \|p\|_B) + A_1\|q\|_B$$

for any $(x, t) \in \bar{G}$, $u, q \in B$ and $p = (p_1, \ldots, p_n) \in B^n$, where

$$\|p\|_B = \sum_{i=1}^{n} \|p_i\|_B$$

and A_1 is the constant introduced in assumption (2.V).

(3.II) There is a constant $A_6 > 0$ and for any $b > e$ (e being the Euler's number) there is a constant $A_7 = A_7(b) > 0$ such that

$$\|f(P, u, p, q) - f(P', u', p', q)\|_B$$
$$\leqslant A_7[d(P, P')]^{\alpha} + A_6(\ln b)^r[\|u' - u\|_B + \|p' - p\|_B]$$

for any $P = (x, t)$, $P' = (x', t') \in \bar{G}$, $q \in B$ and $u, u' \in B$, $p, p' \in B^n$ such that

$$\|u\|_B, \|u'\|_B, \|p\|_B, \|p'\|_B \leqslant b,$$

where $r \in (0, (1-\alpha)(3+\alpha)^{-1})$ is a constant and

$$d(P, P') = (|x - x'|^2 + |t - t'|)^{1/2}.$$

(3.III) For the function $\varphi\colon \Gamma \to B$ there exists an extension

$$\Phi \in C^{(1+\beta)}(\bar{G}, B) \cap C^{(2+\alpha)}(\bar{G}, B),$$

where $\beta \in [\alpha, 1)$ is a constant. For each such a function Φ we have

(3.1) $$(L\varphi)(x, 0) = (F_2 \Phi)(x, 0), \quad x \in \partial E_0 \ (^2),$$

where ∂E_0 is the boundary of E_0.

THEOREM 3.1. *If assumptions* (2.I)–(2.V), (3.I)–(3.III) *are satisfied, then problem* (1.2), (1.3) *has a unique solution u in the set*

$$C^{(1+\alpha)}(\bar{G}, B) \cap C^{(2,1)}(\bar{G}, B).$$

Moreover,

$$u \in C^{(1+\beta)}(\bar{G}, B) \cap C^{(2+\alpha)}(\bar{G}, B).$$

Proof. We use the Bielecki's norms introduced in [13] (p. 455, 456) and the norms

$$\|v\|_{B,[0,T],a} = \sup\{e^{-at}\|v(t)\|_B \colon 0 \leqslant t \leqslant T\}$$

for $v \in C([0, T], B)$ and

$$\|v\|_{B,[0,T],a}^{(\gamma)} = \|v\|_{B,[0,T],a}$$
$$+\sup\{\exp[-a\max(t, t')]\|v(t)-v(t')\|_B |t-t'|^{-\gamma},\ t, t' \in [0, T]\}$$

for $v \in C^{(\gamma)}([0, T], B)$. Taking into consideration relations (2.9), (2.4), assumptions (2.V), (3.I), (3.II), and Lemma 6.3, one can prove the following assertions:

(a) $F_2\colon C^{(1,0)}(\bar{G}, B) \to C(\bar{G}, B)$, $F_2\colon C^{(1+\alpha)}(\bar{G}, B) \to C^{(\alpha)}(\bar{G}, B)$;
(b) there are constants $N_1, N_2 > 0$ such that

$$\|F_2 u\|_{B,\bar{G},a} \leqslant N_1 + N_2 \|u\|_{B,\bar{G},a}^{(1,0)}$$

for any $a \geqslant 0$, $u \in C^{(1,0)}(\bar{G}, B)$;
(c) for any $b > e$ there is a constant $N_3(b) > 0$ such that

$$\|F_2 u_1 - F_2 u_2\|_{B,\bar{G},a} \leqslant N_3(b)\, a^r \|u_1 - u_2\|_{B,\bar{G},a}^{(1,0)}, \quad a \geqslant 0$$

for any $u_1, u_2 \in C^{(1,0)}(\bar{G}, B)$ such that

$$\|u_i\|_{B,\bar{G},a}^{(1,0)} \leqslant b, \quad i = 1, 2;$$

(d) for any $b > e$ there is a constant $N_4(b) > 0$ such that

$$\|F_2 u\|_{B,\bar{G}}^{(\alpha)} \leqslant N_4(b)[1 + \|u\|_{B,\bar{G}}^{(1+\alpha)}]$$

for any $u \in C^{(1+\alpha)}(\bar{G}, B)$ such that $\|u\|_{B,\bar{G}}^{(1+\alpha)} \leqslant b$.

(^{2}) Concerning $(L\varphi)(x, 0)$, we use Remark 1.1 of [11] with L_p replaced by B. Since (2.9), (2.4) and Lemma 6.3 (iv) imply that $(F_2\Phi)(x, 0)$ is independent of Φ (Φ being any extension of φ), equality (3.1) is correct.

It follows from the above assertions that assumptions of Theorem 3.1 imply assumptions of Theorem 4.1 of [13] in relation to problem (2.8), (1.3). Therefore, the assertion of Theorem 2.1 holds true for problem (2.8), (1.3). According to Theorem 2.1 this completes the proof.

4. Problem (1.2), (1.3) **in a linear case.** We consider problem (1.2), (1.3) in the case

$$(4.1)\qquad f(x, t, u, p, q) = \sum_{i=1}^{n} b_i(x, t) p_i + c(x, t) u + c_0(x, t) q + f_0(x, t).$$

We retain assumptions (2.I)–(2.IV), whereas assumptions (2.V), (3.I), (3.II) result from the following one:

(4.I) $b_i,\ c,\ c_0 \in C^{(\alpha)}(\bar{G})$, $f_0 \in C^{(\alpha)}(\bar{G}, B)$ and $|c_0(x, t)| \leqslant A_1$, $(x, t) \in \bar{G}$, where A_1 is the constant introduced in assumption (2.V).

Making use of (2.I)–(2.V), (4.1), (4.I), one can find that for any $u \in C^{(1,0)}(\bar{G}, B)$ the unique solution of equation (2.6) is given by the formula

$$(4.2)\qquad (L_3 u)(t) = g_2(t) \Big\{(L_1 u)(t) + \int_{\check{E}_t} g_1(y, t) \Big[f_0(y, t) + \sum_{i=1}^{n} b_i(y, t) u_{y_i}(y, t) + c(y, t) u(y, t)\Big] dy\Big\},$$

where

$$g_2(t) = \Big[1 - \int_{E_t} g_1(y, t) c_0(y, t)\, dy\Big]^{-1}.$$

THEOREM 4.1. *If assumptions* (2.I)–(2.IV), (4.I) *and* (3.III) *are satisfied, then the assertion of Theorem* 3.1 *is true in case* (4.1). *Moreover, there are constants* $a_0 \geqslant 1$, $A_8, A_9 > 0$ *such that*

$$(4.3)\qquad \|u\|_{B,\bar{G},a}^{(1+\beta)} \leqslant A_8\, a^{-\bar{r}} \big[\|f_0\|_{B,\bar{G},a} + \|\varphi\|_{B,\bar{G},a}^{(2,1)}\big] + 2\|\varphi\|_{B,\bar{G},a}^{(1+\beta)}, \qquad a \geqslant a_0,$$

$$(4.4)\qquad \|u\|_{B,\bar{G}}^{(2+\alpha)} \leqslant A_9 \big[\|f_0\|_{B,\bar{G}}^{(\alpha)} + \|\varphi\|_{B,\bar{G}}^{(2+\alpha)}\big],$$

where $\bar{r} = (1-\beta)(3+\beta)^{-1}$.

Proof. The first assertion of the above theorem is an immediate consequence of Theorem 3.1. To prove estimates (4.3), (4.4) notice that operator F_2 defined by (2.9) has now the following form

$$(4.5)\qquad (F_2 u)(x, t) = [c_0(x, t) - 1](L_3 u)(t) + \sum_{i=1}^{n} b_i(x, t) u_{x_i}(x, t) + c(x, t) u(x, t) + f_0(x, t),$$

where L_3 and L_1 are defined by (4.2) and (2.2), respectively. According to Theorem 2.1, the function u is a solution of problem (2.9), (1.3) and consequently, by Theorem 3.2 of [13], there are constants $a_1 \geqslant 1$, $N_5 > 0$ such that

$$(4.6)\qquad \|u\|^{(1+\beta)}_{B,\bar{G},a} \leqslant N_5\, a^{-\bar{\gamma}} [\|F_2 u\|_{B,\bar{G},a} + \|\varphi\|^{(2,1)}_{B,\bar{G},a}] + \|\varphi\|^{(1+\beta)}_{B,\bar{G},a}, \qquad a \geqslant a_1.$$

Relations (4.5), (4.2), (2.2) and Lemma 6.1 imply that

$$(4.7)\qquad \|F_2 u\|_{B,G,a} \leqslant N_6 [\|u\|^{(1,0)}_{B,\bar{G},a} + \|f_0\|_{B,\bar{G},a}], \qquad a \geqslant 0,$$

$N_6 > 0$ being a constant. It follows from (4.6), (4.7) that there exist constants $a_0 \geqslant a_1$, $A_8 > 0$ such that (4.3) holds true. Using (4.3) with $\beta = \alpha$ and $a = a_0$ we obtain

$$(4.8)\qquad \|u\|^{(1+\alpha)}_{B,\bar{G}} \leqslant N_7 [\|f_0\|_{B,\bar{G}} + \|\varphi\|^{(2,1)}_{B,\bar{G}} + \|\varphi\|^{(1+\alpha)}_{B,\bar{G}}],$$

$N_7 > 0$ being a constant. Relations (4.5), (4.2), (2.2), (4.8) and Lemma 6.2 yield the inequality

$$\|F_2 u\|^{(\alpha)}_{B,\bar{G}} \leqslant N_8 [\|f_0\|_{B,\bar{G}} + \|\varphi\|^{(2,1)}_{B,\bar{G}} + \|\varphi\|^{(1+\alpha)}_{B,\bar{G}}],$$

$N_8 > 0$ being a constant. Hence, by Theorem 2.3 of [12] applied to problem (2.8), (1.3), we obtain (4.4). This completes the proof.

It follows from the above proof that each of the constants a_0, A_8, A_9 is independent of f_0 and φ. In the scalar case (i.e. $B = \boldsymbol{R}$), the estimate (4.4) is the same as that one for solution of the first Fourier problem for the linear parabolic equation (see Theorem 3.6 of [8]).

5. An application of measures of noncompactness. In this section we prove an existence theorem for problem (1.2), (1.3) with the aid of Theorem 2.1 of [14]. For this purpose we use the Hausdorff measures of noncompactness μ, $M^{(a)}$, $M^{(a)}_0$ and $M^{(a)}_{1,0}$ in the Banach spaces

$$B,\ C(\bar{G}, B),\ C([0, T], B),\ C^{(1,0)}(\bar{G}, B)$$

with respect to the norms

$$\|\cdot\|_B,\ \|\cdot\|_{B,\bar{G},a},\ \|\cdot\|_{B,[0,T],a},\ \|\cdot\|^{(1,0)}_{B,\bar{G},a},$$

respectively(3). We recall the definition of μ. For any bounded set $B_0 \subset B$, we define $\mu(B_0)$ as the greatest lower bound of all numbers $s > 0$ such that B_0 can be covered by a finite number of balls of radius s. The remaining measures of noncompactness are defined likewise.

We retain assumptions (2.I)–(2.V), (3.I) and (3.III), whereas instead of (3.II) we introduce the following ones.

(3) Concerning measures of noncompactness see, for instance, monograph [1].

(5.I) For any $b > 0$, there is a constant $A_{10}(b) > 0$ such that

$$\|f(P, u, p, q) - f(P', u', p', q)\|_B \leqslant A_{10}(b)\{[d(P, P')]^\gamma + [\|u-u'\|_B + \|p-p'\|_B]^{\gamma/\alpha}\}$$

for any $P, P' \in \bar{G}$ and $(u, p, q), (u', p', q) \in B^{n+2}$ such that

$$\|u\|_B,\ \|u'\|_B,\ \|p\|_B,\ \|p'\|_B \leqslant b,$$

where $\gamma \in (0, \alpha)$ is a constant.

(5.II) There is a constant $A_{11} > 0$ such that for any $(x, t) \in \bar{G}$, $q \in B$ and any bounded sets $U \subset B$, $P = P_1 \times \ldots \times P_n \subset B^n$ we have

$$\mu(f(x, t, U, P, q)) \leqslant A_{11}[\mu(U) + \sum_{i=1}^{n} \mu(P_i)],$$

where $f(x, t, U, P, q) = \{f(x, t, u, p, q) : u \in U, p \in P\}$.

Note that assumptions (2.V) and (5.II) imply in the standard manner the following condition.

(5.III) For any $(x, t) \in \bar{G}$ and any bounded sets

$$U, Q \subset B, \qquad P = P_1 \times \ldots \times P_n \subset B^n$$

we have

$$\mu(f(x, t, U, P, Q)) \leqslant A_{11}[\mu(U) + \sum_{i=1}^{n} \mu(P_i)] + A_1\mu(Q).$$

THEOREM 5.1. *If assumptions* (2.I)–(2.V), (3.I), (3.III), (5.I) *and* (5.II) *are satisfied, then there exists a solution*

$$u \in C^{(1+\beta)}(\bar{G}, B) \cap C^{(2+\gamma)}(\bar{G}, B)$$

of problem (1.2), (1.3).

Proof. Taking into consideration relations (2.9), (2.4), assumptions (2.V), (3.I), (5.I), condition (5.III), and Lemma 6.4, one can obtain the following assertions for F_2:

(α) $F_2 : C^{(1,0)}(\bar{G}, B) \to C(\bar{G}, B)$, $F_2 : C^{(1+\alpha)}(\bar{G}, B) \to C^{(\gamma)}(\bar{G}, B)$ and assertion (b) from the proof of Theorem 3.1;

(β) for any $b > 0$ there is a constant $N_9(b) > 0$ such that

$$\|F_2 u_1 - F_2 u_2\|_{B,\bar{G}} \leqslant N_9(b)[\|u_1 - u_2\|_{B,\bar{G}}^{(1,0)}]^{\gamma/\alpha}$$

for any $u_1, u_2 \in C^{(1,0)}(\bar{G}, B)$ such that $\|u_i\|_{B,\bar{G}}^{(1,0)} \leqslant b$, $i = 1, 2$;

(γ) for any $b > 0$ there is a constant $N_{10}(b) > 0$ such that

$$\|F_2 u\|_{B,\bar{G}}^{(\gamma)} \leqslant N_{10}(b)[1 + \|u\|_{B,\bar{G}}^{(1+\alpha)}]$$

for any $u \in C^{(1+\alpha)}(\bar{G}, B)$ such that $\|u\|_{B,\bar{G}}^{(1+\alpha)} \leqslant b$;

(δ) there is a constant $N_{11} > 0$ such that for any bounded set $U \subset C^{(1+\alpha)}(\bar{G}, B)$ we have

$$M^{(a)}(F_2 U) \leqslant N_{11} M^{(a)}_{1,0}(U).$$

It follows from the above assertions that assumptions of Theorem 5.1 imply assumptions of Theorem 2.1 of [14] in relation to problem (2.8), (1.3). Therefore, the assertion of Theorem 5.1 holds true for problem (2.8), (1.3). According to Theorem 2.1 this completes the proof.

6. Lemmas. In this section we state and prove lemmas which have been used in the previous sections of this paper. We begin with two lemmas concerning the functions

$$v_1(t) = \int_{E_t} u(y, t)\, p_1(y, t)\, dy, \qquad v_2(t) = \int_{S_t} u(y, t)\, p_2(y, t)\, dy.$$

LEMMA 6.1. *Let $S \in C^{(1)}$, $u \in C(\bar{G}, B)$ and $p_1, p_2 \in C(\bar{G})$. Then*

$$v_i \in C([0, T], B), \qquad \|v_i\|_{B,[0,T],a} \leqslant A_{12} \|u\|_{B,\bar{G},a} \|p_1\|_{\bar{G}}, \qquad i = 1, 2$$

for any $a \in R$, where $A_{12} > 0$ is a constant.

LEMMA 6.2. *We assume that $S \in C^{(1)}$, $u \in C^{(\alpha)}(\bar{G}, B)$, $p_1, p_2 \in C(\bar{G})$ and $p_i(x, t)$ satisfy the uniform Hölder condition of exponent $\alpha/2$ in t. Then*

$$v_i \in C^{(\alpha/2)}([0, T], B), \qquad \|v_i\|^{(\alpha/2)}_{B,[0,T],a} \leqslant A_{13} \|u\|^{(\alpha)}_{B,\bar{G},a}, \qquad i = 1, 2$$

for any $a \in R$, where $A_{13} > 0$ is a constant depending only on S, p_1 and p_2.

Proofs. The assertions of the lemmas concerning the function v_1 can be proved in the standard manner with the aid of the formula

$$\begin{aligned} v_1(t) - v_1(s) = & \int_{E_t \setminus E_s} u(y, t)\, p_1(y, t)\, dy \\ & + \int_{E_t \cap E_s} [u(y, t)\, p_1(y, t) - u(y, s)\, p_1(y, s)]\, dy \\ & + \int_{E_s \setminus E_t} u(y, s)\, p_1(y, s)\, dy \end{aligned}$$

and Lemma 3 of [10] (4).

To prove the assertions of the lemmas concerning the function v_2 we divide the surface S into a finite number k of parts

$$S^i = \{(y, t) \in S:\ (i-1)\delta \leqslant t \leqslant i\delta\}, \qquad \delta = T/k,\ i = 1, \ldots, k.$$

---

(4) Note that Lemma 3 of [10] holds true under assumption $S \in C^{(1)}$.

Under sufficiently large k each surface S^i can be divided into a finite number j_i of surfaces S^{ij} $(j = 1, \ldots, j_i)$ represented by equations of the form

$$y_{r_{ij}} = h_{ij}(\bar{y}_{r_{ij}}, t), \quad \bar{y}_{r_{ij}} \in D_{ij}, \quad t \in [(i-1)\delta, i\delta],$$

where

$$h_{ij} \in C^{(1+\alpha)}(D_{ij} \times [(i-1)\delta, i\delta]).$$

For any $t \in [(i-1)\delta, i\delta]$ we have

$$v_2(t) = \sum_{j=1}^{j_i} \int_{S_t^{ij}} u(y, t) p_2(y, t) dy = \sum_{j=1}^{j_i} \int_{D_{ij}} w_{ij}(\bar{y}_{r_{ij}}, t) d\bar{y}_{r_{ij}},$$

where $S_t^{ij} = \{x: (x, t) \in S^{ij}\}$. Hence it follows that

$$v_2(t) - v_2(s) = \sum_{j=1}^{j_i} \int_{D_{ij}} [w_{ij}(\bar{y}_{r_{ij}}, t) - w_{ij}(\bar{y}_{r_{ij}}, s)] d\bar{y}_{r_{ij}}$$

for any $t, s \in [(i-1)\delta, i\delta]$. The further proceeding is obvious. Thus the proofs of Lemmas 6.1 and 6.2 are completed.

LEMMA 6.3. *Let assumptions* (2.II), (2.IV), (2.V), (3.I) *and* (3.II) *be satisfied and suppose* $S \in C^{(1)}$. *Then the following assertions hold.*

(i) L_3: $C^{(1,0)}(\bar{G}, B) \to C([0, T], B)$ *and for any* $u \in C^{(1,0)}(\bar{G}, B)$ *and* $a \geqslant 0$ *we have*

$$\|L_3 u\|_{B,[0,T],a} \leqslant A_{14} + A_{15} \|u\|_{B,\bar{G},a}^{(1,0)},$$

where A_{14}, A_{15} *are positive constants and* L_3 *is the operator defined in Section* 2.

(ii) *For any* $b > e$ *there is a constant* $A_{16}(b) > 0$ *such that*

$$\|L_3 u_1 - L_3 u_2\|_{B,[0,T],a} \leqslant A_{16}(b) a^r \|u_1 - u_2\|_{B,\bar{G},a}^{(1,0)}, \quad a \geqslant 1$$

for any $u_1, u_2 \in C^{(1,0)}(\bar{G}, B)$ *such that* $\|u_i\|_{B,\bar{G},a}^{(1,0)} \leqslant b$, $i = 1, 2$, *where* r *is the constant introduced in assumption* (3.II).

(iii) L_3: $C^{(1+\alpha)}(\bar{G}, B) \to C^{(\alpha/2)}([0, T], B)$ *and for any* $b > e$ *there is a constant* $A_{17}(b) > 0$ such that

$$\|L_3 u\|_{B,[0,T]}^{(\alpha/2)} \leqslant A_{17}(b) [1 + \|u\|_{B,\bar{G}}^{(1+\alpha)}]$$

for any $u \in C^{(1+\alpha)}(\bar{G}, B)$ *such that* $\|u\|_{B,\bar{G}}^{(1+\alpha)} \leqslant b$.

(iv) *If* $u_1, u_2 \in C^{(1,0)}(\bar{G}, B)$ *and* $u_1 = u_2$ *on* Γ, *then*

$$(L_3 u_1)(0) = (L_3 u_2)(0).$$

Proof. The first part of (i) follows from the definition of L_3, Section 2.

Using (2.1)–(2.5), (3.I) and Lemma 6.1, we find that

$$\|L_2(u, L_3 u)\|_{B,[0,T],a} \leqslant N_{12} + N_{13}\|u\|_{B,\bar{G},a}^{(1,0)} + A_3\|L_3 u\|_{B,[0,T],a},$$

where N_{12}, $N_{13} > 0$ are constants independent of a. Hence, in view of (2.7) and

$$L_3 u = L_2(u, L_3 u), \tag{6.1}$$

we obtain the inequality in (i).

To prove (ii) note that

$$\|u_i(x, t)\|_B,\ \|u_{ix_j}(x, t)\|_B \leqslant be^{aT}, \quad i = 1, 2,\ (x, t)\in\bar{G}.$$

Hence, taking into account (2.1)–(2.5), (2.7), (2.V), (3.II), Lemma 6.1 with $u = u_i$, we obtain (ii).

In order to prove (iii) take any $b > e$ and $u \in C^{(1+\alpha)}(\bar{G}, B)$ such that $\|u\|_{B,\bar{G}}^{(1+\alpha)} \leqslant b$. It follows from (2.3) and (6.1) that

$$v(t) = (L_1 u)(t) + \int_{E_t} g_1(y, t)(F_1(u, v))(y, t)\,dy, \tag{6.2}$$

where $v = L_3 u$. Now, as in the proofs of Lemmas 6.1 and 6.2, we use the formula

$$\begin{aligned} v(t) - v(s) = {} & [(L_1 u)(t) - (L_1 u)(s)] + \int_{E_t\setminus E_s} g_1(y, t)(F_1(u, v))(y, t)\,dy + \\ & + \int_{E_t\cap E_s} [g_1(y, t)(F_1(u, v))(y, t) \\ & \qquad - g_1(y, s)(F_1(u, v))(y, s)]\,dy + \\ & + \int_{E_s\setminus E_t} [-g_1(y, s)](F_1(u, v))(y, s)\,dy = I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{6.3}$$

Lemma 6.2 yields the estimate

$$\|I_1\|_B \leqslant N_{14}|t-s|^{\alpha/2}\|u\|_{B,\bar{G}}^{(1+\alpha)}, \tag{6.4}$$

$N_{14} > 0$ being a constant. Taking into account Lemma 3 of [10], assumption (3.I) and assertion (i), we find that

$$\|I_j\|_B \leqslant |t-s|\,[N_{15} + N_{16}\|u\|_{B,\bar{G}}^{(1,0)}], \quad j = 2, 4, \tag{6.5}$$

N_{15} and N_{16} being positive constants. To estimate I_3 we use the formula

$$\begin{aligned} I_3 = {} & \int_{E_t\cap E_s} [g_1(y, t) - g_1(y, s)](F_1(u, v))(y, t)\,dy \\ & + \int_{E_t\cap E_s} g_1(y, s)[(F_1(u, v))(y, t) - (F_1(u, v))(y, s)]\,dy = I_{31} + I_{32}. \end{aligned} \tag{6.6}$$

Assumptions (2.IV), (2.V), (3.I), (3.II) and assertion (i) imply that

$$\|I_{31}\|_B \leqslant |t-s|^{\alpha/2}\,[N_{17}+N_{18}\,\|u\|_{B,\bar G}^{(1,0)}], \tag{6.7}$$

$$\|I_{32}\|_B \leqslant |t-s|^{\alpha/2}\,N_{19}(b)\,[1+\|u\|_{B,\bar G}^{(1+\alpha)}]+A_3\,\|v(t)-v(s)\|_B, \tag{6.8}$$

where N_{17}, N_{18} and $N_{19}(b)$ are positive constants. Combining relations (6.2)–(6.8) and assertion (i), we obtain assertion (iii).

Now take any $u \in C^{(1,0)}(\bar G, B)$ and $t \in [0, T]$. Taking into consideration the definitions of the operators L_1 and L_3, we conclude that each of the numbers $(L_1 u)(t)$ and $(L_3 u)(t)$ depends only on the functions $u(\cdot, t)$ and $u_x(\cdot, t)$. Hence assertion (iv) follows, which completes the proof.

LEMMA 6.4. *Let assumptions* (2.II), (2.IV), (2.V), (3.I), (5.I), (5.II) *be satisfied and suppose* $S \in C^{(1)}$. *Then the following assertions hold.*

(i) *Assertions* (i) *and* (iv) *of Lemma* 6.3.

(ii) *For any* $b > 0$ *there is a constant* $A_{18}(b) > 0$ *such that*

$$\|L_3 u_1 - L_3 u_2\|_{B,[0,T]} \leqslant A_{18}(b)\,[\|u_1-u_2\|_{B,\bar G}^{(1,0)}]^{(\gamma/\alpha)}$$

for any $u_1, u_2 \in C^{(1,0)}(\bar G, B)$ *such that* $\|u_i\|_{B,\bar G}^{(1,0)} \leqslant b$, $i = 1, 2$.

(iii) L_3: $C^{(1+\alpha)}(\bar G, B) \to C^{(\gamma/2)}([0, T], B)$ *and for any* $b > 0$ *there is a constant* $A_{19}(b) > 0$ *such that*

$$\|L_3 u\|_{B,[0,T]}^{(\gamma/\alpha)} \leqslant A_{19}(b)\,[1+\|u\|_{B,\bar G}^{(1+\alpha)}]$$

for any $u \in C^{(1+\alpha)}(\bar G, B)$ *such that* $\|u\|_{B,\bar G}^{(1+\alpha)} \leqslant b$.

(iv) *There is a constant* $A_{20} > 0$ *such that for any bounded set* $U \subset C^{(1+\alpha)}(\bar G, B)$ *we have*

$$M_0^{(a)}(L_3 U) \leqslant A_{20}\,M_{1,0}^{(a)}(U), \qquad a \geqslant 0.$$

Proof. Assertion (i) follows from Lemma 6.3. The proof of assertions (ii) and (iii) is similar to that of assertions (ii) and (iii) of Lemma 6.3, respectively.

To prove (iv) write

$$(L_4 u)(t) = \int_{E_t} g_1(y, t)(F_1(u, L_3 u))(y, t)\,dy, \qquad u \in U. \tag{6.9}$$

Using (5.III), (2.4) and the main properties of Hausdorff's measures of noncompactness, and proceeding like in [14] (p. 619, 620), we obtain

$$M_0^{(a)}(L_4 U) \leqslant N_{20}\,M_{1,0}^{(a)}(U)+A_3\,M_0^{(a)}(L_3 U), \tag{6.10}$$

where A_3 is defined by (2.7), $N_{20} > 0$ is a constant and

(6.11) $$L_i U = \{L_i u \colon u \in U\}.$$

We also have

(6.12) $$M_0^{(a)}(L_1 U) \leqslant N_{21} M_{1,0}^{(a)}(U),$$

$N_{21} > 0$ being a constant. It follows from (6.2) and (6.9) that

(6.13) $$L_3 u = L_1 u + L_4 u, \quad u \in U.$$

Relations (6.10)–(6.13) yield the inequality

$$M_0^{(a)}(L_3 U) \leqslant (N_{20}+N_{21}) M_{1,0}^{(a)}(U) + A_3 M_0^{(a)}(L_3 U),$$

which implies assertion (iv). This completes the proof.

7. Final remarks. The employment of a Banach space B instead of the Euclidean space R enables us to obtain various classes of equations with parameter. Take, for instance, $B = C[a, b]$. Then problem (1.2), (1.3) has the form

(7.1) $$\sum_{i,j=1}^{n} a_{ij}(x, t) u_{x_i x_j}(x, t, s) - u_t(x, t, s) + K \int_{E_t} g(y, t) u_t(y, t, s)\, dy$$
$$= F\Big(x, t, s, u(x, t, s), u_x(x, t, s), \int_{E_t} g(y, t) u_t(y, t, s)\, dy\Big),$$
$$(x, t) \in \bar{G} \setminus \Gamma,$$

(7.2) $$u(x, t, s) = \varphi(x, t, s), \quad (x, t) \in \Gamma,$$

where $s \in [a, b]$ is a parameter. For a solution u of the above problem we have

(7.3) $$u(x, t, \cdot),\ u_{x_i}(x, t, \cdot),\ u_{x_i x_j}(x, t, \cdot),\ u_t(x, t, \cdot) \in C[a, b]$$

for any $(x, t) \in \bar{G}$, which implies the continuity of functions (7.3), uniform with respect to parameter s. Taking B as a Banach space of differentiable functions, we obtain the differentiability of functions (7.3) with respect to parameter s for any solution u of problem (7.1), (7.2).

Note that problem (7.1), (7.2) involves certain random case. Namely, let $s = \omega \in \Omega$, where $(\Omega, \mathcal{F}, P)$ is a complete probability space. Then B may be taken as the Banach space consisting of all random variables $\lambda \colon \Omega \to R$ with finite norm

$$\|\lambda\|_B = \Big[\int_{\Omega} |\lambda(\omega)|^q P(d\omega)\Big]^{1/q} \quad (q \in [1, \infty) \text{ being a constant})$$

or

$$\|\lambda\|_B = \operatorname*{ess\,sup}_{\omega\in\Omega} |\lambda(\omega)|.$$

In [2] there has been considered problem similar to (7.1), (7.2) in the case where s is a parameter with values in a Banach space. Various results concerning boundary-value problems for diffrential equations with parameter can be found in references of [2].

Now we give an additional comment concerning assumptions (2.V) and (3.III). It is clear that the restriction on the constant $A_1 > 0$ in (2.V) may be replaced by the appropriate restriction on the constant K. Namely, under arbitrary $A_1 > 0$ the condition $A_2 > A_1$ is satisfied for sufficiently large $|K|$.

The right-hand side of (3.1) depends on $(L_3\Phi)(0)$. The value of the last expression is uniquely determined by φ. However, it is possible to determine effectively the above value if a formula for operator L_3 is known (see (4.2)). Now we give two nonlinear cases of equation (1.2) where $(L_3\Phi)(0)$ can be effectively determined. Let the function f in (1.2) be independent of $L_0 u$. Then operator L_3 is defined by the formula

$$(L_3 u)(t) = (L_1 u)(t) + \int_{E_t} g_1(y, t) f\big(y, t, u(y, t), u_y(y, t)\big)\,dy.$$

Now let the function f in (1.2) satisfy the condition

$$f(x, 0, u, p, q) = 0, \qquad x \in \bar{E}_0,\ u, q \in B,\ p \in B^n.$$

Then we have $(L_3 u)(0) = (L_1 u)(0)$, which implies the following formula for (3.1)

$$(L\varphi)(x, 0) = -K(L_1\varphi)(0), \qquad x \in \partial E_0.$$

Finally, one can extend the results of this paper to the general cases which have been considered in papers [13], [14]. Namely, the function f in (1.2) can be replaced by an operator, and the single equation can be replaced by a system of equations (finite or infinite). However, for simplicity of considerations we have not investigated the above general cases.

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