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An Orlicz extension of Cesàro sequence spaces

1. In [1], Jagers determined the Banach dual of the sequence space b_p for $1 < p < \infty$, where

$$b_p = \{x\colon \{\beta_n \sum_{k=1}^{n} |x_k|\}_{n\geq 1} \in l_p\},$$

which is a slight generalization of the Cesàro sequence space ces_p. In this paper, we generalize the l_p space to the Orlicz space l_φ, and also consider the so-called reverse Cesàro sequence space. More precisely, let $|(l_\varphi)_A|$ denote the space of all sequences x such that $A|x| \in l_\varphi$, where $A = (a_{nk})$ is an infinite matrix given by $a_{nk} = \alpha_k \beta_n$, when $1 \leqslant k \leqslant n$, and 0 otherwise. Similarly, $|(l_\varphi)_{\bar{A}}|$ denotes the corresponding sequence space with $\bar{A} = (\bar{a}_{nk})$ given by $\bar{a}_{nk} = \alpha_k \beta_n$ when $k \geqslant n$ and 0 otherwise. We then determine the Köthe duals of $|(l_\varphi)_A|$ and $|(l_\varphi)_{\bar{A}}|$ subject to certain restrictions on the Orlicz function φ and on α_k, β_n. We recall that the Köthe dual or α-dual of a sequence space X, denoted by X^α, is the space of all sequences $y = \{y_k\}$ such that for every $x = \{x_k\} \in X$

$$\sum_{k=1}^{\infty} |x_k y_k| \quad \text{converges.}$$

We shall see that for $|(l_\varphi)_A|$ and $|(l_\varphi)_{\bar{A}}|$ their Banach and Köthe duals coincide.

2. Let φ be an Orlicz function (cf. [3]), i.e., φ is continuous and even on $(-\infty, \infty)$, convex on $(0, \infty)$, vanishes only at 0 and satisfies the following conditions:

$$(0_1) \qquad \lim_{u\to 0} \varphi(u)/u = 0,$$

$$(\infty_1) \qquad \lim_{u\to\infty} \varphi(u)/u = \infty.$$

We assume that φ satisfies the (Δ_2, δ_2) condition, namely,

$$\varphi(2|u|) \leqslant M\varphi(|u|) \quad \text{for every } u.$$

Let ψ denote the complementary Orlicz function of φ, i.e.,

$$\psi(v) = \sup\{|uv| - \varphi(u);\ u \geqslant 0\}.$$

We shall assume that there is a continuous increasing function f: $[0, \infty) \to [0, \infty)$ with inverse f^{-1} such that the following three conditions are satisfied for $u, v \in [0, \infty)$

$$f(u)u = \varphi(f(u)) + \psi(u), \tag{2.1}$$

$$vf^{-1}(v) = \varphi(v) + \psi(f^{-1}(v)), \tag{2.2}$$

$$\lim_{v \to \infty} f^{-1}(v) = \infty. \tag{2.3}$$

Note that if φ' is the derivative of φ, the function f is in fact $(\varphi')^{-1}$. For example, when $\varphi(u) = u^p/p$ and $\psi(v) = v^q/q$, $f(u) = u^{q-1}$.

Now consider the space $|(l_\varphi)_A|$, where $\alpha_k > 0$ for every k, $\beta_n > 0$ for every n and $\{\beta_n\}_{n \geqslant 1} \in l_\varphi$. Using the Orlicz norm in l_φ, with the usual notation $\|\cdot\|_\varphi^0$, the norm in $|(l_\varphi)_A|$ is defined by

$$\|x\| = \|A|x|\|_\varphi^0.$$

We shall also consider the space

$$|(l_\varphi)_{\bar{A}}| = \Big\{x\colon \Big\{\beta_n \sum_{k=n}^{\infty} \alpha_k |x_k|\Big\}_{n \geqslant 1} \in l_\varphi\Big\},$$

where the norm is similarly defined by

$$\|x\| = \|\bar{A}|x|\|_\varphi^0.$$

It may easily be verified that with the norms given above, the spaces $|(l_\varphi)_A|$ and $|(l_\varphi)_{\bar{A}}|$ are BK spaces (cf. [2]). In fact, they are absolute in the sense that if $x \in |(l_\varphi)_A|$ and $|z| = |x|$, then $z \in |(l_\varphi)_A|$ and $\|z\| = \|x\|$. This means that the Köthe dual of $|(l_\varphi)_A|$ coincides with its β-dual (cf. [2]), and similarly for $|(l_\varphi)_{\bar{A}}|$. Also, for $y \in |(l_\varphi)_A|^\alpha$,

$$\sum_{k=1}^{\infty} |x_k y_k| \leqslant \|x\| \, \|y\|^\beta, \tag{2.4}$$

where

$$\|y\|^\beta = \sup\Big\{\Big|\sum_{k=1}^{\infty} x_k y_k\Big|;\ x \in |(l_\varphi)_A| \text{ and } \|x\| \leqslant 1\Big\}$$

$$= \sup\Big\{\sum_{k=1}^{\infty} |x_k y_k|;\ x \in |(l_\varphi)_A| \text{ and } \|x\| \leqslant 1\Big\}.$$

This is also true for $|(l_\varphi)_{\bar{A}}|$ and its Köthe dual.

As for notation, we shall adopt the usual notation used for Orlicz spaces such as

$$\varrho_\varphi(s) = \sum_{k=1}^{\infty} \varphi(|s_k|) \quad \text{and} \quad \varrho_\psi(t) = \sum_{k=1}^{\infty} \psi(|t_k|)$$

and shall use without proof known identities such as

$$\|s\|_\varphi^0 \leqslant \varrho_\varphi(s)+1.$$

3. Now, we shall characterize the Köthe dual of $|(l_\varphi)_A|$. First, for any sequence $y = \{y_k\}$ satisfying $\{\alpha_n^{-1} y_n\}_{n\geqslant 1} \in c_0$, define $y^\sim$ to be the coordinatewise infimum of all $y^* \geqslant |y|$ such that

$$\text{(3.1)} \qquad \{\alpha_n^{-1} y_n^*\}_{n\geqslant 1} \quad \text{is decreasing}$$

$$\text{and} \quad \{\beta_n^{-1} f(t_n^*)\}_{n\geqslant 1} \quad \text{is increasing,}$$

where

$$t_n^* = \beta_n^{-1}(\alpha_n^{-1} y_n^* - \alpha_{n+1}^{-1} y_{n+1}^*).$$

Next, let $t_n^\sim = \beta_n^{-1}(\alpha_n^{-1} y_n^\sim - \alpha_{n+1}^{-1} y_{n+1}^\sim)$ and define the sequence $\{m(k)\}_{k\geqslant 1}$ by

$$m(1) = \min\{n;\ \beta_n^{-1} f(t_n^\sim) > 0\},$$

$$m(k) = \min\{n > m(k-1);\ \beta_n^{-1} f(t_n^\sim) > \beta_{n-1}^{-1} f(t_{n-1}^\sim)\}.$$

Let $I_k = \{m(k), m(k)+1, \ldots, m(k+1)-1\}$ and then, for each k, there is a constant c_k such that

$$\text{(3.2)} \qquad \beta_n^{-1} f(t_n^\sim) = c_k \quad \text{for every } n \in I_k.$$

Note that c_k increases strictly with k and, for $n = m(k)+1, \ldots, m(k+1)$,

$$\text{(3.3)} \qquad \alpha_n^{-1} y_n^\sim = \alpha_{m(k)}^{-1} y_{m(k)}^\sim - \sum_{j=m(k)}^{n-1} \beta_j f^{-1}(c_k \beta_j)$$

and, in particular,

$$\text{(3.4)} \qquad \alpha_{m(k)}^{-1} y_{m(k)}^\sim - \alpha_{m(k+1)}^{-1} y_{m(k+1)}^\sim = \Sigma_k \beta_j f^{-1}(c_k \beta_j),$$

where Σ_k denotes the sum over all j in I_k. If the sequence $\{m(k)\}$ terminates at $m(K)$, equation (3.3) holds, with k replaced by K, for all $n > m(K)$.

3.1. Lemma. *For every k such that $m(k)$ is defined, $y_{m(k)}^\sim = |y_{m(k)}|$.*

Proof. Suppose that for some $k \geqslant 1$, $y_{m(k)}^\sim > |y_{m(k)}|$. By continuity and monotonicity of f, since $\beta_{m(k)}^{-1} f(t_{m(k)}^\sim) > \beta_{m(k)-1}^{-1} f(t_{m(k)-1}^\sim)$, there exists $\lambda \in (0, 1)$ such that

$$\beta_{m(k)}^{-1} f\Big(\beta_{m(k)}^{-1}\big(\alpha_{m(k)}^{-1}(\lambda y_{m(k)}^\sim) - \alpha_{m(k)+1}^{-1} y_{m(k)+1}^\sim\big)\Big)$$

$$> \beta_{m(k)-1}^{-1} f\Big(\beta_{m(k)-1}^{-1}\big(\alpha_{m(k)-1}^{-1} y_{m(k)-1}^\sim - \alpha_{m(k)}^{-1}(\lambda y_{m(k)}^\sim)\big)\Big) > 0$$

and we may ensure that $\lambda y^{\sim}_{m(k)} > |y_{m(k)}|$. Defining y^* by $y^*_j = y^{\sim}_j$ for $j \neq m(k)$ and $y^*_{m(k)} = \lambda y^{\sim}_{m(k)}$, y^* satisfies (3.1) but $y^*_{m(k)} < y^{\sim}_{m(k)}$, which gives a contradiction.

3.2. LEMMA. *For any sequence y satisfying* $\{\alpha_n^{-1} y_n\}_{n\geq 1} \in c_0$,

$$\alpha_n^{-1} y_n^{\sim} \to 0 \qquad as\ n \to \infty.$$

Proof. Since $\{\alpha_n^{-1} y_n^{\sim}\}_{n\geq 1}$ is non-negative and decreasing, it is a convergent sequence. If the sequence $\{m(k)\}$ does not terminate, $\alpha_{m(k)}^{-1} y^{\sim}_{m(k)} = \alpha_{m(k)}^{-1} |y_{m(k)}| \to 0$ as $k \to \infty$. Hence $\{\alpha_n^{-1} y_n^{\sim}\}_{n\geq 1}$ is a null sequence. Now we consider the case where $\{m(k)\}$ terminates at $m(K)$. Assume that $\alpha_n^{-1} y_n^{\sim} \to c > 0$ and let $M > m(k)$ be a positive integer such that $\alpha_n^{-1} |y_n| < c/2$ for $n > M$.

Let g be a function defined on the reals by

$$g(v) = \alpha_M^{-1} y_M - \sum_{j=M}^{\infty} \beta_j f^{-1}(v\beta_j).$$

Since $f^{-1}(v\beta_j)$ is increasing in v for each j, the series is uniformly convergent for v in any finite interval and hence g is continuous in v over any finite interval. Replacing k by K in equation (3.3) and taking limits as $n \to \infty$, we have

$$c = \lim_{n\to\infty} \alpha_n^{-1} y_n^{\sim} = g(c_K)$$

and for v sufficiently large, $g(v) < 0$. Hence there is a positive number $L > c_k$ such that $g(L) = c/2$. Defining y^* by $y^*_n = y^{\sim}_n$ for $n \leqslant M$ and

$$\alpha_n^{-1} y_n^* = \alpha_M^{-1} y_M^* - \sum_{j=M}^{n-1} \beta_j f^{-1}(L\beta_j) \qquad \text{for } n > M$$

we have a sequence y^* with $y_n^* < y_n^{\sim}$ for $n > M$ but y^* satisfying (3.1) which is a contradiction.

3.3 LEMMA. *If* $\{\alpha_n^{-1} y_n\}_{n\geq 1} \in c_0$ *and* $t^{\sim} \in l_\psi$, *and if* $s = A|x| \in l_\varphi$, *then*

$$\beta_n^{-1} \alpha_n^{-1} y_n^{\sim} s_n \to 0 \qquad as\ n \to \infty.$$

Proof. By Lemma 3.2, we may write

$$\beta_n^{-1} \alpha_n^{-1} y_n^{\sim} s_n = \beta_n^{-1} s_n \sum_{k=n}^{\infty} (\alpha_k^{-1} y_k^{\sim} - \alpha_{k+1}^{-1} y_{k+1}^{\sim}) \leqslant \sum_{k=n}^{\infty} s_k t_k^{\sim}.$$

Since $s = A|x| \in l_\varphi$, and $t^{\sim} \in l_\psi$, the last series above converges to 0 and the result follows.

3.4. THEOREM. *Let* $y \in |(l_\varphi)_A|^\alpha$ *with* $\|y\|^\beta \leqslant 1$. *Then*
(i) $\alpha_n^{-1} y_n \to 0$ *as* $n \to \infty$ *and*
(ii) $t^{\sim} = y^{\sim} A^{-1} \in l_\psi$.

Proof. First, define $x(n) = \alpha_n^{-1} e^n$, where e^n is the sequence with 1 in the nth position and 0 elsewhere. Let $\theta = \{\beta_k\}_{k\geq 1}$ and θ^n be its truncation at the nth term. Then $A|x(n)| = \theta - \theta^{n-1}$ and hence $\|x(n)\| = \|\theta - \theta^{n-1}\|_\varphi^0$. Since l_φ is an AK space (cf. [2]), $\|\theta - \theta^{n-1}\|_\varphi^0 \to 0$ as $n \to \infty$ and, substituting $x = x(n)$ into equation (2.4), condition (i) follows.

To prove (ii), first consider the case where $m(k)$ terminates at $m(N)$. Writing $c_0 = 0$, define the sequence x by

$$x_{m(k)} = (c_k - c_{k-1})\alpha_{m(k)}^{-1}, \qquad k = 1, 2, 3, \ldots, N,$$
$$x_i = 0, \qquad \text{otherwise}.$$

Then

$$(A|x|)_n = \begin{cases} \beta_n c_{M_n} & \text{if } M_n < N, \\ \beta_n c_N & \text{if } M_n \geq N, \end{cases}$$

where $M_n = \max\{k;\, m(k) \leq n\}$.

Since $\{\beta_n\}_{n\geq 1} \in l_\varphi$, $A|x| \in l_\varphi$ and so $x \in |(l_\varphi)_A|$. For $n \in I_k$, $M_n = k$ and so, for $k = 1, 2, \ldots, N$,

$$\beta_n c_{M_n} = \beta_n c_k = f(t_n^{\sim})$$

and, for $n \geq m(N)$,

$$\beta_n c_N = f(t_n^{\sim}).$$

Hence $A|x| = \{f(t_n^{\sim})\}_{n\geq 1}$ and thus

$$\varrho_\varphi(\{f(t_n^{\sim})\}_{n\geq 1}) = \varrho_\varphi(A|x|) < \infty. \tag{3.5}$$

Now, substituting $k = N$ in (3.3) and letting $n \to \infty$, by Lemma 3.2,

$$\alpha_{m(N)}^{-1} y_{m((N)}^{\sim} = \sum_{j=m(N)}^{\infty} \beta_j f^{-1}(c_N \beta_j). \tag{3.6}$$

Then, using Lemma 3.1 and equations (3.4) and (3.6),

$$\begin{aligned} \|A|x|\|_\varphi^0 &\geq \|x\|\,\|y\|^\beta \geq \sum_{k=1}^{N} |x_{m(k)}|\, y_{m(k)}^{\sim} \\ &= \sum_{k=1}^{N-1} c_k \Sigma_k \beta_j f^{-1}(c_k\beta_j) + c_N \sum_{j=m(N)}^{\infty} \beta_j f^{-1}(c_N \beta_j) \\ &= \sum_{j=1}^{m(N)-1} [\varphi(f(t_j^{\sim})) + \psi(t_j^{\sim})] + \sum_{j=m(N)}^{\infty} [\varphi(f(t_j^{\sim})) + \psi(t_j^{\sim})], \end{aligned} \tag{3.7}$$

the last equality resulting from equations (2.2) and (3.2).

Since $\varrho_\varphi(\{f(t_j^\sim)\}_{j\geq 1})$ is finite, and $\|A|x|\|_\varphi^0 \leqslant \varrho_\varphi(\{f(t_j^\sim)\}_{j\geq 1})+1$, $\varrho_\varphi(t^\sim) = \sum_{j=1}^{\infty} \psi(t_j^\sim) \leqslant 1$ which gives (ii).

Now, if the sequence $\{m(k)\}$ does not terminate, let $y(N)$ be the sequence y truncated at $m(N)$ and let $t(N)^\sim$ be the corresponding $t^\sim$ sequence. Then for $y(N)$, the $\{m(k)\}$ sequence terminates at $m(N)$ and $y_k^\sim = y(N)_k^\sim$ for $k \leqslant m(N)$. Hence for $j \leqslant m(N)$, $t_j^\sim = t(N)_j^\sim$ and from above,

$$\sum_{j=1}^{m(N)-1} \psi(t_j^\sim) = \sum_{j=1}^{m(N)-1} \psi(t(N)_j^\sim) \leqslant 1.$$

Since this is true for every positive integer N, we obtain $\varrho_\psi(t^\sim) \leqslant 1$ by letting $N \to \infty$ and so $t^\sim \in l_\psi$.

3.5. THEOREM. *$y \in |(l_\varphi)_A|^\alpha$ if and only if* (i) *$\alpha_n^{-1} y_n \to 0$ as $n \to \infty$ and* (ii) *$t^\sim \in l_\psi$, where $t^\sim = (\lambda y)^\sim A^{-1}$ for some $\lambda > 0$.*

Proof. First, we show the sufficiency of (i) and (ii). From (i), $\alpha_n^{-1}(\lambda y_n)^\sim \to 0$ as $n \to \infty$, by Lemma 3.2. Hence, writing $s = A|x| \in l_\varphi$ and applying Lemma 3.3 with λy replacing y to the following:

$$\sum_{k=1}^{n} |x_k \lambda y_k| \leqslant \sum_{k=1}^{n} |x_k| (\lambda y)_k^\sim = \sum_{k=1}^{n-1} s_k t_k^\sim + \beta_n^{-1} \alpha_n^{-1} (\lambda y)_n^\sim s_n;$$

we have $\lambda y \in |(l_\varphi)_A|^\alpha$. Hence $y \in |(l_\varphi)_A|^\alpha$.

Now, conversely, if $y \in |(l_\varphi)_A|^\alpha$ take $\lambda = 1$ if $\|y\|^\beta \leqslant 1$ and $\lambda = (\|y\|^\beta)^{-1}$ if $\|y\|^\beta > 1$. Then conditions (i) and (ii) follow from Lemma 4.1.

4. Now we shall characterize the Köthe dual of $|(l_\varphi)_{\bar{A}}| = \{x\colon \{\beta_n \sum_{k=n}^{\infty} \alpha_k |x_k|\}_{n\geq 1} \in l_\varphi\}$, where $\alpha_k > 0$ for every k, $\beta_n > 0$ for every n. We further assume that for any constant scalar $\gamma > 0$,

$$\sum_{j=1}^{n} \psi(f^{-1}(\gamma \beta_j)) \to \infty \quad \text{as } n \to \infty. \tag{4.1}$$

4.1. LEMMA. *If $y \in |(l_\varphi)_{\bar{A}}|^\alpha$, then there is a positive number B such that, for every n,*

$$|y_n| \leqslant (\alpha_n \sum_{j=1}^{n} \beta_j f^{-1}(B\beta_j)) \|y\|^\beta. \tag{4.2}$$

Proof. First, choose a positive number B such that $\psi(f^{-1}(B\beta_1)) \geqslant 1$. Then, for every positive integer n,

$$\sum_{j=1}^{n} \psi(f^{-1}(B\beta_j)) \geqslant 1. \tag{4.3}$$

Defining the sequence $x(n) = (\alpha_n \sum_{j=1}^{n} \beta_j f^{-1}(B\beta_j))^{-1} e^n$, it may be checked that $\bar{A}|x(n)| \in l_\varphi$ with

$$\|x(n)\| = \|\bar{A}|x(n)|\|_\varphi^0 \leqslant [\sum_{j=1}^{n} \varphi(B\beta_j) + \sum_{j=1}^{n} \psi(f^{-1}(B\beta_j))]^{-1} (\sum_{j=1}^{n} \varphi(B\beta_j) + 1)$$

from which $\|x(n)\| \leqslant 1$ by (4.3). The result follows on substituting $x(n)$ into (2.4).

Now, for any sequence y, let $y^* \geqslant |y|$ satisfy

(4.4) $\{\alpha_n^{-1} y_n^*\}_{n\geqslant 1}$ is increasing and $\{\beta_n^{-1} f(u_n^*)\}_{n\geqslant 1}$ is decreasing,

where

(4.5) $$u_n^* = \beta_n^{-1}(\alpha_n^{-1} y_n^* - \alpha_{n-1}^{-1} y_{n-1}^*) \quad \text{and} \quad \alpha_0^{-1} y_0^* = 0.$$

For any sequence y satisfying the following condition:
There exists a positive number B such that

(4.6) $$|y_n| \leqslant \alpha_n \sum_{j=1}^{n} \beta_j f^{-1}(B\beta_j) \quad \text{for all } n.$$

We may define y_n^* to be the right-hand quantity of inequality (4.6). Such a y^* satisfies (4.4), and hence such y^* exist for sequences y satisfying (4.6). Let $y^\sim$ be the coordinatewise infimum of all such y^* and define $u^\sim$ for $y^\sim$ as in (4.5) with $\alpha_0^{-1} y_0^\sim = 0$. The sequence $\{m(k)\}$ is defined by

$$m(0) = 0, \quad m(k) = \min\{n > m(k-1);\, \beta_n^{-1} f(u_n^\sim) > \beta_{n+1}^{-1} f(u_{n+1}^\sim)\}.$$

Now, for each k such that $m(k)$ is defined, there exists a constant c_k such that

(4.7) $$\beta_n^{-1} f(u_n^\sim) = c_k \quad \text{for } m(k-1) < n \leqslant m(k),$$

where c_k decreases strictly with k. Hence, for $m(k-1) < n \leqslant m(k)$,

$$\alpha_n^{-1} y_n^\sim = \alpha_{m(k-1)}^{-1} y_{m(k-1)}^\sim + \sum_{j=m(k-1)+1}^{n} \beta_j f^{-1}(c_k \beta_j)$$

and in particular,

(4.8) $$\alpha_{m(k)}^{-1} y_{m(k)}^\sim - \alpha_{m(k-1)}^{-1} y_{m(k-1)}^\sim = \Sigma_k \beta_j f^{-1}(c_k \beta_j),$$

where Σ_k denotes summation from $j = m(k-1)+1$ to $j = m(k)$.

Furthermore, if the sequence $\{m(k)\}$ terminates at $m(K)$, then for every $n > m(K)$,

(4.9) $$\beta_n^{-1} f(u_n^\sim) = c_{K+1}$$

and

$$(4.10) \qquad \alpha_n^{-1} y_n^{\sim} = \alpha_{m(K)}^{-1} y_{m(K)}^{\sim} + \sum_{j=m(K)+1}^{n} \beta_j f^{-1}(c_{K+1} \beta_j).$$

4.2. LEMMA. *If* $y \in |(l_\varphi)_{\bar{A}}|^\alpha$ *with* $\|y\|^\beta \leqslant 1$, *then for any positive scalar* γ,

$$\gamma \alpha^{-1} |y_n| \leqslant \sum_{j=1}^{n} \varphi(\gamma \beta_j) + 1 \qquad \text{for all } n.$$

Proof. The result follows by substituting $x(n) = \gamma \alpha_n^{-1} e^n$ into (2.4) since

$$A|x(n)| = \gamma \{\beta_1, \beta_2, \ldots, \beta_n, 0, 0, \ldots\}$$

and

$$\|x(n)\| = \|A|x(n)|\|_\varphi^0 \leqslant \varrho_\varphi(\bar{A}|x(n)|) + 1.$$

4.3. LEMMA. *Let* $y \in |(l_\varphi)_A|^\alpha$ *and* $\|y\|^\beta \leqslant 1$. *If the sequence* $\{m(k)\}_{k \geqslant 1}$ *terminates at* $m(K)$, *then for* $n > m(K)$

$$g(n) = c_{K+1} \alpha_n^{-1} y_n^{\sim} - \sum_{j=1}^{n} \varphi(c_{K+1} \beta_j)$$

increases with n *and* $\lim_{n \to \infty} g(n)$ *exists.*

Proof. Since $g(n) = 0$ if $c_{K+1} = 0$ and the result is trivial, it will be assumed that $c_{K+1} > 0$. First, note that for any $n > m(K)$, by (4.9) and (4.10),

$$\begin{aligned} \sum_{m(K)+1}^{n} \psi(u_j^{\sim}) &= \sum_{m(K)+1}^{n} \psi(f^{-1}(c_{K+1} \beta_j)) \\ &= \sum_{m(K)+1}^{n} c_{K+1} \beta_j f^{-1}(c_{K+1} \beta_j) - \sum_{m(K)+1}^{n} \varphi(c_{K+1} \beta_j) \\ &= c_{K+1}(\alpha_n^{-1} y_n^{\sim} - \alpha_{m(K)}^{-1} y_{m(K)}^{\sim}) - \sum_{m(K)+1}^{n} \varphi(c_{K+1} \beta_j) \\ &= c_{K+1} \alpha_n^{-1} y_n^{\sim} - \sum_{1}^{n} \varphi(c_{K+1} \beta_j) + \sum_{1}^{m(K)} \varphi(c_{K+1} \beta_j) - c_{K+1} \alpha_{m(K)}^{-1} y_{m(K)}^{\sim}. \end{aligned}$$

Hence

$$(4.11) \qquad \sum_{m(K)+1}^{n} \psi(u_j^{\sim}) = g(n) + \sum_{1}^{m(K)} \varphi(c_{K+1} \beta_j) - c_{K+1} \alpha_{m(K)}^{-1} y_{m(K)}^{\sim}$$

and so $g(n)$ increases with $n > m(K)$. Now,

$$(4.12) \qquad g(n) = c_{K+1}\alpha_n^{-1}|y_n| - \sum_{j=1}^{n}\varphi(c_{K+1}\beta_j) + c_{K+1}\alpha_n^{-1}(y_n^{\sim} - |y_n|)$$
$$\leqslant 1 + c_{K+1}\alpha_n^{-1}(y_n^{\sim} - |y_n|),$$

by Lemma 4.2. Now consider two cases:

Case 1. There is a subsequence $\{y_{n(k)}^{\sim}\}_{k\geqslant 1}$ such that $y_{n(k)}^{\sim} = |y_{n(k)}|$. Then, for all k such that $n(k) > m(K)$, $g(n(k)) \leqslant 1$ by (4.12), and since $g(n(k))$ increases with k, $\lim_{k\to\infty} g(n(k))$ exists. But, as $\{g(n)\}_{n\geqslant 1}$ is an increasing sequence, $\lim_{n\to\infty} g(n)$ exists.

Case 2. There is a positive integer $M \geqslant m(K)$ such that $y_M^{\sim} = |y_M|$ and $y_n^{\sim} > |y_n|$ for all $n > M$.

It will be shown that either a y^* may be constructed to obtain a contradiction or that $c_{K+1} = 0$ which is again a contradiction. Now, for each $n > M$, $\alpha_M^{-1}|y_M| + \sum_{M+1}^{n}\beta_j f^{-1}(u\beta_j)$ is continuous and increasing with u. If $\alpha_n^{-1}|y_n| > \alpha_M^{-1}|y_M|$, define $c(n)$ to be the positive number such that

$$(4.13) \qquad \alpha_n^{-1}|y_n| = \alpha_M^{-1}|y_M| + \sum_{M+1}^{n}\beta_j f^{-1}(c(n)\beta_j).$$

If $\alpha_n^{-1}|y_n| \leqslant \alpha_M^{-1}|y_M|$, define $c(n) = 0$.

Now, since

$$\alpha_n^{-1}y_n^{\sim} = \alpha_M^{-1}|y_M| + \sum_{M+1}^{n}\beta_j f^{-1}(c_{K+1}\beta_j),$$

we have $0 \leqslant c(n) < c_{K+1}$ for all $n > M$.

Now define $c = \sup\{c(n);\ n \geqslant M\}$.

Case (i). If $c < c_{K+1}$, define y^* by $y_n^* = y_n^{\sim}$ for $n \leqslant M$ and

$$\alpha_n^{-1}y_n^* = \alpha_M^{-1}|y_M| + \sum_{M+1}^{n}\beta_j f^{-1}(c\beta_j) \qquad \text{for } n > M$$

and obtain a contradiction as y^* satisfies (4.4) but $|y_n| \leqslant y_n^* < y_n^{\sim}$ for $n > M$.

Case (ii). If $c = c_{K+1}$, since $c(n) < c_{K+1}$ for all $n > M$, we must have a countable subset S_1 of positive integers such that $\{c(n)\}_{n\in S_1}$ is convergent to c. We will show that

$$c = 0 \quad \text{and hence} \quad c_{K+1} = c = 0.$$

If $c(n) = 0$ for all large $n \in S_1$, there is nothing to prove. Hence it may be assumed that there is a countable subset S_2 of S_1 such that $c(n) = 0$ for $n \in S_1 \setminus S_2$ and (4.13) holds for $n \in S_2$.

Then, for every $n \in S_2$, using the proof of Lemma 4.2 with $\gamma = 1$,

$$\alpha_M^{-1} |y_M| + \sum_{M+1}^{n} \beta_j f^{-1}(c(n)\beta_j) = \alpha_n^{-1} |y_n| \leqslant \|\{\beta_j\}_{1 \leqslant j \leqslant n}\|_\varphi^0$$

which implies

$$\sum_{M+1}^{n} c(n)\beta_j f^{-1}(c(n)\beta_j) \leqslant \|\{c(n)\beta_j\}_{1 \leqslant j \leqslant n}\|_\varphi^0 \leqslant \sum_{j=1}^{n} \varphi(c(n)\beta_j) + 1$$

and so

$$\sum_{M+1}^{n} \varphi(c(n)\beta_j) + \sum_{M+1}^{n} \varphi(f^{-1}(c(n)\beta_j)) \leqslant \sum_{j=1}^{n} \varphi(c(n)\beta_j) + 1$$

giving

$$\sum_{M+1}^{n} \psi(f^{-1}(c(n)\beta_j)) \leqslant 1 + \sum_{j=1}^{M} \varphi(c(n)\beta_j) \tag{4.14}$$

$$\leqslant 1 + \sum_{j=1}^{M} \varphi(c_{K+1}\beta_j) = H \quad \text{(say)}.$$

Now, if $c(n)$ does not tend to 0 as $n \to \infty$ in S_2, there is a $\gamma > 0$ and a countable subset S_3 of S_2 such that $c(n) > \gamma$ for all $n \in S_3$. Hence for $n \in S_3$,

$$\sum_{M+1}^{n} \psi(f^{-1}(c(n)\beta_j)) \geqslant \sum_{M+1}^{n} \psi(f^{-1}(\gamma\beta_j)).$$

But the right-hand side is divergent as $n \to \infty$ which contradicts (4.14). Hence $\{c(n)\}_{n \in S_2}$ is null sequence. But $c(n) = 0$ for every $n \in S_1 \setminus S_2$. Hence $\{c(n)\}_{n \in S_1}$ is also a null sequence and so $c_{K+1} = c = 0$ which gives a contradiction for case (ii).

Thus Cases 2 (i) and (ii) are impossible. Hence the result holds, i.e., $\lim_{n \to \infty} g(n)$ exists.

4.4. Theorem. *Let* $y \in |(l_\varphi)_{\bar{A}}|^\alpha$ *with* $\|y\|^\beta \leqslant 1$. *Then* y *satisfies*

(i) *there exists* $B > 0$ *such that* $|y_n| \leqslant \alpha_n \sum_{j=1}^{n} \beta_j f^{-1}(B\beta_j)$ *for all* n;

(ii) $y^\sim \bar{A}^{-1} = u^\sim \in l_\psi$.

Proof. The first condition follows from Lemma 4.1. With (i), $y^\sim$ is well defined. Consider first the case where $\{m(k)\}$ terminates, say at $m(K)$. By equation (4.11) in Lemma 4.3,

$$\lim_{n \to \infty} \sum_{m(K)+1}^{n} \psi(u_j^\sim) = \left(\lim_{n \to \infty} g(n)\right) + \sum_{1}^{m(K)} \varphi(c_{K+1}\beta_j) - c_{K+1}\alpha_{m(K)}^{-1} y_{m(K)}^\sim .$$

Therefore, by the result of that lemma $\sum_{m(K)+1}^{\infty} \psi(u_j^{\sim})$ converges and hence $u^{\sim} \in l_{\psi}$.

Now consider the case where $\{m(k)\}$ does not terminate. Let N be a positive integer and define x as follows

$$x_{m(k)} = \alpha_{m(k)}^{-1}(c_k - c_{k+1}) \quad \text{for } k = 1, 2, \ldots, N-1,$$
$$x_{m(N)} = \alpha_{m(N)}^{-1} c_N,$$
$$x_j = 0 \quad \text{otherwise.}$$

Let $M_j = \min\{k; m(k) \geqslant j\}$. Then

$$(\bar{A}|x|)_j = \begin{cases} \beta_j c_{M_j} & \text{if } M_j \leqslant N, \text{ i.e., } j \leqslant m(N), \\ 0 & \text{otherwise.} \end{cases}$$

Hence $|x| \in |(l_\varphi)_{\bar{A}}|$ and

$$\varrho_\varphi(\bar{A}|x|) = \sum_{k=1}^{N} \Sigma_k \varphi(\beta_j c_k) = \sum_{k=1}^{N} \Sigma_k \varphi(f(u_j^{\sim})) = \sum_{j=1}^{m(N)} \varphi(f(u_j^{\sim})).$$

Thus, $\|x\| \|y\|^{\beta} \leqslant \|A|x|\|_{\varphi}^{0} \leqslant \sum_{j=1}^{m(N)} \varphi(f(u_j^{\sim})) + 1$ and furthermore,

$$\begin{aligned} \sum_{k=1}^{\infty} |x_k y_k| &= \sum_{k=1}^{N} (\alpha_{m(k)}^{-1} y_{m(k)}^{\sim} - \alpha_{m(k-1)}^{-1} y_{m(k-1)}^{\sim}) c_k \\ &= \sum_{k=1}^{N} \Sigma_k(\beta_j f^{-1}(c_k \beta_j)) c_k \\ &= \sum_{k=1}^{N} \Sigma_k(\varphi(c_k \beta_j) + \psi(f^{-1}(c_k \beta_j))) \\ &= \sum_{j=1}^{m(N)} \varphi(f(u_j^{\sim})) + \sum_{j=1}^{m(N)} \psi(u_j^{\sim}). \end{aligned}$$

Hence, substituting into (2.4), $\sum_{j=1}^{m(N)} \psi(u_j^{\sim}) \leqslant 1$. Letting $N \to \infty$, $\varrho_\psi(u^{\sim}) \leqslant 1$ and so $u^{\sim} \in l_\psi$.

4.5. THEOREM. *$y \in |(l_\varphi)_{\bar{A}}|^{\alpha}$ if and only if for some $\lambda > 0$,*

(i) *there exists $B > 0$ such that $\lambda|y_n| \leqslant \alpha_n \sum_{j=1}^{n} \beta_j f^{-1}(B\beta_j)$ for all n;*

(ii) $(\lambda y)^{\sim}(\bar{A})^{-1} \in l_\psi$.

Proof. If conditions (i) and (ii) hold for some $\lambda > 0$, since

$$\sum_{k=1}^{n-1} |x_k(\lambda y_k)| \leqslant \sum_{k=1}^{n} u_k^{\sim}(\bar{A}|x|)_k - \beta_n^{-1}\alpha_n^{-1}(\lambda y)_n^{\sim}(\bar{A}|x|)_n \leqslant \|\bar{A}|x|\|_\varphi^0 \|u^{\sim}\|_\psi,$$

where $u = (\lambda y)^{\check{}} \bar{A}^{-1}$, $\lambda y \in |(l_\varphi)_{\bar{A}}|^\alpha$ and so $y \in |(l_\varphi)_{\bar{A}}|^\alpha$. Conversely, if $y \in |(l_\varphi)_{\bar{A}}|^\alpha$, choose $\lambda = 1$ if $\|y\|^\beta \leqslant 1$ and $\lambda = (\|y\|^\beta)^{-1}$ if $\|y\|^\beta \geqslant 1$. Then (i) follows from Lemma 4.1 and (ii) from Lemma 4.4.

5. The results in Sections 3 and 4 charaterize the Köthe duals of $|(l_\varphi)_A|$ and $|(l_\varphi)_{\bar{A}}|$. Of greater interest may be the special cases in which A is the Cesàro matrix C, where $\alpha_k = 1$ and $\beta_n = 1/n$ and when $\bar{A}$ is the so-called reverse Cesàro matrix whence $\alpha_k = 1$ and $\beta_n = n$.

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