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Notes on modular function spaces. II

Abstract. This paper is a continuation of the article *Notes on modular function spaces. I* (the same issue). Section 6 deals with a Δ_2-condition defined for modular function spaces. The case of non-monotone convex function modulars is considered in Section 7. Section 8 is a brief exposition of some special cases and examples; it contains many references to the mathematical literature. Section 9 deals with countably modulared function spaces and Section 10 collects some results about interpolation in modular function spaces.

6. Subspace L^c_ϱ and the condition Δ_2

6.1. DEFINITION. By L^0_ϱ we shall mean a class of all functions $f \in L_\varrho$ such that $\varrho(f, \cdot)$ is order continuous. The smallest linear subspace of L_ϱ which contains L^0_ϱ will be denoted by L^c_ϱ, i.e., $f \in L^c_\varrho$ iff there exists a $\lambda > 0$ such that $\lambda f \in L^0_\varrho$.

The following remark is an immediate consequence of Definitions 4.1 and 6.1.

6.2. PROPOSITION. *$E_\varrho \subset L^0_\varrho \subset L^c_\varrho$ and L^0_ϱ is a linear space if and only if $E_\varrho = L^0_\varrho = L^c_\varrho$.*

Modifying slightly the proof of the Vitali Theorem 4.3 we obtain the following lemma.

6.3. LEMMA. *Let $f_n \in L^0_\varrho$ and $f_n \to 0$ ϱ-a.e.; then the following conditions are equivalent:*

(i) $\varrho(f_n) \to 0$,

(ii) *$\varrho(f_n, \cdot)$ are order equicontinuous.*

6.4. DEFINITION. We say that ϱ satisfies the *Δ_2-condition* if and only if for each sequence $(f_n) \subset L^c_\varrho$ the following implication holds: $\varrho(f_n, \cdot)$ are order equicontinuous implies $\varrho(2f_n, \cdot)$ are order equicontinuous.

It is easy to check that L^0_ϱ is a convex and balanced subset of L_ϱ. If we assume additionally that L^0_ϱ is absorbing in L_ϱ, then clearly $L^c_\varrho = L_\varrho$. In fact, in many special cases these spaces are identical.

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The next theorem is the main result of this section.

6.5. THEOREM. *If L_ϱ^0 is absorbing, then the following conditions are equivalent:*

(a) *ϱ satisfies the Δ_2-condition,*

(b) *L_ϱ^0 is a linear subspace of L_ϱ,*

(c) $E_\varrho = L_\varrho^0 = L_\varrho$,

(d) *the modular convergence is equivalent to the F-norm convergence in L_ϱ.*

Proof. (a)$\Rightarrow$(b) Let $f \in L_\varrho^0$ and $E_k \in \Sigma$, $E_k \searrow \emptyset$. Since $f \in L_\varrho^0$ then $\varrho(f, E_k) \to 0$ and, by the Δ_2-condition $\varrho(2f, E_k) \to 0$, which implies that $2f \in L_\varrho^0$ and consequently L_ϱ^0 is linear.

(b)$\Rightarrow$(c) Evident.

(c)$\Rightarrow$(a) Suppose to the contrary that there exists a sequence of functions $(f_n) \subset L_\varrho$ such that $\varrho(f_n, \cdot)$ are order equicontinuous while $\varrho(2f_n, \cdot)$ are not. It follows from Theorem 5.6 in [6] that $\varrho(2f_n, \cdot)$ are not uniformly exhausting. Thus, passing if necessary to a subsequence we may assume that there exists a sequence of disjoint sets $D_k \in \Sigma$ and a constant $\eta > 0$ such that $\varrho(2f_k, D_k) > \eta$ for every $k \in N$. Observe that $\varrho(f_k, D_k) \to 0$ because $\varrho(f_k, D_k) \leqslant \sup_n \varrho(f_n, D_k) \to 0$. The latter convergence is a consequence of the fact that $\sup_n \varrho(f_n, \cdot)$ is order continuous, hence, exhausting as well. Let (f_{k_i}) be a subsequence of (f_k) such that $\sum_{i=1}^{\infty} \varrho(f_{k_i}, D_{k_i}) \leqslant 1$, therefore, defining $s_m = \sum_{i=1}^{m} f_{k_i} 1_{D_i} \in L_\varrho = E_\varrho$ and $f = \sum_{i=1}^{\infty} f_{k_i} 1_{D_i}$, note that $\varrho(f - s_m) \leqslant \sum_{i=m+1}^{\infty} \varrho(f_{k_i}, D_{k_i}) \to 0$. The function f, therefore, is a member of $L_\varrho = E_\varrho$. Indeed, let $\varepsilon > 0$ be given arbitrarily; for $0 < \varepsilon_k < \frac{1}{2}$, $\varepsilon_k \to 0$, $\lambda_k = 2\varepsilon_k$ we have

$$\varrho(\varepsilon_k f) \leqslant \varrho\big(\lambda_k (f - s_m)\big) + \varrho(\lambda_k s_m) \leqslant \varrho(f - s_m) + \varrho(\lambda_k s_m).$$

We may take m_0 such that $\varrho(f - s_{m_0}) < \varepsilon/2$ and k_0 such that $\varrho(\lambda_k s_m) < \varepsilon/2$ for all $k \geqslant k_0$. Thus, $\varrho(\varepsilon_k f) < \varepsilon$ for $k \geqslant k_0$, i.e., $f \in L_\varrho = E_\varrho$. Since $L_\varrho = E_\varrho$ is linear, it follows from $f \in E_\varrho$ that $2f$ is also a member of E_ϱ. It was observed above, however, that $\varrho(2f, D_k) = \varrho(2f_k, D_k) > \eta$. Therefore, $\varrho(2f, \cdot)$ is a σ-subadditive submeasure which is not exhausting, i.e., $\varrho(2f, \cdot)$ must not be order continuous. The latter means that the function $2f$ does not belong to E_ϱ. This contradiction completes this part of the proof.

(a)$\Rightarrow$(d) It suffices to prove that $\varrho(f_n) \to 0$ implies $\varrho(2f_n) \to 0$ for $f_n \in L_\varrho$ (see [23], p. 18). Assume, therefore, that $f_n \in L_\varrho$ and $\varrho(f_n) \to 0$. There exists a subsequence (g_n) of (f_n) such that $g_n \to 0$ ϱ-a.e. (cf. Propositions 3.2 and 3.3).

By Lemma 6.3, we conclude that $\varrho(g_n, \cdot)$ are order equicontinuous; therefore, $\varrho(2g_n, \cdot)$ are also equicontinuous in virtue of the Δ_2-condition. Using Lemma 6.3 again we get $\varrho(2g_n) \to 0$. Hence, $\varrho(2f_n) \to 0$.

(d) $\Rightarrow$ (a) Let $f_n \in L_\varrho$ and let $\varrho(f_n, \cdot)$ be order equicontinuous. Assume to the contrary that there exists a sequence of sets $E_k \in \Sigma$ such that $E_k \searrow \emptyset$ and $\sup_n \varrho(2f_n, E_k)$ does not tend to zero. Choose an $\varepsilon > 0$ and a subsequence (g_k) of (f_n) such that $\varrho(2g_k, E_k) = \varrho(2g_k 1_{E_k}) > \varepsilon$. On the other hand, $\varrho(g_k, E_k) \leqslant \sup_n \leqslant \varrho(f_n, E_k) \to 0$. By (d) then $\varrho(2g_k, E_k) = \varrho(2g_k 1_{E_k}) \to 0$. Contradiction.

7. The case of non-monotone convex function modulars

In Definition 2.1 we assumed that the function modular was monotone with respect to the norm $|\cdot|$ in the Banach space S (property (P_2)). We may raise a question whether the theory developed through the precedings sections can be applied to the case of non-monotone functionals which have some similar properties to those considered in Sections 2, 3 and 4. This problem is of great importance to us since many "function modulars" like those introduced by Turett in [34] or by Kozek in [14], [15] are non-monotone. The more accurate question can be formulated as follows: is it possible (under some reasonable assumptions) to equip the Banach space S with the norm $\|\cdot\|_\varphi$ equivalent to the previous one such that ϱ will be non-decreasing with respect to the norm $\|\cdot\|_\varphi$?

In this section we shall demonstrate that for the convex case the answer is affirmative.

7.1a. DEFINITION. A functional ϱ: $M(X, S) \times \Sigma \to [0, \infty]$ is called a *non-monotone convex function modular* iff

(a_1) for every $f \in M(X, S)$, $\varrho(f, \cdot)$: $\Sigma \to [0, \infty]$ is a σ-submeasure,

(a_2) $\varrho(\cdot, X)$: $M(X, S) \to [0, \infty]$ is a convex modular,

(a_3) (P_4), (P_5) and (P_6) from Definition 2.1 are satisfied,

(a_4) $\mathrm{cl}\,\mathcal{E} = E_\varrho \subset L_\varrho \subset M(X, S)$.

7.1b. DEFINITION. A non-monotone convex function modular ϱ is said to be *accurate* if and only if there exists a partition (X_i) of X ($X_i \in \mathcal{P}$, X_i are mutually disjoint) such that the modular φ: $S \to [0, \infty]$ defined by $\varphi(r) = \sup_n \varrho(r 1_X, X_n)$ satisfies the following conditions:

(b_1) φ is continuous at zero in S,

(b_2) $\varphi(r) < \infty$ for every $r \in S$,

(b_3) for each $r_1, r_2 \in S$ such that $\varphi(r_1) \leqslant \varphi(r_2)$ there holds $\varphi(\alpha r_1) \leqslant \varphi(\alpha r_2)$ for all $\alpha > 0$,

(b_4) for $f \in M(X, S)$, $E \in \Sigma$ there holds $\varrho(f, E) = \sup\{\varrho(g, E)\colon g \in \mathscr{E}, \varphi(g(x)) \leqslant \varphi(f(x))$ for every $x \in E\}$.

Observe that on account of convexity of φ it follows from (b_2) that for every $r \in S$ the function $R \ni \mapsto \varphi(r\lambda)$ is continuous. Let ϱ be an accurate convex function modular, then we can equip S with a new norm $\|\cdot\|_\varphi$ defined by the formula

$$\|r\|_\varphi = \inf\{\alpha > 0\colon \varphi(r/\alpha) \leqslant 1\}, \qquad r \in S.$$

Since φ is continuous at zero, it follows that the modular space S_φ introduced by φ coincides with S. Note that by (b_3), the fact $\varphi(r_1) \leqslant \varphi(r_2)$ implies $\|r_1\|_\varphi \leqslant \|r_2\|_\varphi$. For a set $A \in \mathscr{P}$ put $\mathscr{C}(A) = \{r1_A\colon r \in S\}$.

7.2. Lemma. *For every $A \in \mathscr{P}$ the set $\mathscr{C}(A)$ is closed in L_ϱ.*

Proof. Let us note that it suffices to prove that $\mathscr{E} \setminus \mathscr{C}(A)$ is open since $\mathscr{E}$ is dense in the closed set E_ϱ. Let us fix a function $f \in \mathscr{E} \setminus \mathscr{C}(A)$; since f is simple, it follows that its range is a finite set $\{r_1, \ldots, r_k\} \subset S$. Let $\delta > 0$ be such that $\|r_1 - r_i\|_\varphi > 2\delta$ for $i = 2, \ldots, k$. Then for given $r \in S$ there holds either $\|r - r_1\|_\varphi > \delta$ or $\|r - r_i\|_\varphi > \delta$ for $i = 2, \ldots, k$. Thus, $\|r - f(x)\|_\varphi > \delta$ or $\|r - r_i\|_\varphi > \delta$ for $i = 2, \ldots, k$. Thus, $\|r - f(x)\|_\varphi > \delta$ for all $x \in E$ or $\|r - f(x)\|_\varphi > \delta$ for all $x \in F$, where $E = f^{-1}(\{r_1\})$, $F = f^{-1}(\{r_1, \ldots, r_k\})$. Let us suppose, for instance, that the first possibility arises. Hence,

$$\inf\left\{\alpha > 0\colon \varphi\left(\frac{r - f(x)}{\alpha}\right) \leqslant 1\right\} > \delta \qquad \text{for all } x \in E.$$

Therefore, for all $x \in E$ there holds

$$\varphi\left(\frac{r - f(x)}{\delta}\right) > 1.$$

Let $t \in \varphi^{-1}(\{1\})$. Then

$$\left\{x \in X\colon \varphi\left(\frac{r1_A(x) - f(x)}{\delta}\right) > 1\right\} = \left\{x \in X\colon \varphi\left(\frac{r1_A(x) - f(x)}{\delta}\right) > \varphi(t)\right\} \supset E.$$

By (a_1), (a_2) and (b_4) we obtain

$$\varrho\left(\frac{r1_A - f}{\delta}\right) \geqslant \varrho\left(\frac{r1_A - f}{\delta}, E\right) \geqslant \varrho(t1_E, E) > 0.$$

Denoting $k_E = \varrho(t1_E, E)$ we observe that two possibilities arise: if $k_E \geqslant 1$, then $\|r_1 - f\|_\varphi > \delta$; if $k_E < 1$, then for all $x \in E$

$$\varrho\left(\frac{r1_A - f}{\delta \cdot k_E}\right) \geqslant \varrho\left(\frac{r1_A - f}{\delta}\right) > 1 \qquad \text{for all } x \in E.$$

The last inequality implies that $\|r1_A - f\|_\varrho > \delta \cdot k_E$.

Finally, $\|r1_A - f\|_\varrho > \min(\delta, \delta \cdot k_E, \delta \cdot k_F) > 0$, where $k_F = \varrho(t1_F, F)$. Consequently, $\mathscr{E} \setminus \mathscr{C}(A)$ is an open subset of L_ϱ.

7.3. Lemma. *$(S, \|\cdot\|_\varphi)$ is complete.*

Proof. Let (r_n) be a Cauchy sequence in $(S, \|\cdot\|_\varphi)$. Let $A = X_k$ for a certain $k \in N$. Denoting $f_n = r_n 1_A \in \mathscr{C}(A)$ we observe that we have for all $\alpha > 0$

$$\varrho(\alpha(f_n - f_m)) = \varphi(\alpha(r_n - r_m)) \to 0 \quad \text{as } n, m \to \infty .$$

L_ϱ is complete, therefore, there exists a function $f \in L_\varrho$ such that $\|f_n - f_m\|_\varrho \to 0$. By Lemma 7.2, however, $\mathscr{C}(A)$ is closed in L_ϱ, then $f = r1_A$ for a certain $r \in S$. Finally, $\varphi(\alpha(r_n - r)) = \varrho(\alpha(f_n - f)) \to 0$.

7.4. Theorem. *The norm $\|\cdot\|_\varphi$ is equivalent to $|\cdot|$.*

Proof. Let us consider the identity map e: $(S, |\cdot|) \to (S, \|\cdot\|_\varphi)$. We shall prove that e is continuous. Indeed, let $|r_n| \to 0$, then for each $\alpha > 0$, $|\alpha r_n| \to 0$. Hence, $\|r_n\|_\varphi \to 0$ and e is continuous. Since e is a linear isomorphism and both spaces $(S, |\cdot|)$ and $(S, \|\cdot\|_\varphi)$ are complete then they are isomorphic, by the open mapping theorem. This completes the proof of the theorem.

As a consequence of properties (b_2) and (b_3) we obtain the next result.

7.5. Proposition. *There holds $\|r_1\|_\varphi \leqslant \|r_2\|_\varphi$ if and only if $\varphi(r_1) \leqslant \varphi(r_2)$.*

Proof. Since $\varphi(r_1) \leqslant \varphi(r_2)$ implies $\|r_1\|_\varphi \leqslant \|r_2\|_\varphi$, it suffices to prove that $\varphi(r_1) = \varphi(r_2)$ if $\|r_1\|_\varphi = \|r_2\|_\varphi$. Let $\|r_1\|_\varphi = \|r_2\|_\varphi = \alpha$. By the continuity of the function $R \ni \lambda \mapsto \varphi(\lambda r)$ for every $r \in S$, we conclude that $\varphi(r_1/\alpha) = \varphi(r_2/\alpha) = 1$ and by (b_3) we obtain $\varphi(r_1) = \varphi(r_2)$.

The following theorem is the main result of this section. It is an immediate consequence of Theorem 7.4, Proposition 7.5 and property (b_4).

7.6. Theorem. *Let ϱ be an accurate non-monotone convex function modular. Then ϱ is monotone with respect to the equivalent norm $\|\cdot\|_\varphi$, i.e.,*

(1) $\varrho(f, E) = \sup\{\varrho(g, E)\colon g \in \mathscr{E},\ \|g(x)\|_\varphi \leqslant f(x)\|_\varphi$ *for all* $x \in E\}$,

which implies also

(2) *if* $f, g \in M(X, S)$ *and* $\|g(x)\|_\varphi \leqslant \|f(x)\|_\varphi$ *for all* $x \in E$, *then* $\varrho(g, E) \leqslant \varrho(f, E)$.

8. Special cases

8.1. Musielak–Orlicz spaces may be regarded as modular function spaces in the sense of Definition 2.1. Following Musielak [23], we shall recall some basic concepts of the theory of Musielak–Orlicz spaces.

Let (X, Σ, μ) be a measure space. Assume μ to be a non-negative σ-

finite measure and denote by $\mathscr{P}$ the δ-ring of all sets of finite measure. Let us consider a function φ: $X \times \boldsymbol{R}_+ \to \boldsymbol{R}_+$ satisfying the following conditions:

(i) for every $x \in X$, $\varphi(x, \cdot)$ is non-decreasing, continuous function such that $\varphi(x, 0) = 0$, $\varphi(x, u) > 0$ for $u > 0$, $\varphi(x, u) \to \infty$ as $u \to \infty$;

(ii) $\varphi(\cdot, u)$ is a Σ-measurable, locally integrable function for all $u \geqslant 0$.

It is easily seen that

$$\text{(iii)} \qquad \varrho(f, E) = \int_E \varphi(x, |f(x)|)\, d\mu$$

is a function modular.

The modular space introduced by ϱ is called the *Musielak–Orlicz space* L^φ.

Our Theorem 3.6 corresponds to Theorem 7.7 in [23], Theorem 4.3 (the Vitali theorem) corresponds to Lemma 9.2 in [23], our Theorem 4.6 (the characterization of E_ϱ by means of simple functions) and Theorem 5.3 (the separability theorem) are similar to Theorems 7.6 and 7.10 in [23].

Let us remark that in the theory of Musielak–Orlicz spaces the subspace E^φ (E_ϱ in our notation) is called the *space of finite elements*, i.e., $f \in E^\varphi$ iff $\varrho(\lambda f) < \infty$ for all $\lambda > 0$. Furthermore, L_0^φ (L_ϱ^0 in our notation) is absorbing; For the modular given by formula (iii) the following Δ_2-condition is considered (see [23], p. 52):

$\varphi(x, 2u) \leqslant K\varphi(x, u) + h(x)$ for all $u \geqslant 0$ and almost every $x \in X$, where h is a non-negative, integrable function in X and K is a positive constant.

Modifying the proof of Theorem 8.4 from [23], one can observe that this condition is equivalent to Δ_2 from our Definition 6.4.

Let us note that our compactness Theorem 4.7 is analogous to Theorem 9.3 from [23] giving similar characterization of compact subsets of E_ϱ.

8.2. L. Drewnowski and A. Kamińska introduced in [7] a modular I_Φ defined on the space of measurable vector-valued functions by the formula

$$I_\Phi(f) = \sup_{\mu \in M} \int_X \Phi(f(x), x)\, d\mu,$$

where Φ: $S \times X \to [0, \infty]$ is an $\mathcal{N}'$-function (see [7], p. 178) and M is a family of countably additive non-negative measures on X.

If we assume additionally that Φ is non-decreasing, i.e., $\Phi(r_1, x) \leqslant \Phi(r_2, x)$ if $|r_1| \leqslant |r_2|$, then I_Φ may be regarded as a function modular in the sense of Section 1 of our paper.

Our Theorem 3.6 and Proposition 3.2 correspond to Theorem 2.2 in [7], Theorem 4.7 is similar to Theorem 1.2 from [12], our Theorem 5.3 to Theorem 3.1 from [7].

Observe that under some supplementary assumptions on the family M

of measures the assumptions of monotonicity of Φ may be omitted and the methods of Section 7 may be applied.

The reader is referred to [7] for interesting list of examples. We want only to stress that Musielak and Waszak in [26] (see also [7], p. 120) investigated similar generalization of Orlicz spaces. We shall deal with such spaces in Section 9.

8.3. As it was stated, Fenchel–Orlicz spaces introduced by Turett in [34] may be regarded as accurate non-monotone convex function modular spaces in the sense of Definition 7.1. The reader is referred to [34] for the definition of Fenchel–Orlicz spaces; we would only note that in Turett's main theorem (Theorem 2.20) there are included special versions of our Proposition 3.2 and Theorem 3.6. Some of ideas from [34] were employed by us in proving results of Section 7.

8.4. Dobrakov considered in [4], [5] linear operator valued measure m, i.e., m: $\mathscr{P} \to L(S, Y)$ is countably additive in the strong operator topology. For a simple function $f = \sum_{i=1}^{n} r_i 1_{E_i}$ $(r_i \in S,\ E_i \in \mathscr{P})$ the integral was defined there as follows: $\int_E f\,dm = \sum_{i=1}^{n} m(E \cap E_i) r_i$, where E belongs to Σ. Then the domain of integration was extended to the space $\mathscr{M}(m)$ of all $f \in M(X, S)$ such that there exists a sequence of simple functions (f_n) converging m-a.e. to f, for which the integrals $\big(\int_{(\cdot)} f_n\,dm\big)$ are uniformly countably additive on Σ. In Part II of his paper Dobrakov defined "the L_1-norm" on $M(X, S)$ by $\hat{m}(g, E) = \sup\{\|\int_E f\,dm\|: f \in \mathscr{E},\ |f(x)| \leqslant |g(x)| \text{ for each } x \in E\}$. Theorem 1 in [5] states that $\hat{m}$ is a function modular in the sense of our paper. Our Proposition 3.3 corresponds to Lemma 4 from [5], the space $L^1(m)$ in the sense of Dobrakov plays an identical role to E_ϱ in the theory presented here, our Theorem 3.6 corresponds to Dobrakov's Theorem 9, the Vitali and Lebesgue convergence theorems correspond to Theorems 16 and 17 and our Theorems 4.6 and 5.3 to Dobrakov's Theorems 8 and 20, respectively. Theorem 5.4 at last corresponds to Theorem 19 from [5].

8.5. Integration with respect to non-linear operator valued measures was considered by many authors in connection with representation of orthogonally additive operators (see e.g. [9] and [1]). We use here terminology and notation taken from recent papers [16] and [32].

Let F be a Banach space. By $N(S, F)$ we denote the space of all mappings U: $S \to F$ such that $U0 = 0$ and U are uniformly continuous on bounded subsets of S. A set function μ: $\mathscr{P} \to N(S, F)$ is said to be an *operator valued measure* if μ has the following properties:

(μ_1) $\mu(\emptyset) = 0$,

(μ_2) μ is countably additive in the point-wise sense,

(μ_3) for each $E \in \mathscr{P}$ and $\alpha > 0$ $sv_\delta(\mu_\alpha, E) \to 0$ as $\delta \to 0$,

(μ_4) for each $r \in S$ the submeasure majorant for μ denoted by $\bar{\mu}$ is order continuous on Σ.

Recall two definitions:

$$sv_\delta(\mu_\alpha, E) = \sup \Big\{ \Big\| \sum_{i=1}^{n} [\mu(E_i) r_i - \mu(E_i) r_i'] \Big\| : \bigcup_{i=1}^{n} E_i \subset E,\ E_i \in \mathscr{P},$$

$$|r_i|,\ |r_i'| \leqslant \alpha,\ |r_i - r_i'| \leqslant \delta,\ 1 \leqslant i \leqslant n,\ n \in N \Big\}$$

and

$$\bar{\mu}(E) r = \sup \{ \|\mu(A) r\| : A \subset E,\ A \in \mathscr{P} \}.$$

Integration of simple functions and extension of integration to the class $\mathscr{M}(\mu)$ can be done similarly as in the linear case. Defining then

$$\varrho(g, E) = \sup \Big\{ \Big\| \int_E f\, d\mu \Big\| : f \in \mathscr{E},\ |f(x)| \leqslant |g(x)| \text{ for each } x \in E \Big\}$$

and assuming additionally

(μ_5) if $E \in \Sigma$, $\int_E f\, d\mu = 0$ for all $f \in \mathscr{E}$ such that $|f(x)| \leqslant \alpha$ in E, then E is μ-null,

we obtain a function modular ϱ. Properties (P_1), (P_2) and (P_3) follow immediately from the definition of ϱ while (μ_3) implies (P_4), (μ_5) implies (P_5) and property (P_6) is a consequence of (μ_4).

The above introduced integral may be regarded as a non-linear operator $\mathscr{M}(\mu) \ni f \mapsto \int_X f\, d\mu \in F$. From this point of view we note two interesting properties of the space E_ϱ and of the class L_ϱ^0.

8.5a. Theorem. $L_\varrho^0 \subset \mathscr{M}(\mu)$.

Proof. We claim that $\|\int_E f\, d\mu\| \leqslant \varrho(f, E)$ for every $f \in \mathscr{M}(\mu)$ and $E \in \Sigma$. Since $f \in \mathscr{M}(\mu)$, it follows that there exists a sequence (s_n) of simple functions such that $s_n(x) \to f(x)$ and $|s_n(x)| \leqslant |f(x)|$ for all $x \in X$ and $\int_E s_n\, d\mu \to \int_E f\, d\mu$. For given $\varepsilon > 0$ we can choose a number $n \in N$ such that

$$\Big\| \int_E f\, d\mu - \int_E s_n\, d\mu \Big\| < \varepsilon.$$

Hence,

$$\Big\| \int_E f\, d\mu \Big\| \leqslant \varepsilon + \Big\| \int_E s_n\, d\mu \Big\| \leqslant \varepsilon + \varrho(f, E),$$

which implies that $\|\int_E f\, d\mu\| \leqslant \varrho(f, E)$ because ε was chosen arbitrarily.

Assume now that $f \in L_\varrho^0 \subset M(X, S)$; let (s_n) be such as it was mentioned above. Let $E_k \in \Sigma$, $E_k \searrow \emptyset$. For every $n, k \in N$ we have then

$$\|\int_{E_k} s_n \, d\mu\| \leqslant \varrho(s_n, E_k) \leqslant \varrho(f, E_k),$$

therefore, $\sup_n \|\int_E s_n \, d\mu\| \leqslant \varrho(f, E_k) \to 0$ because $f \in L_\varrho^0$. Thus, $(\int_{(\cdot)} s_n \, d\mu)$ are order equicontinuous and consequently $f \in \mathscr{M}(\mu)$ (cf. [32], Theorem 2.5).

8.5b. THEOREM. *The integral is a continuous operator acting from E_ϱ into F, i.e., if $f_n, f \in E_\varrho$ and $|f_n - f|_\varrho \to 0$, then, for every $E \in \Sigma$, $\int_E f_n \, d\mu \to \int_E f \, d\mu$.*

Proof. Let $E \in \Sigma$, $f_n, f \in E_\varrho$ and $|f_n - f|_\varrho \to 0$. Then, by Propositions 3.2 and 3.3, there exists a subsequence (g_n) of (f_n) such that $g_n \to f$ ϱ-a.e. By the Egoroff theorem, there exists a sequence (H_k) such that $H_k \in \mathscr{P}$, $H_k \nearrow E$ and $g_n \rightrightarrows f$ on every H_k. For given $\varepsilon > 0$ let us choose a natural number k such that $\varrho(2f, E \setminus H_k) < \varepsilon/8$ and an $N_1 \in N$ such that there holds $\varrho(2(g_n - f)) < \varepsilon/8$ for $n \geqslant N_1$. Thus,

$$\varrho(g_n, E \setminus H_k) \leqslant \varrho(2(g_n - f), E \setminus H_k) + \varrho(2f, E \setminus H_k) < \varepsilon/4.$$

Let us consider the following inequalities:

$$\begin{aligned} \|\int_E g_n \, d\mu - \int_E f \, d\mu\| &\leqslant \|\int_{H_k} g_n \, d\mu - \int_{H_k} f \, d\mu\| + \|\int_{E \setminus H_k} g_n \, d\mu\| + \|\int_{E \setminus H_k} f \, d\mu\| \\ &\leqslant \|\int_{H_k} g_n \, d\mu - \int_{H_k} f \, d\mu\| + \varrho(g_n, E \setminus H_k) + \varrho(f, E \setminus H_k) \\ &\leqslant \|\int_{H_k} g_n \, d\mu - \int_{H_k} f \, d\mu\| + \varepsilon/4 + \varepsilon/8 \leqslant \|\int_{H_k} g_n \, d\mu - \int_{H_k} f \, d\mu\| + \varepsilon/2 \end{aligned}$$

for $n \geqslant N_1$.

Let (G_m) be a sequence of sets from $\mathscr{P}$ such that $G_m \nearrow H_k$ and $f 1_{G_m}$ are bounded for all m. Similarly as it was done above we can take an m and $N_2 \geqslant N_1$ such that $\varrho(2f, H_k \setminus G_m) < \varepsilon/4$ for $n \geqslant N_2$. Since $g_n \rightrightarrows f$ on G_m and all g_n are bounded on G_m, it follows by (μ_3) that

$$\|\int_{G_m} g_n \, d\mu - \int_{G_m} f \, d\mu\| < \varepsilon \quad \text{for } n \geqslant N_3 \geqslant N_2 \geqslant N_1.$$

Hence,

$$\|\int_{H_k} g_n \, d\mu - \int_{H_k} f \, d\mu\| \leqslant \varepsilon + \|\int_{H_k \setminus G_m} g_n \, d\mu\| + \|\int_{H_k \setminus G_m} f \, d\mu\| \leqslant \varepsilon + \varepsilon/2$$

and finally

$$\|\int_E g_n \, d\mu - \int_E f \, d\mu\| < 2\varepsilon.$$

Since (f_n) was an arbitrary sequence converging to f in E_ϱ, it follows that the integral is a continuous operator.

Since many classical operators like Urysohn, Hammerstein or Nemytskii ones may be regarded as integrals with respect to suitable operator measures then Theorem 8.5b gives an answer to the question how to construct an F-space in which the given operator is continuous.

9. Countably modulared function spaces

Let (ϱ_n) be a sequence of function pseudomodulars, i.e.,

$$\varrho_n\colon M(X, S)\times\Sigma\to[0, \infty]$$

which satisfy conditions (P_1), (P_2), (P_3), (P_4) and (P_6). Assume that $f(x) = 0$ for all $x\in E$, if $\varrho_n(f, E) = 0$ for all $n = 1, 2, \ldots$ Following [26], [27], [23] let us define

$$\varrho(f, E) = \sum_{n=1}^{\infty} 2^{-n}\frac{\varrho_n(f, E)}{1+\varrho_n(f, E)}, \qquad \varrho_0(f, E) = \sup_n \varrho_n(f, E);$$

the convention $\infty/(1+\infty) = 1$ has been used here. Then L_ϱ and L_{ϱ_0} are called the *countably modulared space* and the *uniformly countably modulared space*, respectively.

It is well known ([23], Theorem 15.2) that both ϱ and ϱ_0 are modulars in $M(X, S)$, $L_\varrho = \bigcap_{i=1}^{\infty} L_{\varrho_i}$ and $L_{\varrho_0}\subset L_\varrho$. This embedding is continuous both with respect to modular convergence and F-norm convergence. We are, however, interested in a question, whether ϱ and ϱ_0 are the function modulars in the sense of Definition 2.1.

9.1. THEOREM. *The introduced above modular* ϱ *has properties* (P_1)–(P_6).

Proof. It follows immediately from properties of ϱ_n and of the real function $[0, \infty)\ni t\mapsto t(1+t)^{-1}$, that ϱ satisfies conditions (P_1), (P_2), (P_3) and (P_5). In order to prove (P_4) and (P_6), we shall check first that

$$\bar{\varrho}_\alpha(E)\leqslant\sum_{n=1}^{\infty} 2^{-n}\frac{(\bar{\varrho}_n)_\alpha(E)}{1+(\bar{\varrho}_n)_\alpha(E)}. \tag{9.1a}$$

Indeed,

$$\bar{\varrho}_\alpha(E) = \sup\{\varrho(g, E)\colon g\in\mathcal{E}, |g(x)|\leqslant\alpha \text{ for all } x\in E\}$$

$$= \sup\left\{\sum_{n=1}^{\infty} 2^{-n}\frac{\varrho_n(g, E)}{1+\varrho_n(g, E)}\colon g\in\mathcal{E}, |g(x)|\leqslant\alpha \text{ for all } x\in E\right\}$$

$$\leqslant\sum_{n=1}^{\infty} 2^{-n}\sup\left\{\frac{\varrho_n(g, E)}{1+\varrho_n(g, E)}\colon g\in\mathcal{E}, |g(x)|\leqslant\alpha \text{ for all } x\in E\right\}$$

$$\leqslant\sum_{n=1}^{\infty} 2^{-n}\frac{(\bar{\varrho}_n)_\alpha(E)}{1+(\bar{\varrho}_n)_\alpha(E)}.$$

The last inequality is caused by the fact that the function $t \mapsto t(1+t)^{-1}$ is non-decreasing. Since ϱ_n is a function pseudomodular, for every n separately $(\bar{\varrho}_n)_\alpha(E) \to 0$ as $\alpha \to 0^+$. Clearly,

$$\sum_{n=1}^{\infty} 2^{-n} \frac{(\bar{\varrho}_n)_\alpha(E)}{1+(\bar{\varrho}_n)_\alpha(E)} \to 0 \quad \text{as } \alpha \to 0^+;$$

hence, by (9.1a) we obtain that $\bar{\varrho}_\alpha(E) \to 0$ as $\alpha \to 0^+$. This completes the proof of (P_4); (P_6) may be proved similarly.

The next theorem deals with uniformly countably modulared spaces.

9.2. THEOREM. *ϱ_0 is a function modular if and only if the following conditions are satisfied, simultaneously*:

(a) $\sup_k (\bar{\varrho}_k)_\alpha(E_n) \to 0$ *for all* $\alpha > 0$, $E_n \in \mathscr{P}$, $E_n \searrow \emptyset$,

(b) $\sup_k (\bar{\varrho}_k)_\alpha(E) \to 0$ *as* $\alpha \to 0^+$, $E \in \mathscr{P}$.

It is clear that ϱ_0 satisfies conditions (P_1), (P_2), (P_3) and (P_5) while (P_4) and (P_6) are assumed in (a) and (b).

An answer to the question under which conditions both spaces L_ϱ and L_{ϱ_0} are equal may be formulated as follows:

9.3. THEOREM. $L_\varrho = L_{\varrho_0}$ *if and only if condition* (a) *from Theorem* 9.2 *holds and also*

(b′) $\sup_k \varrho_k(\lambda_n f, E) \to 0$ *for* $\lambda_n \to 0^+$, $f \in L_\varrho$, $E \in \mathscr{P}$.

We omit the easy proof and observe that (b′) implies (b) therefore, the following theorem holds:

9.4. THEOREM. *If* $L_\varrho = L_{\varrho_0}$, *then* ϱ_0 *is a function modular.*

Let us note that there are examples of $(\varrho_k)_{k=1}^{\infty}$ such that $\varrho_0 = \sup_k \varrho_k$ is a function modular while $L_{\varrho_0} \neq L_\varrho$.

9.5. EXAMPLE. Let m denote the Lebesgue measure in $X = [0, 1)$ and let $\varphi_1(x) = |x|$; for $k \geq 2$ put

$$\varphi_k(x) = \begin{cases} 0, & x \in [0, 1], \\ 2^{(m-1)^2}, & x \in [m-1, m),\ 2 \leq m \leq k, \\ 2^{k^2}, & x \geq k \end{cases}$$

and let $\varrho_k(f, E) = \int_E \varphi_k(f(x))\,dm$. Put $f = \sum_{n=1}^{\infty} n 1_{I_n}$, where $I_n \subset X$ are mutually disjoint and $m(I_n) = 1/2^n$.

We claim that $f \in L_\varrho = \bigcap_{k=1}^{\infty} L_{\varrho_k}$. Given a sequence $\gamma_m \to 0^+$, let us choose a subsequence (λ_j) such that $\lambda_j \leqslant 1/j$. Then we have $\varrho_1(\lambda_j f) = \lambda_j \sum_{n=1}^{\infty} n\, m(I_n) \leqslant c\lambda_j \to 0$, where $c = \sum_{n=1}^{\infty} n/2^n < \infty$ and for $k \geqslant 2$

$$\varrho_k(\lambda_j f) \leqslant \varrho_k(f/j) \leqslant 2^k \sum_{n=j}^{\infty} m(I_n) = 2^k \cdot \sum_{n=j}^{\infty} 2^{-n} \to 0 \quad \text{as } j \to \infty.$$

We used here the fact that for $n < j$ and $x \in I_n$ there holds $f(x)/j = n/j < 1$ and, consequently, $\varphi_k(f(x)/j) = 0$. On the other hand, for $n \geqslant kj$, $k \geqslant 2$, $x \in I_n$ there holds $f(x)/j = n/j \geqslant k$. Thus, $\varphi_k(f(x)/j) = 2^{k^2}$ and

$$\sup_k \varrho_k(f/j) \geqslant \sup_k 2^{k^2} \sum_{n=kj}^{\infty} m(I_n) = \sup_k 2^{k^2} \sum_{n=kj}^{\infty} 2^{-n}$$

$$= \sup_k 2^{k^2} \cdot 2 \cdot 2^{-kj} = \sup_k 2^{k^2-kj+1} = +\infty.$$

Hence, f does not belong to L_{ϱ_0}.

10. Interpolation of modular spaces

In this section we shall give an outline of an application of the Krbec interpolation method [18] to modular function spaces. Books [2] and [33] deal with general interpolation theory.

For the sake of simplicity we shall restrict our consideration to the case $S = R$ and convex function modulars ϱ_0, ϱ_1. Let σ: $(0, \infty) \to (0, \infty)$ be a measurable function such that $\int_0^{\infty} \min(1, t)\sigma(t)\,dt < \infty$. For given ϱ_0 and ϱ_1 we construct a σ-interpolated convex modular by the formula

$$\varrho_\sigma(f) = \int_0^{\infty} \mathscr{L}(t, f)\sigma(t)\,dt, \tag{10.1}$$

where $\mathscr{L}(t, f) = \inf\{\varrho_0(f_0) + t\varrho_1(f_1): f = f_0 + f_1,\ f_i \in L_{\varrho_i}\}$.

We shall prove that $\varrho_\sigma(f, E) = \int_0^{\infty} \mathscr{L}(t, f1_E)\sigma(t)\,dt$ satisfies conditions (P_1)–(P_6), i.e., it is a function modular.

Let us start with three preliminary results.

10.2. Proposition. *For every $E \in \Sigma$ we have $\varrho_\sigma(0, E) = 0$.*

Proof. Observe that for given $E \in \Sigma$, $01_E \in L_{\varrho_1}$ and $01_E \in L_{\varrho_0}$, therefore,

$$0 \leqslant \mathscr{L}(t, 01_E) \leqslant \varrho_0(01_E) + t\varrho_1(01_E) = \varrho_0(0, E) + t\varrho_1(0, E).$$

Hence, $\mathscr{L}(t, 01_E) = 0$ and consequently, $\varrho_\sigma(0, E) = 0$.

10.3. Proposition. *If $|f(x)| = |g(x)|$ for all $x \in X$, then $\mathscr{L}(t,f) = \mathscr{L}(t, g)$ for all $t > 0$. In particular, there holds $\mathscr{L}(t,f) = \mathscr{L}(t, |f|)$.*

Proof. Define $E_1 = \{x \in X: f(x) = g(x)\}$, $E_2 = \{x \in X: g(x) \neq 0$ and $f(x)/g(x) = -1\}$; then $E_1 \cup E_2 = X$, $E_1 \cap E_2 = \emptyset$, E_1 and E_2 belong to Σ. Let $f = f_0 + f_1$, $f_i \in L_{\varrho_i}$ and put $g_i = f_i 1_{E_1} - f_i 1_{E_2}$ $(i = 0, 1)$.

Thus, $g_i \in L_{\varrho_i}$ and $g_0 + g_1 = g$. Observe that $\varrho_i(h) = \varrho_i(j)$ for measurable functions h and j such that $|h| = |j|$, therefore, $\varrho_0(g_0) + t\varrho_1(g_1) = \varrho_0(f_0) + t\varrho_1(f_1)$. Hence, $\varrho_0(f_0) + t\varrho_1(f_1) \geqslant \mathscr{L}(t, g)$ and taking infimum over all partitions $f = f_0 + f_1$, $f_i \in L_{\varrho_i}$, we have $\mathscr{L}(t,f) \geqslant \mathscr{L}(t, g)$. The inverse inequality may be obtained similarly. Finally, $\mathscr{L}(t,f) = \mathscr{L}(t, g)$.

10.4. Proposition. *Let for each $x \in X$, $0 \leqslant f(x) \leqslant g(x)$; then $\varrho_\sigma(f, X) \leqslant \varrho_\sigma(g, X)$.*

Proof. It can easily be proved that to every partition $h = h_0 + h_1$ of a non-negative function $h \in L_{\varrho_0} + L_{\varrho_1}$ there corresponds a partition $h = \bar{h}_0 + \bar{h}_1$ such that $\bar{h}_i \in L_{\varrho_i}$, $\bar{h}_i \geqslant 0$ and $\varrho_i(\bar{h}_i) \leqslant \varrho_i(h_i)$. Hence, it suffices to take an arbitrary partition $g = g_0 + g_1$, $g_i \in L_{\varrho_i}$ and $g_i \geqslant 0$. Then, we have the following partition of the function f: $f = f_0 + f_1$, $f_0 = \min(g_0, f)$, $f_1 = f - f_0$. Let us note that $0 \leqslant f_i \leqslant g_i$ for $i = 0, 1$ and, therefore, $f_i \in L_{\varrho_i}$. We have

$$\mathscr{L}(t,f) \leqslant \varrho_0(f_0) + t\varrho_1(f_1) \leqslant \varrho_0(g_0) + t\varrho_1(g_1);$$

hence, $\mathscr{L}(t,f) \leqslant \mathscr{L}(t, g)$ for all $t > 0$ and finally, $\varrho_\sigma(f) \leqslant \varrho_\sigma(g)$.

10.5. Theorem. *For every $f \in \mathscr{E}$ the set function $\varrho_\sigma(f, \cdot)\colon \Sigma \to [0, \infty]$ is a σ-subadditive submeasure.*

Proof. It follows immediately from the definition of ϱ_σ and from Proposition 10.4 that $\varrho_\sigma(f, \emptyset) = 0$ and that $\varrho_\sigma(f, A) = \varrho_\sigma(f 1_A) \leqslant \varrho_\sigma(f 1_B) = \varrho_\sigma(f, B)$ for $A, B \in \Sigma$ such that $A \subset B$.

We claim that $\varrho_\sigma(g)$ is σ-subadditive on Σ for every fixed function $g \in \mathscr{E}$. Let us denote $A = \bigcup_{n=1}^{\infty} A_n$, where $A_n \in \Sigma$ are mutually disjoint and $f = g 1_A$. For every $n \in N$ we may choose an arbitrary partition of $f 1_{A_n}$, $f 1_{A_n} = (f 1_{A_n})_0 + (f 1_{A_n})_1$ such that $(f 1_{A_n})_i \in L_{\varrho_i}$ and $(f 1_{A_n})_i \geqslant 0$ $(i = 0, 1)$. Let us define the function f_i by $f_i = \sum_{n=1}^{\infty} (f 1_{A_n})_i$. Let us note that $f_i \in L_{\varrho_i}$. This follows from the Lebesgue dominated convergence theorem for ϱ_i and from the fact that $s_k(x) = \sum_{n=1}^{k} (f 1_{A_n})_i(x) \to f_i(x)$ and $0 \leqslant s_n(x)$ while the function $f \in \mathscr{E} \subset L_{\varrho_i}$ for $i = 0, 1$. Since clearly $f = f_0 + f_1$, then by σ-subadditivity of ϱ_0 and ϱ_1 we obtain

$$\mathcal{L}(t,f) \leqslant \varrho_0(f_0)+t\varrho_1(f_1) \leqslant \sum_{n=1}^{\infty} \left[\varrho_0((f1_{A_n})_0)+t\varrho_1((f1_{A_n})_1)\right].$$

Thus, $\mathcal{L}(t,f) \leqslant \sum_{n=1}^{\infty} \mathcal{L}(t,f1_{A_n})$ and

$$\begin{aligned}\varrho_\sigma(g, A) = \varrho_\sigma(f) &= \int_0^\infty \mathcal{L}(t,f)\sigma(t)\,dt \leqslant \int_0^\infty \Big(\sum_{n=1}^{\infty} \mathcal{L}(t,f1_{A_n})\Big)\sigma(t)\,dt \\ &= \sum_{n=1}^{\infty} \Big(\int_0^\infty \mathcal{L}(t,f1_{A_n})\sigma(t)\,dt = \sum_{n=1}^{\infty} \varrho_\sigma(f, A_n) \\ &= \sum_{n=1}^{\infty} \varrho_\sigma(g, A_n).\end{aligned}$$

Finally we have: $\varrho_\sigma(g, A) \leqslant \sum_{n=1}^{\infty} \varrho_\sigma(g, A_n)$, which is the desired result.

10.6. LEMMA. *There exists a constant $0 < c < \infty$ such that for every $\alpha > 0$ and $E \in \mathcal{P}$ there holds*

$$(\bar{\varrho}_\sigma)_\alpha(E) \leqslant c \max_{i=0,1} (\bar{\varrho}_i)_\alpha(E).$$

Proof. Fix $\alpha > 0$ and $E \in \mathcal{P}$. Since $\alpha 1_E \in \varepsilon$ it follows that $(\bar{\varrho}_\sigma)_\alpha(E) = \varrho_\sigma(\alpha 1_E) = \int_0^\infty \mathcal{L}(t, \alpha 1_E)\sigma(t)\,dt$. It can be easily seen that

$$\mathcal{L}(t, \alpha 1_E) \leqslant \begin{cases} t(\bar{\varrho}_1)_\alpha(E) & \text{for } 0 \leqslant t \leqslant 1, \\ (\bar{\varrho}_0)_\alpha(E) & \text{for } \quad t > 1, \end{cases}$$

$$\leqslant \begin{cases} \min(1, t)(\bar{\varrho}_1)_\alpha(E) \\ \min(1, t)(\bar{\varrho}_0)_\alpha(E) \end{cases} \leqslant \min(1, t) \max_{i=0,1} (\bar{\varrho}_i)_\alpha(E).$$

Thus,

$$(\bar{\varrho}_\sigma)_\alpha(E) = \int_0^\infty \mathcal{L}(t, \alpha 1_E)\sigma(t)\,dt \leqslant \max_{i=0,1} (\bar{\varrho}_i)_\alpha(E) \cdot \int_0^\infty \min(1, t)\sigma(t)\,dt.$$

Since $\int_0^\infty \min(1, t)\sigma(t)\,dt = c < \infty$ then finally

$$(\bar{\varrho}_\sigma)_\alpha(E) \leqslant c \max_{i=0,1} (\bar{\varrho}_i)_\alpha(E).$$

The lemma is completely proved.

From Propositions 10.3, 10.4 and Theorem 10.5 it follows immediately that $\bar{\varrho}_\sigma$ satisfies properties (P_1), (P_2) and (P_3). From Lemma 10.6 it follows that $\bar{\varrho}_\sigma$ satisfies also (P_4), (P_5) and (P_6) since ϱ_i $(i = 0, 1)$ have those properties. Thus we proved the following theorem.

10.7. THEOREM. *If ϱ_0, ϱ_1 are function modulars, then $(L_{\varrho_0}, L_{\varrho_1})_\sigma = L_{\varrho_\sigma}$, where ϱ_σ is given by* (10.1); *ϱ_σ is a function modular in the sense of Definition* 2.1.

10.8. THEOREM. *If ϱ_0, ϱ_1 are orthogonally additive, then ϱ_σ given by* (10.1) *is orthogonally additive as well.*

Proof. It suffices to prove that $\varrho_\sigma(f1_A)+\varrho_\sigma(f1_B) \leqslant \varrho_\sigma(f)$ for $A, B \in \Sigma$ such that $A \cap B = \emptyset$ and $A \cup B = X$. Let $f = f_0+f_1$, $f_i \in L_{\varrho_i}$. Then $f1_A = f_0 1_A + f_1 1_A$ and $f1_B = f_0 1_B + f_1 1_B$. Let us fix a $t > 0$. Compute

$$\mathscr{L}(t, f1_A)+\mathscr{L}(t, f1_B) \leqslant \varrho_0(f_0 1_A)+t\varrho_1(f_1 1_A)+\varrho_0(f_0 1_B)+t\varrho_1(f_1 1_B)$$
$$= \varrho_0(f_0)+t\varrho_1(f_1).$$

Thus, $\mathscr{L}(t, f1_A)+\mathscr{L}(t, f1_B) \leqslant \mathscr{L}(t, f)$, which gives the desired inequality: $\varrho_\sigma(f1_A)+\varrho_\sigma(f1_B) \leqslant \varrho_\sigma(f)$.

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