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Notes on modular function spaces. I

Abstract. Let Σ be a σ -algebra which is local in a δ -ring \mathcal{P} of subsets of a set X such that $X = \bigcup_{i=1}^{\infty} X_i$, $X_i \in \mathcal{P}$. By a function modular we mean a functional $\varrho: \mathcal{E} \times \Sigma \rightarrow [0, \infty]$ (\mathcal{E} denotes the space of all \mathcal{P} -integrable simple functions) such that $\varrho(0, E) = 0$ for $E \in \Sigma$; if $|f| \leq |g|$ in E , then $\varrho(f, E) \leq \varrho(g, E)$; $\varrho(f, \cdot): \Sigma \rightarrow [0, \infty]$ is a σ -subadditive set function. We assume also that $\varrho(\cdot, E): \mathcal{E} \rightarrow [0, \infty]$ is order continuous in \mathcal{P} and continuous with respect to the uniform convergence topology in \mathcal{E} . Having extended the domain of ϱ to all measurable function by the formula $\varrho(f, E) = \sup \{\varrho(g, E): g \in \mathcal{E}, |g| \leq |f| \text{ in } E\}$, we observe that the functional $\varrho(\cdot, X)$ is a modular defined in the space of all measurable functions; the modular space induced by ϱ will be denoted by L_{ϱ} . In Section 3 we shall prove that the modular space L_{ϱ} is complete. Section 4 contains the definition and some basic properties of the subspace E_{ϱ} of finite elements while Section 5 contains the separability theorem.

Introduction

The aim of this paper is to develop a theory of modular spaces of measurable vector-valued functions. This theory is based on the notion of function modulars, i.e., functionals which depend both on measurable functions and measurable sets.

The first attempts to generalize the classical Lebesgue spaces L^p were initiated by Biernbaum and Orlicz in [3], [29], [30]. The theory of Orlicz spaces was later developed by many mathematicians (see Krasnoselskiĭ and Rutickiĭ [17], Musielak [23] for review). A theory of more general class of function spaces was initiated in 1955 by W. A. J. Luxemburg in his PhD thesis [21] and followed by a series of papers of W. A. J. Luxemburg and A. C. Zaanen (see also [22]). Recently a new interest in Orlicz spaces is emerging in connection with problems of convexity in Banach spaces, the Boyd indices and with the theory of rearrangement invariant function spaces (see the books of J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces*, I, II, [19], [20]).

The theory of modular spaces was founded by Nakano [28], then developed by Musielak and Orlicz [24], [25] and their pupils (see [23]). On

the base of the modular theory the same authors developed in [24] the theory of Musielak–Orlicz spaces, i.e., Orlicz spaces with a function φ depending on a parameter. Such spaces were objects of mathematical study for over twenty years (see e.g. [11] and [13]) and were generalized in many directions. Among others, Turett in [34] and Kozek in [14], [15] considered generalizations of Orlicz spaces to the case of vector-valued functions; Musielak and Waszak [26], [27], Rosenberg [31], Drewnowski and Kamińska [7] considered cases of spaces generated by family of Orlicz or Musielak–Orlicz modulars.

Preliminaries

Let us recall some basic concepts of the general theory of modular spaces after [23].

Let \mathcal{X} be a real or complex vector space. A functional $\varrho: \mathcal{X} \rightarrow [0, \infty]$ is called a *pseudomodular* if there holds for arbitrary $x, y \in \mathcal{X}$:

$$(A) \quad \varrho(0) = 0.$$

$$(B) \quad \varrho(\alpha x) = \varrho(x) \text{ for every } \alpha \in K \text{ (} K = C \text{ or } K = R \text{) such that } |\alpha| = 1.$$

$$(C) \quad \varrho(\alpha x + \beta y) \leq \varrho(x) + \varrho(y) \text{ for } \alpha, \beta \geq 0, \alpha + \beta = 1.$$

If in place of (C) there holds

(C') $\varrho(\alpha x + \beta y) \leq \alpha^s \cdot \varrho(x) + \beta^s \cdot \varrho(y)$ for $\alpha, \beta \geq 0, \alpha^s + \beta^s = 1, s \in (0, 1]$, then the pseudomodular ϱ is called *s-convex*; 1-convex pseudomodular is called *convex*.

If we replace (A) by

$$(A') \quad \varrho(0) = 0 \text{ and } \varrho(\lambda x) = 0 \text{ for all } \lambda > 0 \text{ implies } x = 0,$$

then ϱ is called a *semimodular*.

If, moreover,

$$(A'') \quad \varrho(0) = 0 \text{ and } \varrho(x) = 0 \text{ implies } x = 0,$$

then ϱ is called a *modular*.

If ϱ is a pseudomodular in \mathcal{X} , then

$$\mathcal{X}_\varrho = \{x \in \mathcal{X}: \lim_{\lambda \rightarrow 0} \varrho(\lambda x) = 0\}$$

is called a *modular space*; \mathcal{X}_ϱ is a vector subspace of \mathcal{X} . For a pseudomodular ϱ in \mathcal{X} we may define an *F-pseudonorm* by the formula

$$\|x\|_\varrho = \inf \{u > 0: \varrho(x/u) \leq u\}.$$

If ϱ is an *s-convex* pseudomodular, then the functional

$$\|x\|_\varrho^s = \inf \{u > 0: \varrho(x/u^{1/s}) \leq 1\}$$

is an *s-pseudonorm* in \mathcal{X} (a pseudonorm for $s = 1$).

Observe that the previous formulas defined *F-norms* and *s-norms*, respectively, if ϱ is a modular or at least a semimodular. One can check that

$|x_k - x|_\rho \rightarrow 0$ is equivalent to the condition $\rho(\alpha(x_k - x)) \rightarrow 0$ for all $\alpha > 0$. Recall that a sequence (x_k) is called *modular convergent* to $x \in \mathcal{X}_\rho$ if there exists a $\lambda > 0$ such that $\rho(\lambda(x_k - x)) \rightarrow 0$ as $k \rightarrow \infty$.

A modular ρ is called

(a) *right-continuous* if $\lim_{\lambda \rightarrow 1^+} \rho(\lambda x) = \rho(x)$ for all $x \in \mathcal{X}_\rho$,

(b) *left-continuous* if $\lim_{\lambda \rightarrow 1^-} \rho(\lambda x) = \rho(x)$ for all $x \in \mathcal{X}_\rho$,

(c) *continuous* if it is both right- and left-continuous.

Let X be a non-empty set and \mathcal{P} a non-trivial δ -ring (a ring closed with respect to the forming of countable intersections) of subsets of X . Let $\Sigma \supset \mathcal{P}$ be the smallest σ -algebra of subsets of X such that Σ contains \mathcal{P} and

(i) Σ is local in \mathcal{P} , i.e., $E \cap A \in \mathcal{P}$ for every $E \in \mathcal{P}$, $A \in \Sigma$,

(ii) there exists a non-decreasing sequence $X_1 \subset X_2 \subset \dots$, $X_i \in \mathcal{P}$, such that $X = \bigcup_{i=1}^{\infty} X_i$.

If $E \subset X$, then I_E will stand for its characteristic function. By a \mathcal{P} -simple function on X with values in a Banach space S we mean a function of the form

$$g = \sum_{i=1}^n r_i I_{E_i}, \quad r_i \in S, E_i \in \mathcal{P}, E_i \cap E_j = \emptyset \text{ for } i \neq j.$$

The linear space of all \mathcal{P} -simple function will be denoted by $\mathcal{E}(\mathcal{P})$ or, briefly, by \mathcal{E} .

A function $f: X \rightarrow S$ is called *measurable* if there is a sequence of \mathcal{P} -simple functions (f_n) such that $\lim_n f_n(x) = f(x)$ for every $x \in X$. It may be shown that a function f is measurable if and only if it is separable valued in S and for every open $B \subset S$, $f^{-1}(B) \in \Sigma$, cf. [8], III, 6.9. Thus, if f is measurable, then $|f|$ is a scalar measurable function. It follows from the properties of scalar measurable functions that to every measurable function f there corresponds a sequence of \mathcal{P} -simple functions (f_n) such that $\lim_n f_n(x) = f(x)$ and the sequence of norms $(|f_n(x)|)$ is non-decreasing for each $x \in X$, see [10], Theorem B in Section 20. The vector space consisting of all measurable functions $f: X \rightarrow S$ will be denoted by $M(X, S)$.

The following definitions are taken from [6].

A set function $\eta: \Sigma \rightarrow [0, \infty]$ will be called a *submeasure* (*subadditive measure*) on Σ iff

(α) $\eta(\emptyset) = 0$,

(β) $\eta(A \cup B) \leq \eta(A) + \eta(B)$ whenever $A, B \in \Sigma$, $A \cap B = \emptyset$,

(γ) $\eta(A) \leq \eta(B)$ if $A, B \in \Sigma$, $A \subset B$.

If, moreover, $\eta\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \eta(E_n)$ for each sequence $E_n \in \Sigma$, then η will be called σ -subadditive submeasure or simply σ -submeasure.

A submeasure η is said to be order *continuous* if for each sequence $(E_n) \subset \Sigma$ such that $E_n \searrow \emptyset$, $\eta(E_n) \rightarrow 0$ as $n \rightarrow \infty$.

2. Function modulars

2.1. DEFINITION. A functional $\varrho: \mathcal{E} \times \Sigma \rightarrow [0, \infty]$ is called a *function modular* if and only if the following conditions are satisfied:

- (P₁) $\varrho(0, E) = 0$ for each $E \in \Sigma$,
- (P₂) if $E \in \Sigma$, $f, g \in \mathcal{E}$ and $|f(x)| \leq |g(x)|$ for all $x \in E$, then $\varrho(f, E) \leq \varrho(g, E)$,
- (P₃) for every $f \in \mathcal{E}$, $\varrho(f, \cdot): \Sigma \rightarrow [0, \infty]$ is a σ -submeasure,
- (P₄) for every $E \in \mathcal{A}$, $\bar{\varrho}_\alpha(E) \rightarrow 0$ as $\alpha \rightarrow 0^+$, where, for every $E \in \Sigma$,

$$\bar{\varrho}_\alpha(E) = \sup \{ \varrho(g, E) : g \in \mathcal{E}, |g(x)| \leq \alpha \text{ for every } x \in E \},$$

- (P₅) if $\bar{\varrho}_\alpha(E) = 0$ for an $\alpha > 0$, $E \in \Sigma$, then $\bar{\varrho}_\beta(E) = 0$ for every $\beta > 0$,
- (P₆) $\bar{\varrho}_\alpha$ is order continuous on \mathcal{A} for all $\alpha > 0$.

The definition of ϱ is extended to measurable f by defining that

$$\varrho(f, E) = \sup \{ \varrho(g, E) : g \in \mathcal{E}, |g(x)| \leq |f(x)| \text{ for each } x \in E \}.$$

For the sake of simplicity we shall write $\varrho(f)$ instead of $\varrho(f, X)$. The following properties of function modulars are the immediate consequences of the definition.

2.2 PROPOSITION. For $f, g \in M(X, S)$ and $E \in \Sigma$ there holds:

- (a) $|f(x)| \leq |g(x)|$ for all $x \in E$ implies that $\varrho(f, E) \leq \varrho(g, E)$,
- (b) for every $f \in M(X, S)$, $\varrho(f, \cdot): \Sigma \rightarrow [0, \infty]$ is a σ -submeasure,
- (c) if $f(x) = g(x)$ in E , then $\varrho(f, E) = \varrho(g, E)$,
- (d) if $|f(x)| \leq |g(x)|$ for each $x \in X$, then $\varrho(f, E) \leq \varrho(g, E)$,
- (e) if $f(x) = 0$ for each $x \in E$, then $\varrho(f, E) = 0$,
- (f) $\varrho(f, E) = \varrho(f, E \cap \text{supp } f)$, where $\text{supp } f = \{x \in X : f(x) \neq 0\}$,
- (g) $\varrho(f, E) = \varrho(f1_E, E)$.

2.3. DEFINITION. A set $E \in \Sigma$ is said to be ϱ -null iff $\bar{\varrho}_\alpha(E) = 0$ for every $\alpha > 0$.

Obviously, a set $E \in \Sigma$ is ϱ -null if and only if $\varrho(g, E) = 0$ for every $g \in \mathcal{E}$. Furthermore, if E is ϱ -null, then $\varrho(f, E) = 0$ for each $f \in M(X, S)$. In view of (P₅) if $\bar{\varrho}_\alpha(E) = 0$ for an $\alpha > 0$, then E is ϱ -null.

By the use of the obvious fact that a finite (even countable) union of ϱ -null sets is still ϱ -null and that for every $E \in \mathcal{P}$ and $\alpha > 0$ there holds $\varrho(r1_E, E) = 0$ whenever $\bar{\varrho}_\alpha(E) = 0$ and $r \in S$, $|r| = \alpha$, we can easily prove the following result.

2.4. PROPOSITION. The following conditions are equivalent:

- (a) ϱ has property (P₅),
- (b) if $E \in \mathcal{P}$, $f \in \mathcal{E}$ and $\varrho(f, E) = 0$, then $E \cap \text{supp } f$ is ϱ -null.

By the assumption $X = \bigcup_{i=1}^{\infty} X_i$, $X_1 \subset X_2 \subset \dots$, $X_i \in \mathcal{P}$ we can conclude that if ϱ satisfies (P₃), then condition (b) in Proposition 2.4 holds for every $E \in \Sigma$ and $f \in M(X, S)$. Therefore $\text{supp } f$ is ϱ -null (i.e., $f = 0$ ϱ -almost everywhere) if $\varrho(f) = \varrho(f, X) = 0$. Hence, identifying in $M(X, S)$ functions which differ only on ϱ -null sets we get a functional $\varrho: M(X, S) \rightarrow [0, \infty]$ which may be regarded as a modular. Namely, we have the following theorem.

2.5. THEOREM. *The functional $\varrho: M(X, S) \rightarrow [0, \infty]$ is a modular.*

Proof. It is evident that only condition (C) of the definition of modular needs proving. Let $f_1, f_2 \in M(X, S)$ and $0 \leq \alpha, \beta \leq 1$ be such that $\alpha + \beta = 1$. Observe that

$$|\alpha f_1(x) + \beta f_2(x)| \leq \alpha \cdot |f_1(x)| + \beta \cdot |f_2(x)| \leq \max\{|f_1(x)|, |f_2(x)|\}.$$

Put $E_1 = \{x \in X: |f_1(x)| \geq |f_2(x)|\}$, $E_2 = X \setminus E_1$ and $g = f_1 I_{E_1} + f_2 I_{E_2}$. Then $|\alpha f_1(x) + \beta f_2(x)| \leq |g(x)|$ for every $x \in X$ and

$$\begin{aligned} \varrho(\alpha f_1 + \beta f_2) &\leq \varrho(g) \leq \varrho(g, E_1) + \varrho(g, E_2) \\ &= \varrho(f_1, E_1) + \varrho(f_2, E_2) \leq \varrho(f_1) + \varrho(f_2). \end{aligned}$$

According to the general modular theory we can define now a modular space $L_\varrho = \{f \in M(X, S): \varrho(\lambda f) \rightarrow 0 \text{ as } \lambda \rightarrow 0^+\}$. We shall equip L_ϱ with an F -norm $|\cdot|_\varrho$ (norm in the case of convex ϱ) induced by the modular ϱ . It is evident in view of (P₄) that L_ϱ contains all bounded functions with supports from \mathcal{P} ; in particular, $\mathcal{E} \subset L_\varrho$. Therefore (P₄) guarantees the fact that the space L_ϱ is not only non-trivial but it is large enough.

The same condition (P₄) implies another important fact.

2.6. PROPOSITION. *If $f_n, f \in M(X, S)$, $E \in \mathcal{P}$ and $f_n \xrightarrow{\varrho} f$ in E , then $\varrho(\alpha(f_n - f), E) \rightarrow 0$ for all $\alpha > 0$, i.e., $|(f_n - f) I_E|_\varrho \rightarrow 0$.*

On account of the obvious analogy with the classical case we put the following definition.

2.7. DEFINITION. A function modular ϱ is said to have the *Fatou property* iff for every sequence $(f_n)_{n=0}^{\infty}$ of measurable functions such that $f_n \rightarrow f_0$ ϱ -a.e. and $|f_1(x)| \leq |f_2(x)| \leq \dots \leq |f_0(x)|$ for all $x \in X$, there holds $\varrho(f_n) \nearrow \varrho(f_0)$.

2.8. THEOREM. *The following conditions are equivalent:*

- (i) ϱ has the Fatou property,
- (ii) ϱ is a left-continuous modular.

Proof. (i) \Rightarrow (ii). Evident.

(ii) \Rightarrow (i). Suppose $|f_n| \nearrow |f|$ ϱ -a.e. and write $\gamma = \lim_n \varrho(f_n)$. Since $(\varrho(f_n))$ is

non-decreasing sequence, we conclude that γ is well defined and for all natural n there holds $\varrho(f_n) \leq \gamma \leq \varrho(f)$. Thus, it suffices to prove that $\gamma \geq \varrho(f)$.

Let $H \in \Sigma$ be a ϱ -null exceptional set, i.e., $H = \{x \in X: f_n(x) \nrightarrow f(x)\}$ and let $W = X \setminus H$. Let us choose a \mathcal{P} -simple function g such that $|g(x)| \leq |f(x)|$ for every x and let us fix a number $\lambda \in (0, 1)$. Let $E_n = \{x \in W: \lambda |g(x)| \leq |f_n(x)|\}$ and $D_n = W \setminus E_n$; we observe that $D_1 \supset D_2 \supset \dots$. It follows from the definition of D_n that $x \in \bigcap_{n=1}^{\infty} D_n$ if and only if $x \in W$ and $|f_n(x)| < \lambda |g(x)| < |f(x)|$ for all $n = 1, 2, \dots$; therefore, $f_n(x) \nrightarrow f(x)$ which contradicts the fact that $x \in W$. Thus, $\bigcap_{n=1}^{\infty} D_n = \emptyset$ and consequently we have $D_n \searrow \emptyset$. It is evident that $D_n \subset W \cap \text{supp } g$; since $\text{supp } g \in \mathcal{P}$, we see that $D_n \in \mathcal{P}$. Fix $\varepsilon > 0$; by (P_6) there exists a natural n_0 such that $\varrho(\lambda g, D_n) < \varepsilon$ for $n \geq n_0$. Thus,

$$\varrho(\lambda g) \leq \varrho(\lambda g, H) + \varrho(\lambda g, E_n) + \varrho(\lambda g, D_n) \leq \varrho(f_n, E_n) + \varepsilon \leq \varrho(f_n) + \varepsilon \leq \gamma + \varepsilon.$$

Since $\varepsilon > 0$ was chosen arbitrary, we conclude that $\varrho(\lambda g) \leq \gamma$ which implies $\varrho(g) \leq \gamma$, because ϱ is left-continuous. The latter inequality holds for arbitrary $g \in \mathcal{E}$ such that $|g(x)| \leq |f(x)|$, so that $\varrho(f) \leq \gamma$, and the proof is complete.

2.9. THEOREM. *If ϱ has the Fatou property, then for every $f_n \in M(X, S)$, $n = 0, 1, \dots$, $|f_n|_{\varrho} \nearrow |f|_{\varrho}$ whenever $f_n \rightarrow f$ ϱ -a.e. and $|f_1(x)| \leq |f_2(x)| \leq \dots \leq |f(x)|$ for every $x \in X$.*

Proof. It follows from Proposition 2.2 (d) and from the definition of the F -norm that the sequence of F -norm $(|f_n|_{\varrho})$ is non-decreasing. Denoting then $\gamma = \lim_n |f_n|_{\varrho}$, assume to the contrary that $\gamma + \varepsilon < |f|_{\varrho}$ for some $\gamma \geq 0$ and $\varepsilon > 0$. Let $\alpha_n > 0$ be such that $\varrho(f_n/\alpha_n) \leq \alpha_n$ and $\alpha_n < \gamma + \varepsilon$; such α_n exists, because $|f_n|_{\varrho} < \gamma + \varepsilon$ for all $n \in N$. Hence

$$\varrho(f_n/\gamma + \varepsilon) \leq \varrho(f_n/\alpha_n) \leq \alpha_n < \gamma + \varepsilon.$$

Since ϱ has the Fatou property, it follows that $\varrho(f_n/\gamma + \varepsilon) \nearrow \varrho(f/\gamma + \varepsilon)$ so that $\varrho(f/\gamma + \varepsilon) \leq \gamma + \varepsilon$ and consequently $|f|_{\varrho} \leq \gamma + \varepsilon$. Contradiction.

2.10. COROLLARY. *If ϱ has the Fatou property, then for every $f \in M(X, S)$*

$$|f|_{\varrho} = \sup \{|g|_{\varrho}: g \in \mathcal{E}, |g(x)| \leq |f(x)| \text{ for each } x \in X\}.$$

3. Completeness of L_{ϱ}

In the previous section we introduced the submeasure $\bar{\varrho}_{\alpha}: \Sigma \rightarrow [0, \infty]$, $\alpha > 0$; it is an immediate fact that for every bounded measurable function f and $E \in \Sigma$ we have

$$\bar{\varrho}_{\alpha}(E) \leq \varrho(f, E) \leq \bar{\varrho}_{\beta}(E),$$

where $\alpha = \inf_{x \in E} |f(x)|$, $\beta = \sup_{x \in E} |f(x)|$. The notion of $\bar{\varrho}_\alpha$ will be used to define another convergence in $M(X, S)$.

3.1. DEFINITION. Let $f_n, f \in M(X, S)$. We say that (f_n) converges to f in the submeasure $\bar{\varrho}_\alpha$ and write $f_n \rightarrow f(\bar{\varrho}_\alpha)$ if and only if for every $\varepsilon > 0$

$$\bar{\varrho}_\alpha(\{x \in X: |f_n(x) - f(x)| \geq \varepsilon\}) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

We say that (f_n) converges to f in the submeasure $\bar{\varrho}$ and write $f_n \rightarrow f(\bar{\varrho})$ if and only if $f_n \rightarrow f(\bar{\varrho}_\alpha)$ for all $\alpha > 0$.

The definition of Cauchy sequences is analogous.

3.2. PROPOSITION. (a) If (f_n) is a Cauchy sequence in L_ϱ , then (f_n) is a Cauchy sequence in submeasure $(\bar{\varrho})$,

(b) if $f_n, f \in L_\varrho$, $|f_n - f|_\varrho \rightarrow 0$, then $f_n \rightarrow f(\bar{\varrho})$.

Proof. (a) Let us fix $\varepsilon, \alpha > 0$ and denote

$$E_{n,k}(\varepsilon) = \{x \in X; |f_n(x) - f_k(x)| \geq \varepsilon\};$$

then

$$\bar{\varrho}_\varepsilon(E_{n,k}(\varepsilon)) \leq \varrho(f_n - f_k, E_{n,k}(\varepsilon)) \leq \varrho(f_n - f_k).$$

Hence

$$\bar{\varrho}_\alpha(E_{n,k}(\varepsilon)) \leq \varrho\left(\frac{\alpha}{\varepsilon}(f_n - f_k), E_{n,k}(\varepsilon)\right) \leq \varrho\left(\frac{\alpha}{\varepsilon}(f_n - f_k)\right) \rightarrow 0$$

as $n, k \rightarrow \infty$.

Proof of (b) is similar.

3.3. PROPOSITION. (a) Given $\alpha > 0$, let a sequence (f_n) satisfy the Cauchy condition in submeasure $\bar{\varrho}_\alpha$; then there are $f \in M(X, S)$ and a subsequence (g_n) of (f_n) such that $f_n \rightarrow f(\bar{\varrho}_\alpha)$ and $g_n \rightarrow f$ ϱ -a.e.

(b) (The Egoroff Theorem). Let $f_n, f \in M(X, S)$ and $f_n \rightarrow f$ ϱ -a.e. There exists a non-decreasing sequence of sets $H_k \in \mathcal{P}$ such that $\bigcup_{k=1}^{\infty} H_k = X$ and $f_n \rightarrow f$ on every H_k .

We omit the proof of the previous results, because they are rather simple modifications of the standard proof. Nevertheless we want to call the reader's attention to the fact that proving assertion (a) of Proposition 3.3 we conclude that $\bar{\varrho}_\alpha(\{x \in X: g_n(x) \neq f(x)\}) = 0$; it follows from (P₅) that this set is ϱ -null.

The next proposition is also an easy result of (P₅).

3.4. PROPOSITION. If $f_n \rightarrow g_1(\bar{\varrho}_\alpha)$ and $f_n \rightarrow g_2(\bar{\varrho}_\alpha)$, then $g_1 = g_2$ a.e.

3.5. PROPOSITION. If (f_k) is a Cauchy sequence in $(\bar{\varrho})$, then there exists a function $f \in M(X, S)$ such that $f_k \rightarrow f(\bar{\varrho})$.

Proof. In virtue of Proposition 3.3 (a) to every $\alpha > 0$ there exists a function $h_\alpha \in M(X, S)$ such that $f_k \rightarrow h_\alpha(\bar{\varrho}_\alpha)$. Let us choose an arbitrary

sequence of scalars $1 = \alpha_1 < \alpha_2 < \dots$, $\alpha_n \rightarrow \infty$ and define $g_n = h_{\alpha_n}$. If $i \geq j$, then $f_k \rightarrow g_i(\bar{\varrho}_{\alpha_i})$, because $f_k \rightarrow g_i(\bar{\varrho}_{\alpha_i})$ and $\bar{\varrho}_{\alpha_j} \leq \bar{\varrho}_{\alpha_i}$. On the other hand, since $f_k \rightarrow g_j(\bar{\varrho}_{\alpha_j})$, it follows by Proposition 3.4 that $g_i = g_j$ ϱ -a.e. Put $f = g_1$ and take an $\alpha > 0$. For a certain natural n we have $\alpha \leq \alpha_n$. For given $\varepsilon > 0$ we have

$$\begin{aligned} \bar{\varrho}_\alpha(\{x \in X: |f_k(x) - f(x)| \geq \varepsilon\}) &\leq \bar{\varrho}_{\alpha_n}(\{x \in X: |f_k(x) - f(x)| \geq \varepsilon\}) \\ &= \bar{\varrho}_{\alpha_n}(\{x \in X: |f_k(x) - g_n(x)| \geq \varepsilon\}) \rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned}$$

Finally, $f_k \rightarrow f(\bar{\varrho}_\alpha)$. It follows then that $f_n \rightarrow f(\bar{\varrho})$, because α was fixed arbitrary.

Proposition 3.2 (a) and 3.5 are the main tools for proving the main result of this section, the theorem on the completeness of L_ϱ .

3.6. THEOREM. *The linear metric space L_ϱ is complete.*

Proof. Let (f_n) be a Cauchy sequence in $(L_\varrho, |\cdot|_\varrho)$. By Proposition 3.2 (a) (f_n) is also a Cauchy sequence in the sense of convergence in submeasure $\bar{\varrho}$; hence, in virtue of Proposition 3.5, there exists a function $f \in M(X, S)$ such that $f_n \rightarrow f(\bar{\varrho})$.

Let us fix an $\alpha > 0$; then choose a subsequence (g_k) of (f_n) such that $\varrho(2x(g_k - g_{k+n})) < 1/2^{k+1}$ for each $n \in \mathbb{N}$. Let us fix temporarily a natural number k and take arbitrary $h \in \mathcal{E}$ such that $|h(x)| \leq |f(x) - g_k(x)|$ for each $x \in X$. Let us denote $a = \inf_{x \in X} |h(x)|$, $b = \sup_{x \in X} |h(x)|$ and $A_n = \{x \in X: |g_{k+n}(x) - f(x)| > a/2\} \in \Sigma$. Since $g_n \rightarrow f(\bar{\varrho})$, it follows that $\bar{\varrho}_{ab}(A_n) < 1/2^{k+1}$ for n sufficiently large. On the other hand, for each $x \in X \setminus A_n$ we have

$$\begin{aligned} |h(x)| &\leq |f(x) - g_k(x)| \leq |f(x) - g_{k+n}(x)| + |g_{k+n}(x) - g_k(x)| \\ &\leq a/2 + |g_{k+n}(x) - g_k(x)| \leq |h(x)|/2 + |g_{k+n}(x) - g_k(x)|. \end{aligned}$$

Hence,

$$|h(x)| \leq 2|g_{k+n}(x) - g_k(x)| \quad \text{for } x \in X \setminus A_n$$

and finally for n sufficiently large there holds

$$\begin{aligned} \varrho(\alpha h) &\leq \varrho(\alpha h, A_n) + \varrho(\alpha h, X \setminus A_n) \leq \bar{\varrho}_{ab}(A_n) + \varrho(2\alpha(g_{k+n} - g_k), X \setminus A_n) \\ &\leq \bar{\varrho}_{ab}(A_n) + \varrho(2x(g_{k+n} - g_k)) \leq 1/2^{k+1} + 1/2^{k+1} = 1/2^k. \end{aligned}$$

Since h was an arbitrary chosen simple function such that $|h| \leq |f - g_k|$, it follows that $\varrho(\alpha(f - g_k)) < 1/2^k \rightarrow 0$ as $k \rightarrow \infty$, i.e., $|g_k - f|_\varrho \rightarrow 0$. We conclude then that $|f_k - f|_\varrho \rightarrow 0$, because (f_n) is a Cauchy sequence in L_ϱ .

We have to show now that $f \in L_\varrho$. Let $\lambda_n \geq 0$, $\lambda_n \rightarrow 0$ and let $\lambda > 0$ be such that $\lambda_n \leq \lambda$ for n sufficiently large. Fix $\varepsilon > 0$ and take $k \in \mathbb{N}$ such that $\varrho(\lambda(f_k - f)) < \varepsilon$. We have $\varrho(\lambda_n(f_k - f)) \leq \varrho(\lambda(f_n - f)) < \varepsilon$ for n sufficiently large.

The above-mentioned inequality implies that $\varrho(\lambda_n(f_k - f)) \rightarrow 0$ as $n \rightarrow \infty$. Thus $f_k - f \in L_\varrho$ and $f \in L_\varrho$, because $f_k \in L_\varrho$ and L_ϱ is a linear space.

This completes the proof of Theorem 3.6.

4. Subspace E_ϱ

Recall that a function $f \in M(X, S)$ is said to have an *absolutely continuous F -norm* if and only if for every sequence of sets $E_n \in \Sigma$ such that $E_n \searrow \emptyset$ there holds $|f|_{E_n}|_\varrho \rightarrow 0$. Clearly, not all members of L_ϱ have this property, therefore we shall distinguish the class of function having it. Since the condition $|f|_{E_n}|_\varrho \rightarrow 0$ may be equivalently stated in the modular form, we put the following definition.

4.1. DEFINITION. $E_\varrho = \{f \in M(X, S): \varrho(\alpha f, \cdot)$ is order continuous for every $\alpha > 0\}$.

The position of E_ϱ with respect to L_ϱ is described in the next theorem.

4.2. THEOREM. E_ϱ is a closed subspace of L_ϱ .

Proof. Step 1. First we prove that E_ϱ is a subspace of L_ϱ .

Clearly, E_ϱ is a linear space; therefore, it suffices to prove that $E_\varrho \subset L_\varrho$. Let $f \in E_\varrho$ and $0 \leq \lambda_n \rightarrow 0$. Since $f \in M(X, S)$, it follows that there exists a sequence $s_m \in \mathcal{E}$ such that $|s_m(x)| \leq |f(x)|$ and $s_m(x) \rightarrow f(x)$ for each $x \in X$, $m \in N$. By the Egoroff theorem (Proposition 3.3 (b)) we get a sequence (H_i) such that $H_i \in \mathcal{P}$, $H_1 \subset H_2 \subset \dots$, $\bigcup_{i=1}^{\infty} H_i = X$ and $s_m \rightrightarrows f$ on every H_k . Let us choose a $\lambda > 0$ such that $\lambda_n \leq \lambda$ for all natural numbers n and fix an arbitrary number $\varepsilon > 0$. Since $f \in E_\varrho$, there exists an index k_0 such that $\varrho(\lambda f, X \setminus H_{k_0}) < \varepsilon/3$. There holds $s_m \rightrightarrows f$ on H_{k_0} , therefore, we may choose a natural number m_0 such that $\varrho(2\lambda(s_{m_0} - f), H_{k_0}) < \varepsilon/3$. The function s_{m_0} belongs to $\mathcal{E} \subset L_\varrho$; thus, there exists an n_0 such that $\varrho(2\lambda_n s_{m_0}, H_{k_0}) < \varepsilon/3$ for $n \geq n_0$. Finally, for $n \geq n_0$ we have

$$\begin{aligned} \varrho(\lambda_n f) &\leq \varrho(\lambda_n f, X \setminus H_{k_0}) + \varrho(\lambda_n f, H_{k_0}) \\ &\leq \varrho(\lambda f, X \setminus H_{k_0}) + \varrho(2\lambda_n(s_{m_0} - f), H_{k_0}) + \varrho(2\lambda s_{m_0}, H_{k_0}) \\ &\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon, \end{aligned}$$

that is, $f \in L_\varrho$.

Step 2. Now we shall prove that E_ϱ is closed. Let $f_n \in E_\varrho$, $|f_n - f|_\varrho \rightarrow 0$, $f \in L_\varrho$. Take a sequence of sets $E_n \in \Sigma$, $E_n \searrow \emptyset$; fix ε , $\alpha > 0$. We have $\varrho(2\alpha(f - f_{k_0}), E_n) < \varepsilon/2$ for a certain index k_0 . Since $f_{k_0} \in E_\varrho$, it follows that for n sufficiently large $\varrho(2\alpha f_{k_0}, E_n) < \varepsilon/2$. Thus, for n sufficiently large there holds

$$\begin{aligned} \varrho(\alpha f, E_n) &\leq \varrho(2\alpha(f-f_{k_0}), E_n) + \varrho(2\alpha f_{k_0}, E_n) \\ &\leq \varrho(2\alpha(f-f_{k_0})) + \varrho(2\alpha f_{k_0}, E_n) < \varepsilon/2 + \varepsilon/2 = \varepsilon, \end{aligned}$$

that is, $f \in E_\varrho$. Hence E_ϱ is a closed subset of L_ϱ .

The exposition of some basic properties of E_ϱ will be started with the result which is called the *Vitali convergence theorem* because of its similarity to the classical Vitali theorem.

4.3. THEOREM. *Let $f_n \in E_\varrho$ and $f \in L_\varrho$ and $f_n \rightarrow f$ ϱ -a.e.; then the following conditions are equivalent:*

- (i) $f \in E_\varrho$ and $|f_n - f|_\varrho \rightarrow 0$,
- (ii) for every $\alpha > 0$, $\varrho(\alpha f_n, \cdot)$ are order equicontinuous, i.e., if $E_k \in \Sigma$, $E_k \searrow \emptyset$, then $\sup_n \varrho(\alpha f_n, E_k) \rightarrow 0$ as $k \rightarrow \infty$.

Proof. (i) \Rightarrow (ii). Let us choose a sequence of sets $E_k \in \Sigma$ such that $E_k \searrow \emptyset$ and fix arbitrary numbers $\varepsilon, \alpha > 0$. Since $|f_n - f|_\varrho \rightarrow 0$, it follows that there exists n_0 such that $\varrho(2\alpha(f_n - f)) < \varepsilon/2$ for $n \geq n_0$. Let k_0 be a natural number such that

$$(4.3a) \quad \varrho(\alpha f_n, E_k) < \varepsilon \quad \text{for } k \geq k_0 \text{ and } n = 1, 2, \dots, n_0 - 1.$$

Similarly, since $f \in E_\varrho$, there exists $k_1 \geq k_0$ such that for $k \geq k_1$

$$\varrho(2\alpha f, E_k) < \varepsilon/2.$$

Thus,

$$(4.3b) \quad \varrho(\alpha f_n, E_k) \leq \varrho(2\alpha(f_n - f)) + \varrho(2\alpha f, E_k) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for $k \geq k_1$ and $n \geq n_0$. Inequalities (4.3a) and (4.3b) give the desired result.

(ii) \Rightarrow (i). Given $\varepsilon, \alpha > 0$; since $f_n \rightarrow f$ ϱ -a.e., it follows by the Egoroff theorem that there exists a non-decreasing sequence of sets $H_k \in \mathcal{P}$ such that

$$\bigcup_{k=1}^{\infty} H_k = X \text{ and } f_n \rightrightarrows f \text{ on every } H_k. \text{ By the order equicontinuity of modulars}$$

(assumption (ii)) we can take an index k_0 such that $\varrho(2\alpha f_n, X \setminus H_{k_0}) < \varepsilon/4$ for all natural n . Since $f_n \rightrightarrows f$ on H_{k_0} , one can find a natural n_0 such that $\varrho(2\alpha(f_n - f), H_{k_0}) < \varepsilon/4$ for $n \geq n_0$ (cf. Proposition 2.6). Let $n, m \geq n_0$; then we have

$$\begin{aligned} \varrho(\alpha(f_n - f_m)) &\leq \varrho(\alpha(f_n - f_m), X \setminus H_{k_0}) + \varrho(\alpha(f_n - f_m), H_{k_0}) \\ &\leq \varrho(2\alpha f_n, X \setminus H_{k_0}) + \varrho(2\alpha f_m, X \setminus H_{k_0}) + \varrho(2\alpha(f_n - f), H_{k_0}) + \varrho(2\alpha(f_m - f), H_{k_0}) \\ &< \varepsilon/4 + \varepsilon/4 + \varepsilon/4 + \varepsilon/4 = \varepsilon. \end{aligned}$$

Hence, the sequence (f_n) satisfies the Cauchy condition in the sense of F -norm $|\cdot|_\varrho$. Since $f_n \in E_\varrho$ and E_ϱ is complete as a closed subspace of the complete metric space L_ϱ (cf. Theorems 3.6 and 4.2), it follows that there exists a function $g \in E_\varrho$ such that $|f_n - g| \rightarrow 0$. Therefore, by Proposition

3.2 (b), $f_n \rightarrow g(\bar{q})$. It follows by Proposition 3.3 (a) that there exists a subsequence of (f_n) which converges to g q -a.e. On the other hand, $f_n \rightarrow f$ q -a.e., therefore $f = g$ q -a.e. and finally we obtain $|f_n - f|_q \rightarrow 0$, which is the desired result.

Making use of the Vitali theorem and Proposition 3.2, 3.3 one can easily prove the following result.

4.4. COROLLARY. For $f_n, f \in E_q$ the following conditions are equivalent:

- (i) $|f_n - f|_q \rightarrow 0$,
- (ii) $f_n \rightarrow f(\bar{q})$ and $q(\alpha f_n, \cdot)$ are order equicontinuous for every $\alpha > 0$.

As an immediate corollary of the Vitali theorem we obtain the next theorem.

4.5. THEOREM (the Lebesgue dominated convergence theorem). If $f_n \rightarrow f$ q -a.e. ($f_n, f \in M(X, S)$) and there exists a function $g \in E_q$ such that $|f_n(x)| \leq |g(x)|$ q -a.e. for every natural n , then $|f_n - f|_q \rightarrow 0$.

We shall use this theorem to determine the space E_q .

4.6. THEOREM. E_q is a closure (in the sense of $|\cdot|_q$) of the space of all \mathcal{P} -simple functions, i.e., $E_q = \text{cl}(\mathcal{L})$.

Proof. Step 1. We shall prove first that $\text{cl}(\mathcal{L}) \subset E_q$. Take a function $g \in \mathcal{L}$, a number $\lambda > 0$ and a sequence $E_n \subset \Sigma$ such that $E_n \searrow \emptyset$. Write $\alpha = \sup_{x \in X} |\lambda g(x)|$; observe that

$$q(\lambda g, E_n) = q(\lambda g, \text{supp } g \cap E_n) \leq \bar{q}_\alpha(\text{supp } g \cap E_n) \rightarrow 0$$

because $\text{supp } g \cap E_n \in \mathcal{P}$. Thus $\mathcal{L} \subset E_q$. Since E_q is closed in L_q (Theorem 4.2), it follows that $\text{cl}(\mathcal{L}) \subset E_q$ as well.

Step 2. Now the inverse inclusion will be proved. Let $f \in E_q$ and $\varepsilon > 0$ be given. Since $f \in E_q$ and X is a countable union of sets from \mathcal{P} , it follows that there exists a set $E \in \mathcal{P}$ such that $|f 1_{X \setminus E}|_q < \varepsilon/2$. Let us choose a sequence (g_n) of \mathcal{P} -simple functions such that $\text{supp } g_n \subset E$ for all n , $|g_n(x)| \leq |f(x)|$ for each $x \in E$ and $g_m(x) \rightarrow f(x)$ for every $x \in E$. By the Lebesgue dominated theorem (Theorem 4.5) we get $|g_n - f 1_E|_q \rightarrow 0$, therefore, $|g_{n_0} - f 1_E|_q < \varepsilon/2$ for a certain n_0 . Finally,

$$|g_{n_0} - f|_q \leq |g_{n_0} - f 1_E|_q + |g_{n_0} 1_{X \setminus E}|_q + |f 1_{X \setminus E}|_q \leq \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

i.e., $E_q \subset \text{cl}(\mathcal{L})$.

We can now establish the major step towards the characterization of compact subsets of E_q .

4.7. THEOREM. A set $D \subset E_q$ is conditionally compact iff the following condition are satisfied:

- (i) for every $\alpha > 0$, $\{q(\alpha f, \cdot) : f \in D\}$ are order equicontinuous,

(ii) for every sequence (f_n) of elements from D there exists a subsequence (f_{n_k}) and a function $f \in E_\varrho$ such that $f_{n_k} \rightarrow f(\bar{\varrho})$.

Proof. Sufficiency. Let $f_n \in D$ for all n . By (ii) we can choose a subsequence (f_{n_k}) and a function $f \in E_\varrho$ such that $f_{n_k} \rightarrow f(\bar{\varrho})$. By (i) $\varrho(\alpha f_{n_k}, \cdot)$ are order equicontinuous for every $\alpha > 0$. From Corollary 4.4 it follows that $|f_{n_k} - f|_\varrho \rightarrow 0$.

Necessity. To prove (i) let us fix an $\varepsilon > 0$ and sets $E_k \supset \emptyset$. By the conditional compactness of D we may find a set $\{f_1, \dots, f_n\}$ of elements from E_ϱ such that to every $f \in D$ there corresponds an $i \in \{1, \dots, n\}$ for which $|f - f_i|_\varrho < \varepsilon/2$. Since all $f_i \in E_\varrho$ ($1 \leq i \leq n$), it follows that there exists a natural number k_0 such that $|f_i 1_{E_k}|_\varrho < \varepsilon/2$ for $k \geq k_0$, $1 \leq i \leq n$. Hence,

$$|f 1_{E_k}|_\varrho < |f - f_i|_\varrho + |f_i 1_{E_k}|_\varrho < \varepsilon \quad \text{for } k \geq k_0.$$

This completes the proof of (i). To prove (ii) let us choose an arbitrary $(f_n) \subset D$. Since D is conditionally compact, it follows that there exists a subsequence (f_{n_k}) of (f_n) and a function $f \in E_\varrho$ such that $|f_{n_k} - f|_\varrho \rightarrow 0$. Thus, $f_{n_k} \rightarrow f(\bar{\varrho})$ in view of Proposition 3.2.

5. Separability of E_ϱ and L_ϱ

5.1. DEFINITION. Given $Z \subset M(X, S)$, we say that the modular ϱ is *separable on Z* if and only if $|f 1_{(\cdot)}|_\varrho$ is a separable set function for each $f \in Z$. The latter means that there exists a countable $\mathcal{A} \subset \mathcal{P}$ such that to every $A \in \mathcal{P}$ there corresponds a sequence (A_k) of elements \mathcal{A} such that $\varrho(\alpha f, A \dot{-} A_k) \rightarrow 0$ for all $\alpha > 0$; the symbol $\dot{-}$ denotes the symmetric difference.

We start with an interesting result.

5.2. PROPOSITION. *If a linear space $L \subset L_\varrho$ is separable and \mathcal{P} -solid (i.e., $f \in L, A \in \mathcal{P}$ imply $f 1_A \in L$), then ϱ is separable on L .*

Proof. Suppose to contrary that ϱ is not separable on L ; there exists then a function $f \in L$, a number $\varepsilon > 0$ and an uncountable family of sets $\mathcal{B} \subset \mathcal{P}$ such that $|f 1_{B \dot{-} B'}|_\varrho > \varepsilon$ for arbitrary $B, B' \in \mathcal{B}$ such that $B \neq B'$. Thus, $\varepsilon < |f 1_{B \dot{-} B'}|_\varrho = |f 1_B - f 1_{B'}|_\varrho$. Put $W = \{f 1_B : B \in \mathcal{B}\}$, then observe that W is an uncountable subset of L and there holds $|g - h|_\varrho > \varepsilon$ for $g, h \in W, g \neq h$. The latter fact contradicts the separability of L .

The following separability theorem is the main result of this section.

5.3. THEOREM. *Two following conditions are equivalent:*

- (i) E_ϱ is separable,
- (ii) S is separable and ϱ is separable on \mathcal{E} .

Proof. (i) \Rightarrow (ii). Let us fix a set $A \in \mathcal{P}$ which is not ρ -null. Write $W = \{r1_A : r \in S\} \subset E_\rho$ and observe that W is separable. Define a one-to-one mapping $\varphi: W \rightarrow S$ by the formula $\varphi(r1_A) = r$. We conclude that φ is continuous; indeed, $|r_m 1_A - r 1_A|_\rho \rightarrow 0$ implies $r_m 1_A \rightarrow r 1_A(\bar{Q})$ (Proposition 3.2 (b)), i.e., for every $\varepsilon, \alpha > 0$ there holds $\bar{Q}_\alpha(\{x \in X : |r_n - r|_\rho \geq \varepsilon\} \cap A) \rightarrow 0$. Thus, $|r_n - r| < \varepsilon$ for n sufficiently large. Finally, S is a continuous image of the separable set W and therefore S is separable itself.

Since \mathcal{E} is a \mathcal{P} -solid subspace of L_ρ and is a subset of the separable space E_ρ , it follows from Proposition 5.2 that ρ is separable on \mathcal{E} .

(ii) \Rightarrow (i). Let us denote by Q a countable dense subset of S and by $\mathcal{B}_{n,r}$ a countable subfamily of sets dense in $\{X_n \cap A : A \in \mathcal{P}\}$ with respect to the pseudometric $\mathcal{P} \times \mathcal{P} \ni (A, B) \mapsto |r1_{X_n \cap (A-B)}|_\rho$. Let us put $\mathcal{B} = \bigcup_{\substack{n \in \mathbb{N} \\ r \in Q}} \mathcal{B}_{n,r}$. It is

enough to prove that $\{r1_B : r \in Q, B \in \mathcal{B}\}$ is dense in $\{s1_A : s \in S, A \in \mathcal{P}\}$. Given $s \in S, A \in \mathcal{P}$ and $\varepsilon > 0$, by the density of Q in S and by Proposition 2.6 we can choose an $r \in Q$ such that $|s1_A - r1_A|_\rho < \varepsilon/3$. Since $A, X_n \in \mathcal{P}$ and $X_n \nearrow X$ it follows that $A \setminus X_n \searrow \emptyset$ and then by (P₆) we have $|r1_{A \setminus X_n}|_\rho < \varepsilon/3$ for n sufficiently large. It follows from the assumptions that there exists a set $B \in \mathcal{B}_{n,r}$ such that $|r1_{A \cap X_n} - r1_B|_\rho = |r1_{X_n \cap (A-B)}|_\rho < \varepsilon/3$. Finally,

$$|s1_A - r1_B|_\rho \leq |s1_A - r1_A|_\rho + |r1_{A \cap X_n} - r1_B|_\rho + |r1_{A \setminus X_n}|_\rho < \varepsilon.$$

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