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On Fourier's first quasi-linear and linear iterated problems and on Fourier's second linear iterated problem in the $(n+1)$-dimensional time-space cube

Abstract. Constructions of solutions of the Fourier's first quasi-linear and linear iterated problems and a construction of a solution of the Fourier's second linear iterated problem in the domain $\left(\bigtimes_{i=1}^{n}(-c_i, c_i)\right)\times(0, T]$, $T<\infty$, are given.

1. Introduction. In this paper we construct solutions of the Fourier's first quasi-linear and linear iterated problems in the domain $D = \left(\bigtimes_{i=1}^{n}(-c_i, c_i)\right)\times\times(0, T]$, $T<\infty$, and we construct a solution of the Fourier's second linear iterated problem in D. For this purpose we use the Green's method, the method of heat iterated potentials, the Picard method of successive approximations and a similar transformation to H. Blook's transformation from [10]. To construct the solutions of the problems considered, we use [4]–[9]. This paper is a continuation of those papers and bases mainly on [9]. We may apply [6]–[8] since all the results given in those papers in the domain $\left(\bigtimes_{i=1}^{n}(-c_i, c_i)\right)\times(0, T)$, $T\leqslant\infty$, hold also in the domain D.

The results obtained here contain the results from [1], [2], [4], [5], [9] and [14]. The results of this paper are direct generalizations of those given by the author in [4], [5], [9], indirect generalizations of those given by Barański and by Musiałek in [1], [2], and generalizations and indirect generalizations of those given by Milewski in [13] and [14], respectively.

2. Preliminaries. Throughout the paper we use the following notations:

$$R_- = (-\infty, 0),\quad R_+ = (0, \infty),\quad R = (-\infty, \infty),$$

$$N = \{1, 2, \ldots\},\quad N_0 = N\cup\{0\},$$

$$R^n = R\times\ldots\times R,\quad N_0^n = N_0\times\ldots\times N_0\quad (n\text{-times}),$$

$$I_n = \{1, 2, \ldots, n\},\quad \tilde{I}_n = I_n\cup\{0\}\quad (n\in N),$$

$$x = (x_1, \ldots, x_n), \qquad y = (y_1, \ldots, y_n),$$
$$x^i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \quad (i \in \boldsymbol{I}_n),$$
$$x^{i,j} = (x_1, \ldots, x_{i-1}, (-1)^j c_i, x_{i+1}, \ldots, x_n) \quad (i \in \boldsymbol{I}_n,\ j \in \boldsymbol{I}_2),$$
$$D_0 = \bigtimes_{i=1}^{n} (-c_i, c_i), \quad \partial D_0 = \bar{D}_0 \setminus D_0, \quad S_0 = D_0 \times \{0\},$$
$$D_i = \bigtimes_{\substack{k=1 \\ k \neq i}}^{n} (-c_k, c_k) \quad (i \in \boldsymbol{I}_n),$$
$$D_i^j = (-c_1, c_1) \times \ldots \times (-c_{i-1}, c_{i-1}) \times \{(-1)^j c_i\} \times (-c_{i+1}, c_{i+1}) \times \ldots$$
$$\ldots \times (-c_n, c_n) \quad (i \in \boldsymbol{I}_n,\ j \in \boldsymbol{I}_2),$$
$$D = D_0 \times (0, T], \quad S_i^j = D_i^j \times (0, T], \quad \tilde{S}_i^j = \bar{D}_i^j \times (0, T],$$
$$T < \infty \quad (i \in \boldsymbol{I}_n,\ j \in \boldsymbol{I}_2),$$
$$Z_i = \partial(\bar{D}_i \times [0, T]) \setminus \{(x^i, t)\colon t = 0\} \quad (i \in \boldsymbol{I}_n),$$
$$P_{x,t} = \Delta_x - D_t, \quad \bar{P}_{y,s} = \Delta_y + D_s, \quad \tilde{P}_{x,t} = P_{x,t} - c(t), \quad a = \prod_{i=1}^{n} a_i,$$

where $\Delta_x = \sum_{i=1}^{n} a_i D_{x_i}^2$, c is a function defined on the interval $[0, T]$ and $a_i \in \boldsymbol{R}_+$ for $i \in \boldsymbol{I}_n$.

By Δ_x^k, $P_{x,t}^k$, $\bar{P}_{y,s}^k$ and $\tilde{P}_{x,t}^k$ we denote the k-iterations of the operators Δ_x, $P_{x,t}$, $\bar{P}_{y,s}$ and $\tilde{P}_{x,t}$, respectively. As long as it does not lead to misunderstanding, the operators Δ_x, $P_{x,t}$, $\bar{P}_{y,s}$ and $\tilde{P}_{x,t}$ will be denoted by the symbols: Δ, P, $\bar{P}$ and $\tilde{P}$.

For each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \boldsymbol{N}_0^n$, $x \in \boldsymbol{R}^n$ we put: $|\alpha| = \sum_{i=1}^{n} \alpha_i$, $\alpha! = \prod_{i=1}^{n} \alpha_i!$, $a^\alpha = \prod_{i=1}^{n} (a_i)^{\alpha_i}$ and $D_x^\alpha = D_{x_1}^{\alpha_1} \ldots D_{x_n}^{\alpha_n}$. Moreover, $D_{x,t}^\alpha := D_x^{\tilde{\alpha}} D_t^{\alpha_*}$, where $\alpha = (\tilde{\alpha}, \alpha_*)$, $\tilde{\alpha} \in \boldsymbol{N}_0^n$, $\alpha_* \in \boldsymbol{N}_0$, $x \in \boldsymbol{R}^n$ and $t \in [0, T]$.

We assume that m is an arbitrary fixed natural number.

We consider here only real functions and we use the concept of local uniform convergence of considered integrals in the sense of [12].

Let $a_i \in \boldsymbol{R}_+$ for $i \in \boldsymbol{I}_n$. For every fixed index $i \in \boldsymbol{I}_n$ we define the function $\mathscr{U}\colon \boldsymbol{R}^2 \setminus \{0\} \to \boldsymbol{R}$ by the formula

$$\mathscr{U}(\xi, \tau; a_i) = \begin{cases} (4\pi a_i \tau)^{-1/2} \exp\left(-(4a_i\tau)^{-1} \xi^2\right) & \text{for } \xi \in \boldsymbol{R},\ \tau \in \boldsymbol{R}_+, \\ 0 \quad \text{for } \xi \in \boldsymbol{R},\ \tau \in \boldsymbol{R}_- \text{ or } \xi \in \boldsymbol{R} \setminus \{0\},\ \tau = 0. \end{cases}$$

Now, for all $x \in \boldsymbol{R}^n$, $y \in \boldsymbol{R}^n$, $0 \leqslant s < t$, $i \in \boldsymbol{I}_n$, $j \in \boldsymbol{I}_2$ and $k \in \boldsymbol{N}_0$, we define the functions $U_{i,k}^{(j)}$, U_i by the formulae

$$\begin{aligned} U_{i,k}^{(j)}(x_i, t, y_i, s) &= \mathscr{U}(y_i - x_{i,k}^{(j)}, t - s; a_i), \\ U_i(x_i, t, y_i, s) &= U_{i,0}^{(j)}(x_i, t, y_i, s), \end{aligned} \tag{2.1}$$

where $x_{i,k}^{(j)} = (-1)^k \left(x_i + (-1)^{j+1} 2kc_i\right)$.

Next, for every $x \in R^n$, $y \in R^n$, $0 \leqslant s < t \leqslant T$ and for every fixed natural number q, we define the function G^q by the formula

$$G^q(x, t, y, s) = \frac{(-1)^{q-1}}{(q-1)!}(t-s)^{q-1} G(x, t, y, s), \tag{2.2}$$

where

$$G(x, t, y, s) = \prod_{i=1}^{n} G_i(x_i, t, y_i, s), \tag{2.3}$$

$$\begin{aligned} &G_i(x_i, t, y_i, s) \\ &= U_i(x_i, t, y_i, s) + \sum_{k=1}^{\infty} (-1)^k \big(U_{i,k}^{(1)}(x_i, t, y_i, s) + U_{i,k}^{(2)}(x_i, t, y_i, s)\big) \end{aligned} \tag{2.4}$$

and the functions U_i, $U_{i,k}^{(j)}$ $(i \in I_n,\ j \in I_2,\ k \in N)$ are given by formulae (2.1). If $q = 1$, then we apply the symbol G in place of the symbol G^1.

In the sequel, we shall need the following lemmas:

LEMMA 2.1 ([6]). *Let $0 \leqslant s < t$, $\alpha \in N_0$, $\varkappa > -1$ and $i \in I_n$. Then there exist positive constants A_α and $B_{\alpha,\varkappa}$ such that*

(a) $|D_\xi^\alpha \mathscr{U}(\xi, t-s; a_i)| \leqslant A_\alpha (t-s)^{-(\alpha+1)/2}$ *for* $\xi \in R$,

(b) $\int_R |D_\xi^\alpha \mathscr{U}(\xi, t-s, a_i)|\, d\xi \leqslant \sqrt{8\pi A}\, A_\alpha (t-s)^{-\alpha/2}$, *where* $A = \max\{a_1, \ldots, a_n\}$,

(c) $|D_\xi^\alpha \mathscr{U}(\xi, t-s; a_i)| \leqslant B_{\alpha,\varkappa} |\xi|^{-\alpha-\varkappa-1} (t-s)^{\varkappa/2}$ *for* $\xi \in R \setminus \{0\}$.

Particularly,

(d) $|D_{x_i}^\alpha U_{i,k}^{(j)}(x_i, t, y_i, s)| \leqslant (2c_i)^{-\alpha-\varkappa-1} (k-1)^{-\alpha-\varkappa-1} B_{\alpha,\varkappa} (t-s)^{\varkappa/2}$ *for* x_i, $y_i \in [-c_i, c_i]$, $j \in I_2$, $k \in N \setminus \{1\}$.

LEMMA 2.2 ([7]). *Let q be an arbitrary fixed natural number, and let G and G^q be the functions defined by formulae (2.2)–(2.4). Then:*

(a) *The function $G^q(x, t, y, s)$ and the derivatives $D_{x,t}^\alpha G^q(x, t, y, s)$, $D_{y,s}^\alpha G^q(x, t, y, s)$ $(\alpha \in N_0^{n+1},\ |\alpha| \neq 0)$ are continuous for all $(x, t) \in \bar{D}_0 \times (0, T]$, $(y, s) \in \bar{D}$, $s < t$.*

(b)
$$\begin{aligned} &P_{x,t}^k G^q(x, t, y, s) \\ &= \bar{P}_{y,s}^k G^q(x, t, y, s) = \begin{cases} G^{q-k}(x, t, y, s) & \text{for } k = 0, 1, \ldots, q-1, \\ 0 & \text{for } k = q, q+1, \ldots, \end{cases} \end{aligned}$$

where $(x, t) \in \bar{D}_0 \times (0, T]$, $(y, s) \in \bar{D}$, $s < t$.

(c)
$$D_t^{\alpha_*} G(x, t, y, s) = \sum_{\substack{\beta \in N_0^n \\ |\beta| = \alpha_*}} \frac{\alpha_*!}{\beta!} a^\beta D_x^{2\beta} G(x, t, y, s),$$

where $(x, t) \in \bar{D}_0 \times (0, T]$, $(y, s) \in \bar{D}$, $s < t$ and α_* is a natural number.

(d) $P_{x,t}^k G^q(x, t, y, s) = 0$ *for* $(x, t) \in \tilde{S}_i^j$ $(i \in I_n,\ j \in I_2,\ k \in N_0)$, $(y, s) \in \bar{D}$, $s < t$.

(e) $P^k_{x,t} D_{y_p} G^q(x, t, y^{p,r}, s) = 0$ *for* $(x, t) \in \tilde{S}^j_i$ $(i, p \in I_n,\ j, r \in I_2,\ k \in N_0)$, $(y, s) \in \bar{D}$, $s < t$.

(f) $\lim_{s \to t} D^k_t G^q(x, t, y, s) = 0$ *for* $(x, t) \in D$, $(y, s) \in \bar{D}$, $x \neq y$, $s < t$ $(k \in \tilde{I}_{q-1})$.

In this paper we shall denote by C and $\tilde{C}$ the following constants:

$$C = \Big(\max_{i \in I_n, j \in \tilde{I}_{2m-1}} \{3\sqrt{8\pi A}\, A_j;\ 2(2c_i)^{-j-1} B_{j,1} \sum_{k=1}^{\infty} k^{-2}\}\Big)^n,$$

$$\tilde{C} = \Big(\max_{i \in I_n, j \in \tilde{I}_{2m-2}} \{3A_j;\ 2(2c_i)^{-j-2} B_{j,1} \sum_{k=1}^{\infty} k^{-2}\}\Big)^n,$$

where A_j, $B_{j,1}$ $(j \in \tilde{I}_{2m-1})$ are the constants from Lemma 2.1.

Now we shall prove the following:

LEMMA 2.3. *Assume that c is a continuous function in the interval $[0, T]$, $T < \infty$, γ is the function given by formula*

$$\gamma(t) = \exp\Big(-\int_0^t c(\tau)\, d\tau\Big), \quad t \in [0, T], \tag{2.5}$$

and $\Omega = \Omega_0 \times (0, T]$, where Ω_0 is an arbitrary domain in R^n. Then:

(a) *If v is a function such that there exist the derivatives $D^\alpha_{x,t} v$ $(\alpha = (\tilde{\alpha}, \alpha_*)$, $\tilde{\alpha} \in N^n_0$, $\alpha_* \in N_0$, $0 < |\tilde{\alpha}| + 2\alpha_* \leqslant 2i$, $i \in N)$ in Ω, then*

$$\tilde{P}^k\big(\gamma(t) v(x, t)\big) = \gamma(t) P^k v(x, t) \tag{2.6}$$

for $(x, t) \in \Omega$, $k \in I_i$, $i \in N$.

(b) *If v is a function such that there exist the derivatives $D^i_t v$ $(i \in I_{m-1})$ in $\Omega_0 \times \{0\}$ and if there exist the derivatives $c^{(r)}(0) = 0$ $(r \in \tilde{I}_{m-2},\ m \geqslant 2)$, then*

$$D^k_t\big(\gamma(t) v(x, t)\big) = D^k_t v(x, t) \tag{2.7}$$

for $(x, t) \in \Omega_0 \times \{0\}$, $k \in I_{m-1}$.

Proof. (a) Put $i = 1$. Then

$$\begin{aligned} &\tilde{P}\big(\gamma(t) v(x, t)\big) \\ &\quad = \gamma(t) \Delta_x v(x, t) + c(t)\gamma(t) v(x, t) - \gamma(t) D_t v(x, t) - c(t)\gamma(t) v(x, t) \\ &\quad = \gamma(t) P v(x, t) \quad \text{for } (x, t) \in \Omega. \end{aligned}$$

Suppose that for an arbitrary fixed natural number i formulae (2.6) hold. Consequently

$$\begin{aligned} &\tilde{P}^{k+1}\big(\gamma(t) v(x, t)\big) = \tilde{P}\big(\gamma(t) P^k v(x, t)\big) \\ &= \gamma(t) \Delta_x P^k v(x, t) + c(t)\gamma(t) P^k v(x, t) - \gamma(t) D_t P^k v(x, t) - c(t)\gamma(t) P^k v(x, t) \\ &= \gamma(t) P^{k+1} v(x, t) \quad \text{for } (x, t) \in \Omega,\ k \in I_i. \end{aligned}$$

Then, from the mathematical induction principle, assertion (a) is true.
(b) By the Leibniz theorem

$$D_t^k\big(\gamma(t)v(x,t)\big) = \sum_{j=0}^{k-1}\binom{k}{j}D_t^{k-j}\gamma(t)\,D_t^j v(x,t)+\gamma(t)\,D_t^k v(x,t)$$

for $(x, t)\in\Omega_0\times\{0\}$, $k\in \boldsymbol{I}_{m-1}$, $m\geqslant 2$.

Since the derivatives $D_t^j\gamma(t)$ ($j\in \boldsymbol{I}_k$, $k\in \boldsymbol{I}_{m-1}$, $m\geqslant 2$) are linear combinations of such products that at each of them there is at least one derivative of the form $c^{(r)}(t)$ ($r\in\tilde{\boldsymbol{I}}_{j-1}$, $j\in \boldsymbol{I}_k$, $k\in \boldsymbol{I}_{m-1}$, $m\geqslant 2$), it follows that, by the equations $c^{(r)}(0)=0$ ($r\in\tilde{\boldsymbol{I}}_{m-2}$, $m\geqslant 2$) and $\gamma(0)=1$, the proof of assertion (b) is complete.

3. Formulations of Fourier's first quasi-linear and linear iterated problems of type (C_f^m) and $(\tilde{C}_F^m)$. A continuous function u in $\bar{D}$ is called a *quasi-(m)-regular* [*(m)-regular*] in D if the derivatives $D_{x,t}^\alpha u$ ($\alpha=(\tilde{\alpha},\alpha_*)$, $\tilde{\alpha}\in N_0^n$, $\alpha_*\in N_0$, $0<|\tilde{\alpha}|+2\alpha_*\leqslant 2m-1$ [$\leqslant 2m$]) are continuous in D.

Given the functions f, f_k, $f_{i,k}^j$ ($i\in \boldsymbol{I}_n$, $j\in \boldsymbol{I}_2$, $k\in\tilde{\boldsymbol{I}}_{m-1}$), Fourier's first quasi-linear [linear] iterated problem of type (C_f^m) in D consists in finding a quasi-(m)-regular [(m)-regular] function u in D, satisfying the equation

$$P^m u(x,t) = f\big(x,t,u(x,t)\big) \quad \text{for } (x,t)\in D \tag{3.1}$$

$$[\;P^m u(x,t) = f(x,t) \quad \text{for } (x,t)\in D], \tag{3.1'}$$

satisfying the initial conditions

$$D_t^k u(x,t) = \begin{cases} f_0(x) & \text{for } (x,t)\in\bar{S}_0,\ k=0,\\ f_k(x) & \text{for } (x,t)\in S_0,\ k\in \boldsymbol{I}_{m-1}\end{cases} \tag{3.2}$$

and satisfying the boundary conditions

$$P^k u(x,t) = \begin{cases} f_{i,0}^j(x^i,t) & \text{for } (x,t)\in\bar{S}_i^j,\ i\in \boldsymbol{I}_n,\ j\in \boldsymbol{I}_2,\ k=0,\\ f_{i,k}^j(x^i,t) & \text{for } (x,t)\in S_i^j,\ i\in \boldsymbol{I}_n,\ j\in \boldsymbol{I}_2,\ k\in \boldsymbol{I}_{m-1}.\end{cases} \tag{3.3}$$

A function u with the foregoing properties is called a *quasi-(m)-regular* [*(m)-regular*] *solution* in D of the above problem, and this problem is called shortly the (C_f^m) *quasi-linear* [(C_f^m) *linear*] *problem.*

Given the functions F, F_k, $F_{i,k}^j$ ($i\in \boldsymbol{I}_n$, $j\in \boldsymbol{I}_2$, $k\in\tilde{\boldsymbol{I}}_{m-1}$), Fourier's first quasi-linear [linear] iterated problem of type $(\tilde{C}_F^m)$ in D consists in finding a quasi-(m)-regular [(m)-regular] function u in D, satisfying the equation

$$\tilde{P}^m u(x,t) = F\big(x,t,u(x,t)\big) \quad \text{for } (x,t)\in D \tag{3.4}$$

$$[\;\tilde{P}^m u(x,t) = F(x,t) \quad \text{for } (x,t)\in D], \tag{3.4'}$$

satisfying the initial conditions

$$D_t^k u(x,t) = \begin{cases} F_0(x) & \text{for } (x,t)\in\bar{S}_0,\ k=0,\\ F_k(x) & \text{for } (x,t)\in S_0,\ k\in \boldsymbol{I}_{m-1}\end{cases} \tag{3.5}$$

and satisfying the boundary conditions

$$(3.6)\quad \tilde{P}^k u(x,t) = \begin{cases} F^j_{i,0}(x^i, t) & \text{for } (x,t)\in \bar{S}^j_i,\ i\in I_n,\ j\in I_2,\ k=0,\\ F^j_{i,k}(x^i, t) & \text{for } (x,t)\in S^j_i,\ i\in I_n,\ j\in I_2,\ k\in I_{m-1} \text{ (}^1\text{)}.\end{cases}$$

A function u with the foregoing properties is called a *quasi-(m)-regular* [*(m)-regular*] *solution* in D of the above problem, and this problem is called shortly the $(\tilde{C}^m_F)$ *quasi-linear* [$(\tilde{C}^m_F)$ *linear*] *problem*.

4. Properties of heat volume iterated potentials. Let

$$(4.1)\qquad X^\alpha(x,t;w) = \int_0^t Y^\alpha(x,t,s;w)\,ds,$$

where

$$(4.2)\qquad Y^\alpha(x,t,s;w) = -\int_{D_0} f\big(y,s,w(y,s)\big) D^\alpha_{x,t} G^m(x,t,y,s)\,dy,$$

$w = w(y,s)$ is a function defined for $(y,s)\in\bar{D}$, f is the given function, G^m is the function given by formulae (2.2)–(2.4), $\alpha = (\tilde{\alpha}, \alpha_*)$, $\tilde{\alpha} = (\alpha_1, \ldots, \alpha_n)\in N_0^n$, $\alpha_*\in N_0$.

The integral X^0 is called the *heat volume iterated potential* of the domain D.

LEMMA 4.1. *If $w = w(y,s)$ is a function defined for $(y,s)\in\bar{D}$ and such that the composite function $f(y,s,w(y,s))$ is measurable and bounded in the domain D, then:*

(a) *The integrals X^α ($|\tilde{\alpha}|+2\alpha_* \leqslant 2m-1$) are locally uniformly convergent in the domain $\bar{D}_0\times(0,T]$. Moreover, the integrals Y^α ($|\tilde{\alpha}|+2\alpha_*\leqslant 2(m-1)$) are locally uniformly convergent, as the functions of the variable (x,t), in the domain $\bar{D}_0\times(0,T]$.*

(b) $\lim_{s\to t} Y^{\alpha[r]}(x,t,s;w) = 0$ *for* $(x,t)\in\bar{D}_0\times(0,T]$, $0\leqslant s<t$, *where* $\alpha[r] := (\tilde{\alpha}, \alpha_*-r-1)$, $r\in\tilde{I}_{\alpha_*-1}$, $\alpha_*\in N$, $0<|\tilde{\alpha}|+2\alpha_*\leqslant 2m$.

(c) *For every point $(x,t)\in\bar{D}_0\times(0,T]$ there exist the derivatives $D^\alpha_{x,t}X^0$ ($0<|\tilde{\alpha}|+2\alpha_*\leqslant 2m-1$) and $D^\alpha_{x,t}X^0(x,t;w) = X^\alpha(x,t;w)$ ($0<|\tilde{\alpha}|+2\alpha_*\leqslant 2m-1$) for all $(x,t)\in\bar{D}_0\times(0,T]$.*

Proof. (a) To prove this assertion for the integrals X^α ($|\tilde{\alpha}|+2\alpha_*\leqslant 2m-1$), let us fix a multi-index $\alpha = (\tilde{\alpha},\alpha_*)$ such that $|\tilde{\alpha}|+2\alpha_*\leqslant 2m-1$ and observe next that by the Leibniz theorem on the differentiation and by assertion (c) of Lemma 2.2 we obtain the equation

$$X^\alpha(x,t;w) = \frac{(-1)^m\alpha_*!}{(m-1)!}\sum_{j=0}^{\alpha_*}\frac{(m-1)(m-2)\ldots(m-j)}{j!}\sum_{|\alpha^j|=\alpha_*-j}\frac{a^{\alpha^j}}{\alpha^j!} I_{\beta^j}(x,t;w),$$

(1) The left-hand sides of equations (3.2), (3.3), (3.5) and (3.6) are meant in the limit sense.

where

$$I_{\beta j}(x, t; w) = \int_0^t \int_{D_0} f(y, s, w(y, s))(t-s)^{m-j-1} D_x^{\beta^j} G(x, t, y, s)\, dy\, ds,$$

$(x, t) \in \bar{D}_0 \times (0, T]$, $\alpha^j = (\alpha_1^j, \ldots, \alpha_n^j)$, $\beta^j = (\beta_1^j, \ldots, \beta_n^j)$, $\beta_k^j = \alpha_k + 2\alpha_k^j$ $(k \in I_n)$, $|\beta^j| = |\tilde{\alpha}| + 2\alpha_* - 2j$ $(j \in \tilde{I}_{\alpha_*})$. Consequently, to prove the first part of assertion (a), it is sufficient to show that the integrals $I_{\beta j}$ $(|\beta^j| = |\tilde{\alpha}| + 2\alpha_* - 2j,\ j \in \tilde{I}_{\alpha_*})$ are locally uniformly convergent in the domain $\bar{D}_0 \times (0, T]$. For this purpose, observe that by assertions (b) and (d) of Lemma 2.1

$$|I_{\beta j}(x, t; w)| \leqslant C \sup_D |f|\, \varphi_{\beta j}(t) \quad \text{for } (x, t) \in \bar{D}_0 \times (0, T],$$

where

$$\varphi_{\beta j}(t) = \int_0^t (t-s)^{m-j-1} \prod_{r=1}^{n} [(t-s)^{-\beta_r^j/2} + (t-s)^{1/2}]\, ds,$$

$\beta^j = (\beta_1^j, \ldots, \beta_n^j)$, $|\beta^j| = |\tilde{\alpha}| + 2\alpha_* - 2j$ $(j \in \tilde{I}_{\alpha_*})$. Since $m - j - 1 - \frac{1}{2}|\beta^j| \geqslant \frac{1}{2}$ $(j \in \tilde{I}_{\alpha_*})$, the integrals $\varphi_{\beta j}$ $(|\beta^j| = |\tilde{\alpha}| + 2\alpha_* - 2j,\ j \in \tilde{I}_{\alpha_*})$ are the sums of the finite number of the integrals

$$\int_0^t \frac{ds}{(t-s)^{\varkappa}} \quad (\varkappa < 1).$$

Therefore, the integrals $I_{\beta j}$ $(|\beta^j| = |\tilde{\alpha}| + 2\alpha_* - 2j,\ j \in \tilde{I}_{\alpha_*})$ are locally uniformly convergent in the domain $\bar{D}_0 \times (0, T]$ (see [10], § 59.4), and by the fact that the multi-index α is arbitrary the proof of the first part of assertion (a) is complete.

To prove assertion (a) for the integrals Y^α $(|\tilde{\alpha}| + 2\alpha_* \leqslant 2(m-1))$, let us fix a multi-index $\alpha = (\tilde{\alpha}, \alpha_*)$ such that $|\tilde{\alpha}| + 2\alpha_* \leqslant 2(m-1)$ and observe that as in the proof of the first part of assertion (a) we obtain the equation

$$Y^\alpha(x, t, s; w) = \frac{(-1)^m \alpha_*!}{(m-1)!} \sum_{j=0}^{\alpha_*} \frac{(m-1)(m-2)\ldots(m-j)}{j!} \sum_{|\alpha^j| = \alpha_* - j} \frac{a^{\alpha^j}}{\alpha^j!} J_{\beta j}(x, t, s; w),$$

where

$$J_{\beta j}(x, t, s; w) = \int_{D_0} f(y, s, w(y, s))(t-s)^{m-j-1} D_x^{\beta^j} G(x, t, y, s)\, dy, \tag{4.3}$$

$(x, t) \in \bar{D}_0 \times (0, T]$, $0 \leqslant s < t$, $\alpha^j = (\alpha_1^j, \ldots, \alpha_n^j)$, $\beta^j = (\beta_1^j, \ldots, \beta_n^j)$, $\beta_k^j = \alpha_k + 2\alpha_k^j$ $(k \in I_n)$, $|\beta^j| = |\tilde{\alpha}| + 2\alpha_* - 2j$ $(j \in \tilde{I}_{\alpha_*})$. Consequently, to prove the second part of assertion (a), it is sufficient to show that for arbitrary fixed $j \in \tilde{I}_{\alpha_*}$ and $\tilde{T} > 0$ such that $0 \leqslant s + \tilde{T} < t \leqslant T$ the integral $J_{\beta j}$ $(|\beta^j| = |\tilde{\alpha}| + 2\alpha_* - 2j)$ is locally uniformly convergent in the domain $\bar{D}_0 \times (\tilde{T}, T]$. For this purpose let x_0 be an arbitrary fixed point belonging to the set $\bar{D}_0$, ε be an arbitrary fixed positive number and $K_\eta(x_0)$ be a sphere with the center x_0 and a radius η. Now, by assertions (a) and (d) of Lemma 2.1, we have

$$|K_{\beta^j,\eta}(x, t, s; w)| \leqslant \tilde{C} \sup_D |f| \, \psi_{\beta^j}(t-s) \int_{K_\eta(x_0)} dy$$
$$\text{for } (x, t) \in \bar{D}_0 \times (0, T],\ 0 \leqslant s < t,$$

where

$$K_{\beta^j,\eta}(x, t, s; w) = \int_{D_0 \cap K_\eta(x_0)} f(y, s, w(y, s))(t-s)^{m-j-1} D_x^{\beta^j} G(x, t, y, s)\, dy,$$

$$\psi_{\beta^j}(t-s) = (t-s)^{m-j-1} \prod_{r=1}^{n} [(t-s)^{-(\beta_r^j+1)/2} + (t-s)^{1/2}],$$

$\beta^j = (\beta_1^j, \ldots, \beta_n^j)$, $|\beta^j| = |\tilde{\alpha}| + 2\alpha_* - 2j$. Therefore

$$|K_{\beta^j,\eta}(x, t, s; w)| \leqslant \tilde{C} \sup_D |f| \, T^{m-j-1} \prod_{r=1}^{n} [\tilde{T}^{-(\beta_r^j+1)/2} + T^{1/2}] \tau_n \eta^n,$$

where $(x, t) \in \bar{D}_0 \times (\tilde{T}, T]$, $0 \leqslant s + \tilde{T} < t$ and τ_n is the volume of the n-dimensional unit sphere. If η satisfies the inequality

$$\eta < \left[\frac{\varepsilon}{C_2 \sup_D |f| \, T^{m-j-1} \prod_{r=1}^{n} [\tilde{T}^{-(\beta_r^j+1)/2} + T^{1/2}] \tau_n} \right]^{1/n},$$

then

$$|K_{\beta^j,\eta}(x, t, s; w)| \leqslant \varepsilon \quad \text{for } (x, t) \in \bar{D}_0 \times (\tilde{T}, T],\ 0 \leqslant s + \tilde{T} < t.$$

Since the multi-index α, the index j, the point x_0 and the number $\tilde{T}$ are arbitrary, it follows that the proof of assertion (a) is complete.

(b) Let us fix a multi-index $\alpha = (\tilde{\alpha}, \alpha_*)$ such that $\alpha_* \in N$ and $0 < |\tilde{\alpha}| + 2\alpha_* \leqslant 2m$. By the Leibniz theorem on the differentiation and by assertion (c) of Lemma 2.2, we obtain the equations

$$Y^{\alpha[r]}(x, t, s; w) = \frac{(-1)^m (\alpha_* - r - 1)!}{(m-1)!} \sum_{j=0}^{\alpha_* - r - 1} \frac{(m-1)(m-2)\ldots(m-j)}{j!} \times$$
$$\times \sum_{|\alpha^j| = \alpha_* - r - j - 1} \frac{a^{\alpha^j}}{\alpha^j!} J_{\beta^j}(x, t, s; w),$$

where J_{β^j} are given by formula (4.3), $(x, t) \in \bar{D}_0 \times (0, T]$, $0 \leqslant s < t$, $\alpha^j = (\alpha_1^j, \ldots, \alpha_n^j)$, $\beta^j = (\beta_1^j, \ldots, \beta_n^j)$, $\beta_k^j = \alpha_k + 2\alpha_k^j$ $(k \in \boldsymbol{I}_n)$, $|\beta^j| = |\tilde{\alpha}| + 2\alpha_* - 2r - 2j - 2$, $j \in \tilde{\boldsymbol{I}}_{\alpha_* - r - 1}$, $r \in \tilde{\boldsymbol{I}}_{\alpha_* - 1}$. Simultaneously, by assertions (b) and (d) of Lemma 2.1, we have

$$|J_{\beta^j}(x, t, s; w)| \leqslant C \sup_D |f| (t-s)^{m-j-1} \prod_{r=1}^{n} [(t-s)^{-\beta_r^j/2} + (t-s)^{1/2}],$$

where $(x, t) \in \bar{D}_0 \times (0, T]$, $0 \leqslant s < t$, $\beta^j = (\beta_1^j, \ldots, \beta_n^j)$, $\beta_k^j = \alpha_k + 2\alpha_k^j$ $(k \in \boldsymbol{I}_n)$, $|\beta^j| = |\tilde{\alpha}| + 2\alpha_* - 2r - 2j - 2$, $j \in \tilde{\boldsymbol{I}}_{\alpha_* - r - 1}$, $r \in \tilde{\boldsymbol{I}}_{\alpha_* - 1}$. Consequently, assertion (b) holds since $m - j - 1 - \frac{1}{2}|\beta^j| \geqslant r + 1 > 0$ and $m - j - 1 + \frac{1}{2}n > 0$ for all possible j.

(c) Arguing analogously as in the proof of assertion (iii) of Lemma 6.1, given in [9], we obtain

$$D_{x,t}^{\alpha} X^0(x, t; w) = X^{\alpha}(x, t; w) + \sum_{r=0}^{\alpha_* - 1} D_t^r \left(\lim_{s \to t} Y^{\alpha[r]}(x, t, s; w)\right)$$

for $(x, t) \in \bar{D}_0 \times (0, T]$, $\alpha = (\tilde{\alpha}, \alpha_*)$, $\tilde{\alpha} \in N_0^n$, $\alpha_* \in N$, $0 < |\tilde{\alpha}| + 2\alpha_* \leqslant 2m - 1$.

The above equations and assertion (b) imply assertion (c).

LEMMA 4.2. *If the assumptions of Lemma* 4.1 *are satisfied and if* $x_0 \in \bar{D}_0$ *is an arbitrary fixed point, then the function* X^0 *given by formulae* (4.1) *and* (4.2) *satisfies the equation*

$$(4.4) \qquad P^{m-1} X^0(x, t; w) = -\int_0^t \int_{D_0} f(y, s, w(y, s)) G(x, t, y, s)\, dy\, ds$$

for $(x, t) \in \bar{D}_0 \times (0, T]$,

the initial conditions

$$(4.5) \qquad D_t^k X^0(x, t; w) \to 0, \quad \text{as } (x, t) \to (x_0, 0^+),\ (x, t) \in D,\ k \in \tilde{\boldsymbol{I}}_{m-1}$$

and the boundary conditions

$$(4.6) \qquad P^k X^0(x, t; w) = 0, \quad \text{as } (x, t) \in \tilde{S}_i^j,\ i \in \boldsymbol{I}_n,\ j \in \boldsymbol{I}_2,\ k \in \tilde{\boldsymbol{I}}_{m-1}.$$

Proof. Equation (4.4) is a consequence of assertion (c) of Lemma 4.1 and of assertion (b) of Lemma 2.2.

Simultaneously, by an analogous argumentation as in the proof of assertion (a) of Lemma 4.1, we obtain the following estimations:

$$|D_t^k X^0(x, t; w)| \leqslant \frac{Ck!}{(m-1)!} \sum_{j=0}^{k} \frac{(m-1)(m-2)\ldots(m-j)}{j!} \cdot \sum_{|\alpha^j| = k-j} \frac{a^{\alpha^j}}{\alpha^j!} \times$$

$$\times \sup_D |f| \int_0^t (t-s)^{m-j-1} \prod_{r=1}^{n} [(t-s)^{-\alpha_r^j} + (t-s)^{1/2}]\, ds,$$

where $(x, t) \in \bar{D}_0 \times (0, T]$, $k \in \tilde{\boldsymbol{I}}_{m-1}$. Since $m - j - 1 - |\alpha^j| \geqslant 0$ for $j \in \tilde{\boldsymbol{I}}_k$ and $k \in \tilde{\boldsymbol{I}}_{m-1}$, conditions (4.5) hold.

Conditions (4.6) are a consequence of assertion (c) of Lemma 4.1 and of assertion (d) of Lemma 2.2.

LEMMA 4.3. *Assume that*:

(a) *The function* $f(y, s, z)$ *is continuous for* $(y, s) \in \bar{D}$, $z \in \boldsymbol{R}$.

(b) *The functions* $\partial f(y, s, z)/\partial y_i$ $(i \in \boldsymbol{I}_n)$, $\partial f(y, s, z)/\partial z$ *are continuous for* $(y, s) \in D$, $z \in \boldsymbol{R}$.

(c) w *is an arbitrary function continuous in* $\bar{D}$ *and such that the derivatives* $\partial w(y, s)/\partial y_i$ $(i \in \boldsymbol{I}_n)$ *are continuous in* D.

Then the function

$$v(x, t; w) := -\int_0^t \int_{D_0} f(y, s, w(y, s)) G(x, t, y, s) \, dy \, ds$$

satisfies the equation

$$Pv(x, t; w) = f(x, t, w(x, t)) \quad \text{for } (x, t) \in D. \tag{4.7}$$

Proof. Let $w = w(y, s)$ be an arbitrary fixed function with the properties from assumption (c) and let

$$\tilde{f}(y, s) := f(y, s, w(y, s)) \quad \text{for } (y, s) \in \bar{D}.$$

Hence, by assumption (a), the function $\tilde{f}$ is continuous in $\bar{D}$ and since

$$\frac{\partial \tilde{f}(y, s)}{\partial y_i} = \frac{\partial f(y, s, w(y, s))}{\partial y_i} + \frac{\partial f(y, s, w(y, s))}{\partial w} \cdot \frac{\partial w(y, s)}{\partial y_i} \quad \text{for } (y, s) \in D, \ i \in \boldsymbol{I}_n,$$

it follows, by assumptions (b) and (c) that the functions $\partial \tilde{f}/\partial y_i$ $(i \in \boldsymbol{I}_n)$ are continuous in D. Now, applying to the function $\tilde{f}$ Szarski's theorem (see [10], p. 523), we obtain, by the fact that w is an arbitrary function, equation (4.7).

THEOREM 4.1. *If assumptions* (a)–(c) *of Lemma* 4.3 *are satisfied and if the function* X^0 *is given by formulae* (4.1) *and* (4.2), *then:*

(A) *The derivatives* $D^{\alpha}_{x,t} X^0$ $(|\tilde{\alpha}| + 2\alpha_* \leqslant 2m-1)$ *are continuous in the domain* $\bar{D}_0 \times (0, T]$ *and the derivatives* $D^{\alpha}_{x,t} X^0$ $(|\tilde{\alpha}| + 2\alpha_* = 2m)$ *are continuous in* D. *If, moreover,* $X^0(x, 0; w) := 0$ *for* $x \in \bar{D}_0$, *then the function* X^0 *is continuous in* $\bar{D}$.

(B) *The function* X^0 *satisfies the equation*

$$P^m X^0(x, t; w) = f(x, t, w(x, t)) \quad \text{for } (x, t) \in D, \tag{4.8}$$

the initial conditions

$$D^k_t X^0(x, t; w) = 0 \quad \text{for } (x, t) \in \bar{S}_0, \ k \in \tilde{\boldsymbol{I}}_{m-1} \tag{4.9}$$

and the boundary conditions

$$P^k X^0(x, t; w) = 0 \quad \text{for } (x, t) \in \tilde{S}^j_i, \ i \in \boldsymbol{I}_n, \ j \in \boldsymbol{I}_2, \ k \in \tilde{\boldsymbol{I}}_{m-1}. \tag{4.10}$$

Proof. (A) The continuity of the derivatives $D^{\alpha}_{x,t} X^0$ $(|\tilde{\alpha}| + 2\alpha_* \leqslant 2m-1)$ in $\bar{D}_0 \times (0, T]$ is a consequence of Lemma 4.1 and the continuity of the function X^0 in $\bar{D}$ is a consequence of assertion (a) of Lemma 4.1 and of conditions (4.5) for $k = 0$.

To prove assertion (A) for the derivatives $D^{\alpha}_{x,t} X^0$ ($|\tilde{\alpha}|+2\alpha_* = 2m$), it is sufficient to show that the derivatives $D_{x_p x_q} D^{\alpha}_{x,t} X^0$ and $D_t D^{\alpha}_{x,t} X^0$ ($|\tilde{\alpha}|+$ $+2\alpha_* \leqslant 2(m-1)$; $p, q \in \boldsymbol{I}_n$) are continuous in D. First, we shall prove that the derivatives $D_{x_p x_q} D^{\alpha}_{x,t} X^0$ ($|\tilde{\alpha}|+2\alpha_* \leqslant 2(m-1)$; $p, q \in \boldsymbol{I}_n$) have this property. To this purpose let x_0 be an arbitrary fixed point belonging to the domain D_0 and let $K_\eta(x_0)$ be a sphere with the center x_0 and a radius η such that $\overline{K_\eta(x_0)} \subset D_0$. By the fact that the function $G^m(x, t, y, s)$ and the derivatives $D^{\alpha}_{x,t} G^m(x, t, y, s)$ ($\alpha \in N_0^{n+1}$, $|\alpha| \neq 0$) are continuous for $(x, t) \in \bar{D}_0 \times (0, T]$, $(y, s) \in \bar{D}$, $s < t$ and by the fact that

$$\lim_{s \to t} D_t^k G^m(x, t, y, s) = 0 \quad \text{for } (x, t) \in D,\ (y, s) \in \bar{D},\ x \neq y,\ s < t,\ k \in \tilde{\boldsymbol{I}}_{m-1}$$

(see assertions (a) and (f) of Lemma 2.2), the derivatives

$$D_{x_p x_q} D^{\alpha}_{x,t} \Big[-\int_0^t \int_{D_0 \setminus K_\eta(x_0)} f(y, s, w(y, s)) G^m(x, t, y, s)\, dy\, ds\Big]$$
$$(|\tilde{\alpha}|+2\alpha_* \leqslant 2(m-1);\ p, q \in \boldsymbol{I}_n)$$

are continuous at each point (x_0, t), where $t \in (0, T]$. Therefore to prove that the derivatives $D_{x_p x_q} D^{\alpha}_{x,t} X^0$ ($|\tilde{\alpha}|+2\alpha_* \leqslant 2(m-1)$; $p, q \in \boldsymbol{I}_n$) are continuous in D it is sufficient to show that the derivatives

$$D_{x_p x_q} D^{\alpha}_{x,t} \Big[-\int_0^t \int_{K_\eta(x_0)} f(y, s, w(y, s)) G^m(x, t, y, s)\, dy\, ds\Big] \tag{4.11}$$
$$(|\tilde{\alpha}|+2\alpha_* \leqslant 2(m-1);\ p, q \in \boldsymbol{I}_n)$$

are continuous at each point (x_0, t), where $t \in (0, T]$. For this purpose fix $\alpha = (\tilde{\alpha}, \alpha_*)$ such that $|\tilde{\alpha}|+2\alpha_* \leqslant 2(m-1)$, fix $p, q \in \boldsymbol{I}_n$ and observe that, by assertion (c) of Lemma 4.1 and by an analogous argument as in the proof of assertion (a) of Lemma 4.1, we obtain the equations

$$D_{x_p x_q} D^{\alpha}_{x,t} \left[-\int_0^t \int_{K_\eta(x_0)} f(y, s, w(y, s)) G^m(x, t, y, s)\, dy\, ds\right]$$
$$= D_{x_p} \left[-\int_0^t \int_{K_\eta(x_0)} f(y, s, w(y, s)) D_{x_q} D^{\alpha}_{x,t} G^m(x, t, y, s)\, dy\, ds\right]$$
$$= \frac{(-1)^m \alpha_*!}{(m-1)!} \sum_{j=0}^{\alpha_*} \frac{(m-1)(m-2)\ldots(m-j)}{j!} \sum_{|\alpha^j| = \alpha_* - j} \frac{a^{\alpha^j}}{\alpha^j!} D_{x_p} L_{\beta^j, \eta}(x, t; w),$$

where

$$L_{\beta^j, \eta}(x, t; w) = \int_0^t \int_{K_\eta(x_0)} f(y, s, w(y, s)) (t-s)^{m-j-1} D_{x_q} D_x^{\beta^j} G(x, t, y, s)\, dy\, ds,$$

$(x, t) \in \bar{D}_0 \times (0, T]$, $\alpha^j = (\alpha_1^j, \ldots, \alpha_n^j)$, $\beta^j = (\beta_1^j, \ldots, \beta_n^j)$, $\beta_k^j = \alpha_k + 2\alpha_k^j$ $(k \in I_n)$, $|\beta^j| = |\tilde{\alpha}| + 2\alpha_* - 2j$ $(j \in \tilde{I}_{\alpha_*})$. Consequently, to prove that the derivatives (4.11) are continuous at each point (x_0, t), where $t \in (0, T]$, it is sufficient to show that the derivatives $D_{x_p} L_{\beta^j, \eta}$ $(|\beta^j| = |\tilde{\alpha}| + 2\alpha_* - 2j,\ j \in \tilde{I}_{\alpha_*})$ are continuous at each such point. For this purpose observe that, by the formulae $D_{x_i} U_{i,k}^{(j)} = (-1)^k D_{y_i} U_{i,k}^{(j)}$ $(i \in I_n,\ j \in I_2,\ k \in N_0)$ and by the formula for integration by parts (see [12]), we have the equations

$$\begin{aligned} &D_{x_p} L_{\beta^j, \eta}(x, t; w) \\ &= D_{x_p}\Big[-\int_0^t \int_{K_\eta(x_0)} f(y, s, w(y, s))(t-s)^{m-j-1}(D_{x_q}^{\beta_q^j} D_{y_q} \hat{G}_q)\Big(\prod_{\substack{r=1\\ r\neq q}}^n D_{x_r}^{\beta_r^j} G_r\Big)\,dy\,ds\Big] \\ &= D_{x_p} M_{\beta^j, \eta}(x, t; w) - D_{x_p} N_{\beta^j, \eta}(x, t; w), \end{aligned}$$

where

$$\hat{G}_q = U_q + \sum_{k=1}^{\infty} (U_{q,k}^{(1)} + U_{q,k}^{(2)}),$$

$$M_{\beta^j, \eta}(x, t; w) = \int_0^t \int_{K_\eta(x_0)} D_{y_q} f(y, s, w(y, s))(t-s)^{m-j-1}(D_{x_q}^{\beta_q^j} \hat{G}_q)\Big(\prod_{\substack{r=1\\ r\neq q}}^n D_{x_r}^{\beta_r^j} G_r\Big)\,dy\,ds,$$

$$\begin{aligned} &N_{\beta^j, \eta}(x, t; w) \\ &= \int_0^t \int_{\partial K_\eta(x_0)} f(y, s, w(y, s))(t-s)^{m-j-1}(D_{x_q}^{\beta_q^j} \hat{G}_q)\Big(\prod_{\substack{r=1\\ r\neq q}}^n D_{x_r}^{\beta_r^j} G_r\Big)\cos(\vec{n}, x_q)\,d\sigma_{(y,s)}, \end{aligned}$$

$(x, t) \in \bar{D}_0 \times (0, T]$, $|\beta^j| = |\tilde{\alpha}| + 2\alpha_* - 2j$ $(j \in \tilde{I}_{\alpha_*})$, $\vec{n}$ is the exterior normal vector and $d\sigma_{(y,s)}$ is a surface element in R^{n+1} taken with respect to (y, s). Since the derivatives $D_{x_p} M_{\beta^j, \eta}$; $D_{x_p} N_{\beta^j, \eta}$ $(|\beta^j| = |\tilde{\alpha}| + 2\alpha_* - 2j;\ j \in \tilde{I}_{\alpha_*})$ are of order at most $2m-1$, by analogous arguments as in the proof of Lemma 4.1 and as in the proof of Lemma 6.1 from [9] we obtain that the functions $D_{x_p} M_{\beta^j, \eta}$; $D_{x_p} N_{\beta^j, \eta}$ $(|\beta^j| = |\tilde{\alpha}| + 2\alpha_* - 2j;\ j \in \tilde{I}_{\alpha_*})$ are locally uniformly convergent at each point (x_0, t), where $t \in (0, T]$, and consequently are continuous at each such point. Therefore, the derivatives $D_{x_p x_q} D_{x,t}^{\alpha} X^0$ $(|\tilde{\alpha}| + 2\alpha_* \leqslant 2(m-1))$ are continuous in D.

To prove that the derivatives $D_t D_{x,t}^{\alpha} X^0$ $(|\tilde{\alpha}| + 2\alpha_* \leqslant 2(m-1))$ are continuous in D observe that, by assertion (c) of Lemma 4.1, we have

$$\begin{aligned} &D_t D_{x,t}^{\alpha} X^0(x, t; w) \\ &= \frac{(-1)^m}{(m-1)!} D_t \Big[\int_0^t \int_{D_0} f(y, s, w(y, s)) D_x^{\tilde{\alpha}} D_t^{\alpha_*}(t-s)^{m-1} G(x, t, y, s)\,dy\,ds\Big] \end{aligned}$$

for $(x, t) \in \bar{D}_0 \times (0, T]$, $|\tilde{\alpha}| + 2\alpha_* \leqslant 2(m-1)$. If $\alpha_* = 0$, then the formulae $D_{x_i} U_{i,k}^{(j)} = (-1)^k D_{y_i} U_{i,k}^{(j)}$ ($i \in \boldsymbol{I}_n$, $j \in \boldsymbol{I}_2$, $k \in \boldsymbol{N}_0$), the formula for integration by parts, and a similar argument as in the proof of the fact that the derivatives $D_{x_p x_q} D_{x,t}^{\alpha} X^0$ ($|\tilde{\alpha}| + 2\alpha_* \leqslant 2(m-1)$; $p, q \in \boldsymbol{I}_n$) are continuous in D, imply that the derivatives $D_t D_{x,t}^{\alpha} X^0$ ($|\tilde{\alpha}| + 2\alpha_* \leqslant 2(m-1)$) are continuous in D. If $\alpha_* \neq 0$, then the Leibniz theorem on the differentiation, assertion (c) of Lemma 2.2 and an analogous argument as in the proof of the case where $\alpha_* = 0$ prove that the derivatives $D_t D_{x,t}^{\alpha} X^0$ ($|\tilde{\alpha}| + 2\alpha_* \leqslant 2(m-1)$) are continuous in D. Therefore the proof of assertion (A) is complete.

(B) Formulae (4.4) and (4.7) imply formula (4.8), and formulae (4.5) and (4.6) imply formulae (4.9) and (4.10), respectively.

5. Theorem on the existence of the quasi-(m)-regular solution of the (C_f^m) quasi-linear problem. In this section we shall prove the theorem on the existence of the quasi-(m)-regular solution of the (C_f^m) quasi-linear problem. For this purpose, we shall use Theorem 4.1 from this paper and Theorem 7.1 from [9], and we shall apply the Picard method of successive approximations (see [11], Sections 72.1–72.3).

THEOREM 5.1. *Assume that*:

(A) *The functions* $D^{\alpha^i} f_i$ ($\alpha^i \in \boldsymbol{N}_0^n$, $|\alpha^i| \leqslant 2m-2i-2$, $i \in \tilde{\boldsymbol{I}}_{m-1}$) *are continuous and bounded in* D_0, *and, additionally, the function* f_0 *is continuous in* $\bar{D}_0$ *and such that* $f_0|_{\partial D_0} = 0$.

(B) *The functions* $f_{i,q}^j$ ($i \in \boldsymbol{I}_n$, $j \in \boldsymbol{I}_2$, $q \in \boldsymbol{I}_{m-1}$) *are continuous and bounded in the domains* $D_i \times (0, T]$, *respectively, and the functions* $f_{i,0}^j$ ($i \in \boldsymbol{I}_n$, $j \in \boldsymbol{I}_2$) *are continuous in the domains* $\bar{D}_i \times [0, T]$, *respectively, and satisfy the equations*

$$f_{i,0}^j(x^i, t) = 0 \quad \text{for } (x^i, t) \in Z_i \cup (\bar{D}_i \times \{0\}) \; (i \in \boldsymbol{I}_n, \; j \in \boldsymbol{I}_2). \tag{5.1}$$

(C) *The function* $f(y, s, z)$ *is continuous for* $(y, s) \in \bar{D}$, $z \in \boldsymbol{R}$.

(D) *The functions* $\partial f(y, s, z)/\partial y_i$ ($i \in \boldsymbol{I}_n$), $\partial f(y, s, z)/\partial z$ *are continuous for* $(y, s) \in D$, $z \in \boldsymbol{R}$.

(E) *The function* f *satisfies the Lipschitz condition*

$$|f(y, s, z) - f(y, s, \tilde{z})| \leqslant L|z - \tilde{z}| \quad \text{for } (y, s) \in D, \; z, \tilde{z} \in \boldsymbol{R},$$

where

$$\begin{cases} 0 < L < [K_0(T)]^{-1} \quad \text{and} \\ K_0(t) := \dfrac{C}{(m-1)!} \displaystyle\int_0^t (t-s)^{m-1} \left(1 + (t-s)^{1/2}\right)^n ds \quad \text{for } t \in [0, T]. \end{cases} \tag{5.2}$$

Then the function

$$v(x, t) = \lim_{i \to \infty} v_i(x, t) \quad \text{for } (x, t) \in \bar{D}, \tag{5.3}$$

where

$$
(5.4) \qquad v_{i+1}(x, t) = \begin{cases} v_0(x, t) - \int_0^t \int_{D_0} f(y, s, v_i(y, s)) G^m(x, t, y, s)\, dy\, ds & \\ \qquad\qquad \text{for } (x, t) \in \bar{D}_0 \times (0, T],\ i \in N_0, & \\ 0 \qquad\qquad \text{for } (x, t) \in \bar{S}_0,\ i \in N_0, & \end{cases}
$$

and v_0 is the (m)-regular solution in D of the (C^m) linear problem from [9], *is the quasi-(m)-regular solution in D of the (C^m_f) quasi-linear problem.*

Proof. First, we shall prove that the function v given by formulae (5.3) and (5.4) is quasi-(m)-regular in D. Since, by Theorem 7.1 from [9], the function v_0 is (m)-regular in D, and since, by assertion (A) of Theorem 4.1, the functions v_i $(i \in N)$ are (m)-regular in D, so to prove that the function v is quasi-(m)-regular in D it is sufficient to show that

$$
(5.5) \qquad v_i(x, t) \underset{i \to \infty}{\rightrightarrows} v(x, t) \quad \text{for } (x, t) \in \bar{D}
$$

and

$$
(5.6) \qquad D^\alpha_{x,t} v_i(x, t) \underset{i \to \infty}{\rightrightarrows} D^\alpha_{x,t} v(x, t) \quad \text{for } (x, t) \in D,
$$

where $\alpha = (\tilde{\alpha}, \alpha_*)$, $\tilde{\alpha} \in N^n_0$, $\alpha_* \in N_0$, $0 < |\tilde{\alpha}| + 2\alpha_* \leqslant 2m-1$.

Observe now that for an arbitraty fixed natural number k

$$
v_k(x, t) = \sum_{i=0}^{k-1} \big(v_{i+1}(x, t) - v_i(x, t)\big) \quad \text{for } (x, t) \in \bar{D}
$$

and

$$
D^\alpha_{x,t} v_k(x, t) = \sum_{i=0}^{k-1} \big(D^\alpha_{x,t} v_{i+1}(x, t) - D^\alpha_{x,t} v_i(x, t)\big) \quad \text{for } (x, t) \in D,
$$

where $\alpha = (\tilde{\alpha}, \alpha_*)$, $\tilde{\alpha} \in N^n_0$, $\alpha_* \in N_0$, $0 < |\tilde{\alpha}| + 2\alpha_* \leqslant 2m-1$.

Consequently, to prove conditions (5.5) and (5.6) it is sufficient to show that the series

$$
(5.7) \qquad \sum_{i=0}^{\infty} \big(v_{i+1}(x, t) - v_i(x, t)\big)
$$

and

$$
(5.8) \qquad \sum_{i=0}^{\infty} \big(D^\alpha_{x,t} v_{i+1}(x, t) - D^\alpha_{x,t} v_i(x, t)\big),
$$

$$
\alpha = (\tilde{\alpha}, \alpha_*),\ \tilde{\alpha} \in N^n_0,\ \alpha_* \in N_0,\ 0 < |\tilde{\alpha}| + 2\alpha_* \leqslant 2m-1,
$$

are absolutely and uniformly convergent in the domains $\bar{D}$ and D, respectively.

First, we shall show this assertion for series (5.7). To this purpose we shall prove that

(5.9) $$|v_{i+1}(x,t)-v_i(x,t)| \leqslant MK_0(T)\big(LK_0(T)\big)^i \quad \text{for } (x,t)\in\bar{D},\ i\in N_0,$$

where $M = \sup\limits_{(x,t)\in\bar{D}} |f(x,t,v_0(x,t))|$ (2).

If $(x,t)\in\bar{D}_0\times\{0\}$, then formulae (5.9) are a consequence of formulae (5.4). Therefore, it is sufficient to prove that

(5.10) $$|v_{i+1}(x,t)-v_i(x,t)| \leqslant MK_0(T)\big(LK_0(T)\big)^i \quad \text{for } (x,t)\in\bar{D}_0\times(0,T],\ i\in N_0.$$

For $i=0$ the above inequalities are a consequence of formulae (5.4), (2.2)–(2.4) and of inequalities (b) and (d) from Lemma 2.1. Indeed,

$$|v_1(x,t)-v_0(x,t)| = \Big|\int_0^t \int_{D_0} f\big(y,s,v_0(y,s)\big)G^m(x,t,y,s)\,dy\,ds\Big| \leqslant MK_0(T) \quad \text{for } (x,t)\in\bar{D}_0\times(0,T].$$

Assume now that for an arbitrary fixed natural number i

$$|v_i(x,t)-v_{i-1}(x,t)| \leqslant MK_0(T)\big(LK_0(T)\big)^{i-1} \quad \text{for } (x,t)\in\bar{D}_0\times(0,T].$$

Then formulae (5.4), the Lipschitz condition from assumption (E), the above inequality, formulae (2.2)–(2.4) and inequalities (b) and (d) from Lemma 2.1 imply that

$$\begin{aligned}|v_{i+1}(x,t)-v_i(x,t)| &\leqslant L\int_0^t \int_{D_0} |v_i(y,s)-v_{i-1}(y,s)|\,|G^m(x,t,y,s)|\,dy\,ds\\ &\leqslant LMK_0(T)\big(LK_0(T)\big)^{i-1}\int_0^t \int_{D_0} |G^m(x,t,y,s)|\,dy\,ds\\ &\leqslant MK_0(T)\big(LK_0(T)\big)^i \quad \text{for } (x,t)\in\bar{D}_0\times(0,T].\end{aligned}$$

Therefore, by the mathematical induction principle, estimations (5.10) are true. This completes the proof of estimations (5.9).

Estimations (5.9) and formula (5.2) imply that the series (5.7) is absolutely and uniformly convergent in the domain $\bar{D}$ to the function $v(x,t)$ and so the sequence $\{v_i(x,t)\}_{i\in N_0}$ is uniformly convergent in $\bar{D}$ to this function, which is consequently continuous in $\bar{D}$.

Now we shall prove that the series (5.8) are absolutely and uniformly

---

(2) $M<\infty$ since, by assertion (C) of Theorem 5.1 and by Theorem 7.1 from [9], the function $f(x,t,v_0(x,t))$ is continuous in $\bar{D}$.

convergent in the domain D. To this purpose, observe that by formulae (5.4), by the Lipschitz condition from assumption (E), by inequalities (5.9), by assertion (c) of Lemma 2.2 and by inequalities (b) and (d) from Lemma 2.1, we have

$$(5.11)\quad \begin{aligned} &|D^{\alpha}_{x,t} v_{i+1}(x,t) - D^{\alpha}_{x,t} v_i(x,t)| \\ &\leqslant L\int_0^t \int_{D_0} |v_i(y,s) - v_{i-1}(y,s)|\,|D^{\alpha}_{x,t} G^m(x,t,y,s)|\,dy\,ds \\ &\leqslant LMK_0(T)\big(LK_0(T)\big)^{i-1} \int_0^t \int_{D_0} |D^{\alpha}_{x,t} G^m(x,t,y,s)|\,dy\,ds \\ &\leqslant M\big(LK_0(T)\big)^i \sum_{j=0}^{\alpha_*} \frac{\alpha_*!}{j!(m-j-1)!} \sum_{|\beta^j| = \alpha_* - j} \frac{a^{\beta^j}}{\beta^j!} \times \\ &\quad \times \int_0^t \Big[(t-s)^{m-j-1} \int_{D_0} |D_x^{\tilde{\alpha}+2\beta^j} G(x,t,y,s)|\,dy\Big]\,ds \\ &\leqslant M\big(LK_0(T)\big)^i K_\alpha(t), \end{aligned}$$

where

$$\begin{aligned} &K_\alpha(t) \\ &:= C\sum_{j=0}^{\alpha_*} \frac{\alpha_*!}{j!(m-j-1)!} \sum_{|\beta^j|=\alpha_*-j} \frac{a^{\beta^j}}{\beta^j!} \int_0^t (t-s)^{m-j-1} \prod_{r=1}^n \big[(t-s)^{-(\alpha_r+2\beta_r^j)/2} + \\ &\qquad\qquad + (t-s)^{1/2}\big]\,ds, \end{aligned}$$

$$(x,t)\in D,\ i\in N_0,\ \alpha = (\tilde{\alpha}, \alpha_*),\ \tilde{\alpha}\in N_0^n,\ \alpha_*\in N_0,\ 0 < |\tilde{\alpha}| + 2\alpha_* \leqslant 2m-1,$$
$$\beta^j = (\beta_1^j, \ldots, \beta_n^j).$$

Since $m-j-1-\frac{1}{2}\sum_{r=1}^n (\alpha_r + 2\beta_r^j) \geqslant m-j-1-\frac{1}{2}(2m-1-j) = -\frac{1}{2}$, we have $K_\alpha(t) \leqslant K_\alpha(T)$ for $t\in(0,T]$, $\alpha = (\tilde{\alpha}, \alpha_*)$, $\tilde{\alpha}\in N_0^n$, $\alpha_*\in N_0$, $0 < |\tilde{\alpha}| + 2\alpha_* \leqslant 2m-1$. Therefore, by inequalities (5.11), we obtain

$$(5.12)\qquad |D^{\alpha}_{x,t} v_{i+1}(x,t) - D^{\alpha}_{x,t} v_i(x,t)| \leqslant MK_\alpha(T)\big(LK_0(T)\big)^i$$

for $(x,t)\in D$, $i\in N_0$, $\alpha = (\tilde{\alpha}, \alpha_*)$, $\tilde{\alpha}\in N_0^n$, $\alpha_*\in N_0$, $0 < |\tilde{\alpha}| + 2\alpha_* \leqslant 2m-1$.

Inequalities (5.12) and formula (5.2) imply that the series (5.8) are absolutely and uniformly convergent in the domain D to the functions $D^{\alpha}_{x,t} v(x,t)$ ($\alpha = (\tilde{\alpha}, \alpha_*)$, $\tilde{\alpha}\in N_0^n$, $\alpha_*\in N_0$, $0 < |\tilde{\alpha}| + 2\alpha_* \leqslant 2m-1$), respectively, and therefore the sequences $\{D^{\alpha}_{x,t} v_i(x,t)\}_{i\in N_0}$ are uniformly convergent in D to these functions, respectively, which are consequently continuous in D. This completes the proof of assertion that the function v given by formulae (5.3) and (5.4) is quasi-(m)-regular in D.

Observe now that formula (5.5) and assumption (C) imply that the function v satisfies the equation

$$v(x, t) = \begin{cases} v_0(x, t) - \int\limits_0^t \int\limits_{D_0} f(y, s, v(y, s)) G^m(x, t, y, s)\, dy\, ds & \text{for } (x, t) \in \bar{D}_0 \times (0, T], \\ 0 & \text{for } (x, t) \in \bar{S}_0. \end{cases}$$

Hence, by Theorem 7.1 from [9] and by Theorem 4.1, the function v satisfies the (C_f^m) quasi-linear problem.

Consequently, the function v given by formulae (5.3) and (5.4) is the quasi-(m)-regular solution in D of the (C_f^m) quasi-linear problem.

6. Theorem on the existence of the quasi-(m)-regular solution of the $(\tilde{C}_F^m)$ quasi-linear problem. In this section we shall prove the theorem on the existence of the quasi-(m)-regular solution of the $(\tilde{C}_F^m)$ quasi-linear problem. For this purpose we shall use Theorem 5.1 and Lemma 2.3.

THEOREM 6.1. *Assume that:*

(A)–(E) *The functions* F_i $(i \in \tilde{I}_{m-1})$, $F_{i,q}^j$ $(i \in I_n, j \in I_2, q \in \tilde{I}_{m-1})$ *and* F *satisfy assumptions* (A)–(E) *of Theorem* 5.1 *instead of functions* f_i $(i \in \tilde{I}_{m-1})$, $f_{i,q}^j$ $(i \in I_n, j \in I_2, q \in \tilde{I}_{m-1})$ *and* f, *respectively.*

Suppose, moreover, that:

(F) c *is a continuous function on the interval* $[0, T]$ *such that the derivatives* $c^{(i)}(0) = 0$ $(i \in \tilde{I}_{m-2}, m \geq 2)$.

Then the function

(6.1) $$u(x, t) = \gamma(t) v(x, t) \quad \textit{for } (x, t) \in \bar{D},$$

where

(6.2) $$v(x, t) = \lim_{i \to \infty} v_i(x, t) \quad \textit{for } (x, t) \in \bar{D},$$

(6.3) $$v_{i+1}(x, t) = \begin{cases} v_0(x, t) - \int\limits_0^t \int\limits_{D_0} (\gamma(s))^{-1} F(y, s, \gamma(s) v_i(y, s)) G^m(x, t, y, s)\, dy\, ds & \textit{for } (x, t) \in \bar{D}_0 \times (0, T],\ i \in N_0, \\ 0 & \textit{for } (x, t) \in \bar{S}_0,\ i \in N_0, \end{cases}$$

(6.4) $$v_0(x, t) = v^1(x, t) + v^2(x, t) \quad \textit{for } (x, t) \in \bar{D},$$

(6.5) $$v^1(x, t) = \begin{cases} \sum\limits_{i=0}^{m-1} \sum\limits_{j=0}^{i} (-1)^{i-j} \binom{i}{j} \int\limits_{D_0} \Delta^j F_{i-j}(y) G^{i+1}(x, t, y, 0)\, dy & \textit{for } (x, t) \in \bar{D}_0 \times (0, T], \\ F_0(x) & \textit{for } (x, t) \in S_0, \\ 0 & \textit{for } (x, t) \in \partial D_0 \times \{0\}, \end{cases}$$

(6.6) $$v^2(x,t)=\sum_{i=1}^{n}\sum_{k=0}^{m-1}\left(v_{i,k}^1(x,t)+v_{i,k}^2(x,t)\right)\quad \textit{for } (x,t)\in\bar{D},$$

(6.7) $$v_{i,k}^j(x,t)$$
$$=\begin{cases} -2a_i\int_0^t\int_{D_i}(\gamma(s))^{-1}F_{i,k}^j(y^i,s)\,D_{y_i}G^{k+1}(x,t,y,s)\big|_{y_i=(-1)^j c_i}\,dy^i\,ds & \\ & \textit{for } (x,t)\in(\bar{D}\times(0,T])\setminus\tilde{S}_i^j,\\ G_{i,k}^j(x^i,t) & \textit{for } (x,t)\in\tilde{S}_i^j,\\ 0 & \textit{for } (x,t)\in\bar{S}_0\end{cases}$$

for $i\in\boldsymbol{I}_n,\ j\in\boldsymbol{I}_2,\ k\in\tilde{\boldsymbol{I}}_{m-1}$,

(6.8) $$G_{i,k}^j(x^i,t)$$
$$=\begin{cases}(\gamma(t))^{-1}F_{i,0}^j(x^i,t) & \textit{for } (x^i,t)\in\bar{D}_i\times(0,T],\ i\in\boldsymbol{I}_n,\ j\in\boldsymbol{I}_2,\ k=0,\\ 0 & \textit{for } (x^i,t)\in\bar{D}_i\times(0,T],\ i\in\boldsymbol{I}_n,\ j\in\boldsymbol{I}_2,\ k\in\boldsymbol{I}_{m-1},\end{cases}$$

and γ *is the function defined by formula* (2.5), *is the quasi-*(m)*-regular solution in* D *of the* ($\tilde{C}_F^m$) *quasi-linear problem.*

Proof. We shall seek the quasi-(m)-regular solution in D of the ($\tilde{C}_F^m$) quasi-linear problem in the form

(6.9) $$u(x,t)=\gamma(t)\,v(x,t)\quad \text{for } (x,t)\in\bar{D},$$

where γ is the function given by formula (2.5) and v is a quasi-(m)-regular function in D. By Lemma 2.3 and assumption (F), we obtain that if the function v is the quasi-(m)-regular solution in D of the (C_f^m) quasi-linear problem, where:

(6.10) $$f_i(x)=\begin{cases}F_0(x) & \text{for } x\in\bar{D}_0,\ i=0,\\ F_i(x) & \text{for } x\in D_0,\ i\in\boldsymbol{I}_{m-1},\end{cases}$$

(6.11) $$f_{i,q}^j(x^i,t)$$
$$=\begin{cases}(\gamma(t))^{-1}F_{i,0}^j(x^i,t) & \text{for } (x^i,t)\in\bar{D}_i\times[0,T],\ i\in\boldsymbol{I}_n,\ j\in\boldsymbol{I}_2,\ q=0,\\ (\gamma(t))^{-1}F_{i,q}^j(x^i,t) & \text{for } (x^i,t)\in\bar{D}_i\times(0,T],\ i\in\boldsymbol{I}_n,\ j\in\boldsymbol{I}_2,\ q\in\boldsymbol{I}_{m-1}\end{cases}$$

and

(6.12) $$f(x,t,z)=(\gamma(t))^{-1}F(x,t,\gamma(t)z)\quad \text{for } (x,t)\in D,\ z\in\boldsymbol{R},$$

then the function u, given by formula (6.9), is the quasi-(m)-regular solution in D of the ($\tilde{C}_F^m$) quasi-linear problem and vice versa.

To find the form of the function v we may apply Theorem 5.1 since functions (6.10)–(6.12) satisfy all the assumptions of Theorem 5.1. Indeed, formulae (6.10)–(6.12) and assumptions (A)–(E) of Theorem 6.1 imply assumptions (A)–(E) of Theorem 5.1, respectively. Particularly, by the following conditions:

$$\sup_{D_i\times(0,T]}|f^j_{i,k}|\leqslant \exp\Big(\int_0^T c(s)\,ds\Big)\sup_{D_i\times(0,T]}|F^j_{i,k}|\qquad (i\in \boldsymbol{I}_n,\ j\in \boldsymbol{I}_2,\ k\in \boldsymbol{I}_{m-1}),$$

$$f^j_{i,0}(x^i,t)=\big(\gamma(t)\big)^{-1}F^j_{i,0}(x^i,t)=0\qquad \text{for } (x^i,t)\in Z_i\cup(\bar{D}_i\times\{0\})\ (i\in\boldsymbol{I}_n,\ j\in\boldsymbol{I}_2),$$

$$\frac{\partial f(y,s,z)}{\partial y_i}=\big(\gamma(t)\big)^{-1}\frac{\partial F\big(y,s,\gamma(t)z\big)}{\partial y_i}\qquad \text{for } (y,s)\in D,\ z\in \boldsymbol{R}\ (i\in\boldsymbol{I}_n),$$

$$\frac{\partial f(y,s,z)}{\partial z}=\big(\gamma(t)\big)^{-1}\frac{\partial F\big(y,s,\gamma(t)z\big)}{\partial z}=\frac{\partial F\big(y,s,\gamma(t)z\big)}{\partial\big(\gamma(t)z\big)}\qquad \text{for } (y,s)\in D,\ z\in\boldsymbol{R}$$

and

$$\begin{aligned}|f(y,s,z)-f(y,s,\tilde{z})|&=\big(\gamma(t)\big)^{-1}\big|F\big(y,s,\gamma(t)z\big)-F\big(y,s,\gamma(t)\tilde{z}\big)\big|\\&\leqslant L|z-\tilde{z}|\qquad \text{for } (y,s)\in D,\ z,\tilde{z}\in\boldsymbol{R},\end{aligned}$$

we obtain that the functions $f^j_{i,k}$ $(i\in\boldsymbol{I}_n,\ j\in\boldsymbol{I}_2,\ k\in\boldsymbol{I}_{m-1})$ are bounded in $D_i\times(0,T]$, respectively, and satisfy equations (5.1), the function f satisfy the Lipschitz condition together with the constant L from (5.2) and the functions $\partial f(y,s,z)/\partial y_i$ $(i\in\boldsymbol{I}_n)$, $\partial f(y,s,z)/\partial z$ are continuous for $(y,s)\in D$, $z\in\boldsymbol{R}$.

Then, by Theorem 5.1, we obtain that the function

$$v(x,t)=\lim_{i\to\infty}v_i(x,t)\qquad \text{for } (x,t)\in\bar{D}, \tag{6.13}$$

where

$$v_{i+1}(x,t)=\begin{cases}v_0(x,t)-\int_0^t\int_{D_0}f\big(y,s,v_i(y,s)\big)G^m(x,t,y,s)\,dy\,ds\\ \qquad\qquad \text{for } (x,t)\in\bar{D}_0\times(0,T],\ i\in\boldsymbol{N}_0,\\ 0\qquad\qquad \text{for } (x,t)\in\bar{S}_0,\ i\in\boldsymbol{N}_0,\end{cases} \tag{6.14}$$

$$v_0(x,t)=v^1(x,t)+v^2(x,t)\qquad \text{for } (x,t)\in\bar{D}, \tag{6.15}$$

$$v^1(x,t)=\begin{cases}\sum_{i=0}^{m-1}\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}\int_{D_0}\Delta^j f_{i-j}(y)\,G^{i+1}(x,t,y,0)\,dy\\ \qquad\qquad \text{for } (x,t)\in\bar{D}_0\times(0,T],\\ f_0(x)\qquad\qquad \text{for } (x,t)\in S_0,\\ 0\qquad\qquad \text{for } (x,t)\in\partial D_0\times\{0\},\end{cases} \tag{6.16}$$

$$(6.17)\qquad v^2(x,t) = \sum_{i=1}^{n}\sum_{k=0}^{m-1}\left(v_{i,k}^1(x,t)+v_{i,k}^2(x,t)\right) \quad \text{for } (x,t)\in\bar{D},$$

$$(6.18)\qquad v_{i,k}^j(x,t) = \begin{cases} -2a_i \int\limits_0^t \int\limits_{D_i} f_{i,k}^j(y^i,s)\, D_{y_i} G^{k+1}(x,t,y,s)\Big|_{y_i=(-1)^j c_i} dy^i\, ds & \\ & \text{for } (x,t)\in(\bar{D}_0\times(0,T])\setminus\tilde{S}_i^j, \\ g_{i,k}^j(x^i,t) & \text{for } (x,t)\in\tilde{S}_i^j, \\ 0 & \text{for } (x,t)\in\bar{S}_0 \end{cases}$$

for $i\in \boldsymbol{I}_n$, $j\in\boldsymbol{I}_2$, $k\in\tilde{\boldsymbol{I}}_{m-1}$ and

$$(6.19)\qquad g_{i,k}^j(x^i,t) = \begin{cases} f_{i,0}^j(x^i,t) & \text{for } (x^i,t)\in\bar{D}_i\times(0,T],\ i\in\boldsymbol{I}_n,\ j\in\boldsymbol{I}_2,\ k=0, \\ 0 & \text{for } (x^i,t)\in\bar{D}_i\times(0,T],\ i\in\boldsymbol{I}_n,\ j\in\boldsymbol{I}_2,\ k\in\boldsymbol{I}_{m-1}, \end{cases}$$

is the quasi-(m)-regular solution in D of the (C_f^m) quasi-linear problem, where the functions $f, f_k, f_{i,k}^j$ ($i\in\boldsymbol{I}_n$, $j\in\boldsymbol{I}_2$, $k\in\tilde{\boldsymbol{I}}_{m-1}$) are given by formulae (6.10)–(6.12).

Consequently, by formulae (6.9), (6.13)–(6.19) and (6.10)–(6.12), we get that the function u given by formulae (6.1)–(6.8) is the quasi-(m)-regular solution in D of the $(\tilde{C}_F^m)$ quasi-linear problem.

7. Theorems on the existence of solutions of Fourier's linear iterated problems.

7.1. Theorem on the existence of the (m)-regular solution of the Fourier's first linear iterated problem of type $(\tilde{C}_F^m)$. As a consequence of Theorem 4.1 from this paper, Theorem 7.1 from [9] and an analogous argument as in the proof of Theorem 6.1 from this paper, we obtain the following:

THEOREM 7.1. *Assume that asumptions* (A), (B) *and* (F) *of Theorem* 6.1 *are satisfied. Suppose additionally that the function* $F(y,s)$ *is continuous for* $(y,s)\in\bar{D}$ *and the functions* $\partial F(y,s)/\partial y_i$ $(i\in\boldsymbol{I}_n)$ *are continuous for* $(y,s)\in D$. *Then the function*

$$(7.1)\qquad u(x,t) = \gamma(t)\, v(x,t) \quad \textit{for } (x,t)\in\bar{D},$$

where

$$(7.2)\qquad v(x,t) = \sum_{i=1}^{3} v^i(x,t) \quad \textit{for } (x,t)\in\bar{D},$$

the functions v^1 *and* v^2 *are defined by formulae* (6.5)–(6.8) *and*

$v^3(x, t)$

$$= \begin{cases} -\int\limits_0^t \int\limits_{D_0} (\gamma(s))^{-1} F(y, s) G^m(x, t, y, s)\, dy\, ds & \text{for } (x, t) \in \bar{D}_0 \times (0, T], \\ 0 & \text{for } (x, t) \in \bar{S}_0, \end{cases}$$

is the (m)*-regular solution in* D *of the* $(\tilde{C}_F^m)$ *linear problem.*

Remark 7.1. If all the assumptions of Theorem 7.1 are satisfied for $m = 1$, then Section 22.7 from [10] or Theorem 2.1 from [3] imply that the function u given by formulae (7.1) and (7.2) is the only one (1)-regular solution in D of the $(\tilde{C}_F^1)$ linear problem.

Remark 7.2. If $c(t) = 0$ for $t \in [0, T]$, then Theorem 7.1 and Remark 7.1 refer to the Fourier's first linear iterated problem of type (C_f^m).

7.2. Theorem on the existence of a solution of Fourier's second linear iterated problem of type $(\tilde{C}_F^m)$. For all $x \in \boldsymbol{R}^n$, $y \in \boldsymbol{R}^n$, $0 \leqslant s < t \leqslant T$ and for every fixed natural number q we define the function G^q by formulae (2.2), (2.3) and

$$(7.3)\qquad G_i(x_i, t, y_i, s) = U_i(x_i, t, y_i, s) + \sum_{k=1}^{\gamma} \left(U_{i,k}^{(1)}(x_i, t, y_i, s) + U_{i,k}^{(2)}(x_i, t, y_i, s)\right),$$

where the functions U_i, $U_{i,k}^{(j)}$ $(i \in \boldsymbol{I}_n,\ j \in \boldsymbol{I}_2,\ k \in \boldsymbol{N})$ are given by formulae (2.1).

Applying similar arguments as in papers [4], [6]–[9] and in this paper, and using results from [5], we obtain the following theorems:

THEOREM 7.2. *Let* q *be an arbitrary fixed natural number, and let* G^q *be the function defined by formulae* (2.2), (2.3) *and* (7.3). *Then*:

(A) *The function* $G^q(x, t, y, s)$ *and the derivatives* $D_{x,t}^{\alpha} G^q(x, t, y, s)$, $D_{y,s}^{\alpha} G^q(x, t, y, s)$ $(\alpha \in N_0^{n+1},\ |\alpha| \neq 0)$ *are continuous for all* $(x, t) \in \bar{D}_0 \times (0, T]$, $(y, s) \in \bar{D}$, $s < t$.

(B) $P_{x,t}^k G^q(x, t, y, s) = \bar{P}_{y,s}^k G^q(x, t, y, s)$

$$= \begin{cases} G^{q-k}(x, t, y, s) & \text{for } k = 0, 1, \ldots, q-1, \\ 0 & \text{for } k = q, q+1, \ldots, \end{cases}$$

where $(x, t) \in \bar{D}_0 \times (0, T]$, $(y, s) \in \bar{D}$, $s < t$.

(C) $P_{x,t}^k D_{x_i} G^q(x, t, y, s) = 0$ *for* $(x, t) \in \tilde{S}_i^j$ $(i \in \boldsymbol{I}_n,\ j \in \boldsymbol{I}_2,\ k \in \boldsymbol{N}_0)$, $(y, s) \in \bar{D}$, $s < t$.

(D) $\lim\limits_{s \to t} D_t^k G^q(x, t, y, s) = 0$ *for* $(x, t) \in D$, $(y, s) \in \bar{D}$, $x \neq y$, $s < t$ $(k \in \tilde{\boldsymbol{I}}_{q-1})$.

THEOREM 7.3. *Assume that:*

(A) *c is a continuous function on the interval $[0, T]$, $T < \infty$, such that the derivatives $c^{(i)}(0) = 0$ ($i \in \tilde{I}_{m-1}$, $m \geqslant 2$), γ is the function defined by formula (2.5), and the functions G^i ($i \in I_n$) are given by formulae (2.2), (2.3) and (7.3).*

(B) *The functions $D^{\alpha^i} F_i$ ($\alpha^i \in N_0^n$, $|\alpha^i| \leqslant 2m-2i-2$, $i \in \tilde{I}_{m-1}$) are continuous and bounded in D_0.*

(C) *The functions $F^j_{i,q}$ ($i \in I_n$, $j \in I_2$, $q \in \tilde{I}_{m-1}$) are continuous and bounded in the domains $D_i \times (0, T]$, respectively.*

(D) *The function $F(y, s)$ is continuous for $(y, s) \in \bar{D}$ and the functions $\partial F(y, s)/\partial y_i$ ($i \in I_n$) are continuous for $(y, s) \in D$.*

Then the function u of the form

$$u(x, t) = \sum_{i=1}^{3} u^i(x, t) \quad \text{for } (x, t) \in D,$$

where

$$u^1(x, t) = \sum_{i=0}^{m-1} \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} \gamma(t) \int_{D_0} \Delta^j F_{i-j}(y) G^{i+1}(x, t, y, 0)\, dy,$$

$$u^2(x, t) = \sum_{i=1}^{n} \sum_{k=0}^{m-1} \left(u^1_{i,k}(x, t) + u^2_{i,k}(x, t)\right),$$

$$u^j_{i,k}(x, t) = -2a_i \gamma(t) \int_0^t \int_{D_i} (\gamma(s))^{-1} F^j_{i,k}(y^i, s) G^{k+1}(x, t, y, s)\Big|_{y_i = (-1)^j c_i} dy^i\, ds,$$

$$u^3(x, t) = -\gamma(t) \int_0^t \int_{D_0} (\gamma(s))^{-1} F(y, s) G^m(x, t, y, s)\, dy\, ds$$

is continuous together with the derivatives $D^{\alpha}_{x,t} u$ ($\alpha = (\tilde{\alpha}, \alpha_*)$, $\tilde{\alpha} \in N_0^n$, $\alpha_* \in N_0$, $0 < |\tilde{\alpha}| + 2\alpha_* \leqslant 2m$) in the domain D and satisfies the following Fourier's second linear iterated problem of type ($\tilde{C}^m_F$):

$$\begin{aligned} \tilde{P}^m u(x, t) &= F(x, t) && \text{for } (x, t) \in D, \\ D^k_t u(x, t) &= F_k(x) && \text{for } (x, t) \in S_0,\ k \in \tilde{I}_{m-1}, \\ \tilde{P}^k D_{x_i} u(x, t) &= F^j_{i,k}(x^i, t) && \text{for } (x, t) \in S^j_i,\ i \in I_n,\ j \in I_2,\ k \in \tilde{I}_{m-1}. \end{aligned}$$

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