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On the homogeneous with respect to the differential equation Fourier's first linear iterated problem in the $(n+1)$-dimensional time-space cube

Abstract. A construction of a solution of the homogeneous with respect to the differential equation of the Fourier's first linear iterated problem in the domain $(\bigtimes_{i=1}^{n}(-c_i, c_i))\times(0, T]$, $T < \infty$, is given.

1. Introduction. In the paper we construct a solution of the homogeneous with respect to the differential equation of the Fourier's first linear iterated problem in the domain $D = (\bigtimes_{i=1}^{n}(-c_i, c_i))\times(0, T]$, $T < \infty$. For this purpose, we use the Green's method and we apply the method of heat iterated potentials. To construct the solution of the problem considered, we use results of papers [3]–[6]. This paper is a continuation of those ones. We may apply [4]–[6] since all the results given in those papers in the domain $(\bigtimes_{i=1}^{n}(-c_i, c_i))\times(0, T)$, $T \leqslant \infty$, hold also in the domain D.

We consider, therefore, the homogeneous with respect to the differential equation the Fourier's first linear iterated problem in the domain D but not in the domain $(\bigtimes_{i=1}^{n}(-c_i, c_i))\times(0, T)$, $T < \infty$, because we must obtain the solution of this problem in a class of functions which are continuous not only in D and also in $\bar{D}$. It will be seen here that D is more convenient for this purpose than $(\bigtimes_{i=1}^{n}(-c_i, c_i))\times(0, T)$, $T < \infty$. Therefore, we must get the continuous in $\bar{D}$ solution of the problem considered, because we shall use results of this paper in the next paper [7], by the author, on constructions, among other things, of solutions of the Fourier's first quasi-linear iterated problems in D. Since for the construction of the solutions of those quasi-linear problems from [7] we shall apply also the Poisson theorem (see [8], p. 523), the solution of the problem from this paper must be continuous in $\bar{D}$.

The results obtained here are generalizations of those given by the author in [3], by Barański and by Musiałek in [1], and by Milewski in [11].

2. Preliminaries. Throughout the paper we use the following notations:

$$R_- = (-\infty, 0), \quad R_+ = (0, \infty), \quad R = (-\infty, \infty),$$

$$N = \{1, 2, \ldots\}, \quad N_0 = N \cup \{0\},$$

$$R^n = R \times \ldots \times R, \quad N_0^n = N_0 \times \ldots \times N_0 \quad (n\text{-times}),$$

$$I_n = \{1, 2, \ldots, n\}, \quad \tilde{I}_n = I_n \cup \{0\} \quad (n \in N),$$

$$x = (x_1, \ldots, x_n), \quad x_0 = (x_1^0, \ldots, x_n^0), \quad x_* = (x_1^*, \ldots, x_n^*), \quad y = (y_1, \ldots, y_n),$$

$$x^i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \quad x_*^i = (x_1^*, \ldots, x_{i-1}^*, x_{i+1}^*, \ldots, x_n^*) \quad (i \in I_n),$$

$$x^{i,j} = (x_1, \ldots, x_{i-1}, (-1)^j c_i, x_{i+1}, \ldots, x_n),$$

$$x_*^{i,j} = (x_1^*, \ldots, x_{i-1}^*, (-1)^j c_i, x_{i+1}^*, \ldots, x_n^*) \quad (i \in I_n,\ j \in I_2),$$

$$D_0 = \mathop{\times}_{i=1}^{n} (-c_i, c_i), \quad \partial D_0 = \bar{D}_0 \setminus D_0, \quad S_0 = D_0 \times \{0\}, \quad D_i = \mathop{\times}_{\substack{k=1 \\ k \neq i}}^{n} (-c_k, c_k) \quad (i \in I_n),$$

$$D_i^j = (-c_1, c_1) \times \ldots \times (-c_{i-1}, c_{i-1}) \times \{(-1)^j c_i\} \times (-c_{i+1}, c_{i+1}) \times \ldots \times (-c_n, c_n) \quad (i \in I_n,\ j \in I_2),$$

$$D = D_0 \times (0, T], \quad S_i^j = D_i^j \times (0, T], \quad \tilde{S}_i^j = \bar{D}_i^j \times (0, T], \quad T < \infty \quad (i \in I_n,\ j \in I_2),$$

$$Z_i = \partial(\bar{D}_i \times [0, T]) \setminus \{(x^i, t)\colon t = 0\} \quad (i \in I_n),$$

$$P_{x,t} = \Delta_x - D_t \quad \text{and} \quad a = \prod_{i=1}^{n} a_i,$$

where $\Delta_x = \sum_{i=1}^{n} a_i D_{x_i}^2$ and $a_i \in R_+$ for $i \in I_n$.

By Δ_x^k and by $P_{x,t}^k$ we denote the k-iterations of the operators Δ_x and $P_{x,t}$, respectively. As long as it does not lead to misunderstanding, the operators Δ_x and $P_{x,t}$ will be denoted by Δ and P, respectively.

For each $\alpha = (\alpha_1, \ldots, \alpha_n) \in N_0^n$, $x \in R^n$, we put:

$$|\alpha| = \sum_{i=1}^{n} \alpha_i, \quad \alpha! = \prod_{i=1}^{n} \alpha_i!, \quad a^\alpha = \prod_{i=1}^{n} (a_i)^{\alpha_i}, \quad D_x^\alpha = D_{x_1}^{\alpha_1} \ldots D_{x_n}^{\alpha_n}.$$

Moreover, $D_{x,t}^\alpha := D_x^{\tilde{\alpha}} D_t^{\alpha_*}$, where $\alpha = (\tilde{\alpha}, \alpha_*)$, $\tilde{\alpha} \in N_0^n$, $\alpha_* \in N_0$, $x \in R^n$ and $t \in [0, T]$.

We assume that m is an arbitrary fixed natural number.

We consider here only real functions. It is understood that, if it does not lead to misunderstanding, then for an arbitrary function f the symbol f will be used also in the sense: $f(x)$, i.e., in the sense of the value of a function f at a point x.

We use the concept of local uniform convergence of considered integrals in the sense of [10].

Let $a_i \in \boldsymbol{R}_+$ for $i \in \boldsymbol{I}_n$. For every fixed index $i \in \boldsymbol{I}_n$, we define the function $\mathscr{U}$: $\boldsymbol{R}^2 \setminus \{0\} \to \boldsymbol{R}$ by the formula

$$\mathscr{U}(\xi, \tau; a_i) = \begin{cases} (4\pi a_i \tau)^{-1/2} \exp\left(-(4a_i \tau)^{-1} \xi^2\right) & \text{for } \xi \in \boldsymbol{R},\ \tau \in \boldsymbol{R}_+, \\ 0 \quad \text{for } \xi \in \boldsymbol{R},\ \tau \in \boldsymbol{R}_- \text{ or } \xi \in \boldsymbol{R} \setminus \{0\},\ \tau = 0. \end{cases}$$

Now, for all $x \in \boldsymbol{R}^n$, $y \in \boldsymbol{R}^n$, $0 \leqslant s < t$, $i \in \boldsymbol{I}_n$, $j \in \boldsymbol{I}_2$ and $k \in \boldsymbol{N}_0$ we define the functions $U_{i,k}^{(j)}$, U_i by the formulae

$$\text{(2.1)} \qquad \begin{aligned} U_{i,k}^{(j)}(x_i, t, y_i, s) &= \mathscr{U}(y_i - x_{i,k}^{(j)}, t-s; a_i), \\ U_i(x_i, t, y_i, s) &= U_{i,0}^{(j)}(x_i, t, y_i, s), \end{aligned}$$

where $x_{i,k}^{(j)} = (-1)^k (x_i + (-1)^{j+1} 2kc_i)$.

Next, for every $x \in \boldsymbol{R}^n$, $y \in \boldsymbol{R}^n$, $0 \leqslant s < t \leqslant T$ and for every fixed natural number q we define the function G^q by the formula

$$\text{(2.2)} \qquad G^q(x, t, y, s) = \frac{(-1)^{q-1}}{(q-1)!} (t-s)^{q-1} G(x, t, y, s),$$

where

$$\text{(2.3)} \qquad G(x, t, y, s) = \prod_{i=1}^{n} G_i(x_i, t, y_i, s),$$

$$\text{(2.4)} \qquad \begin{aligned} &G_i(x_i, t, y_i, s) \\ &= U_i(x_i, t, y_i, s) + \sum_{k=1}^{\infty} (-1)^k \left(U_{i,k}^{(1)}(x_i, t, y_i, s) + U_{i,k}^{(2)}(x_i, t, y_i, s)\right) \end{aligned}$$

and the functions U_i, $U_{i,k}^{(j)}$ ($i \in \boldsymbol{I}_n$, $j \in \boldsymbol{I}_2$, $k \in \boldsymbol{N}$) are given by formulae (2.1). If $q = 1$, then we apply the symbol G in place of the symbol G^1.

In the sequel we shall need the following lemmas:

LEMMA 2.1 ([4]). *Let $0 \leqslant s < t$, $\alpha \in \boldsymbol{N}_0$, $\varkappa > -1$ and $i \in \boldsymbol{I}_n$. Then there exist positive constants A_α and $B_{\alpha,\varkappa}$ such that*

(i) $|D_\xi^\alpha \mathscr{U}(\xi, t-s; a_i)| \leqslant A_\alpha (t-s)^{-(\alpha+1)/2}$ *for* $\xi \in \boldsymbol{R}$,

(ii) $\int_{\boldsymbol{R}} |D_\xi^\alpha \mathscr{U}(\xi, t-s; a_i)| \, d\xi \leqslant \sqrt{8\pi A}\, A_\alpha (t-s)^{-\alpha/2}$,

where $A = \max\{a_1, \ldots, a_n\}$,

(iii) $|D_\xi^\alpha \mathscr{U}(\xi, t-s; a_i)| \leqslant B_{\alpha,\varkappa} |\xi|^{-\alpha-\varkappa-1} (t-s)^{\varkappa/2}$ *for* $\xi \in \boldsymbol{R} \setminus \{0\}$.

In particular,

(iv) $|D_{x_i}^\alpha U_{i,k}^{(j)}(x_i, t, y_i, s)| \leqslant (2c_i)^{-\alpha-\varkappa-1} (k-1)^{-\alpha-\varkappa-1} B_{\alpha,\varkappa} (t-s)^{\varkappa/2}$

for $x_i,\ y_i \in [-c_i, c_i]$, $j \in \boldsymbol{I}_2$, $k \in \boldsymbol{N} \setminus \{1\}$.

LEMMA 2.2 ([5]). *Let q be an arbitrary fixed natural number, and let G and G^q be the functions defined by formulae* (2.2)–(2.4). *Then*:

(i) $$P^k_{x,t}G^q(x,t,y,s) = \begin{cases} G^{q-k}(x,t,y,s) & \text{for } k = 0, 1, \ldots, q-1, \\ 0 & \text{for } k = q, q+1, \ldots, \end{cases}$$

where $(x,t)\in \bar{D}_0 \times (0, T]$, $(y,s)\in \bar{D}$, $s<t$.

(ii) $$D_t^{\alpha_*}G(x,t,y,s) = \sum_{\substack{\beta\in N_0^n\\ |\beta|=\alpha_*}} \frac{\alpha_*!}{\beta!}a^\beta D_x^{2\beta}G(x,t,y,s),$$

where $(x,t)\in \bar{D}_0 \times (0, T]$, $(y,s)\in \bar{D}$, $s<t$ *and* α_* *is a natural number.*

(iii) $P^k_{x,t}G^q(x,t,y,s) = 0$ *for* $(x,t)\in \tilde{S}_i^j$ $(i\in I_n, j\in I_2, k\in N_0)$, $(y,s)\in\bar{D}$, $s<t$.

(iv) $P^k_{x,t}D_{y_p}G^q(x,t,y^{p,r},s) = 0$ *for* $(x,t)\in\tilde{S}_i^j$ $(i, p\in I_n,\ j, r\in I_2,\ k\in N_0)$, $(y,s)\in\bar{D}$, $s<t$.

Let Ω_0 be an arbitrary domain in R^n. Put

$$g_i(x) = \sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}\Delta^j f_{i-j}(x) \quad \text{for } x\in\Omega_0,\ i\in\tilde{I}_{m-1}. \tag{2.5}$$

LEMMA 2.3. *If the functions $D^{\alpha^k}f_k$ $(\alpha^k\in N_0^n,\ |\alpha^k|\leqslant 2m-2k-2,\ k\in\tilde{I}_{m-1})$ are defined in Ω_0, and if the functions g_i $(i\in\tilde{I}_{m-1})$ are defined by formulae* (2.5), *then*:

(i) *The functions $D^{\beta^i}g_i$ $(\beta^i\in N_0^n,\ |\beta^i|\leqslant 2m-2i-2,\ i\in\tilde{I}_{m-1})$ are defined in Ω_0.*

(ii) *The equations*

$$f_i(x) = \sum_{j=0}^{i}(-1)^j\binom{i}{j}\Delta^{i-j}g_j(x) \quad (i\in\tilde{I}_{m-1}) \tag{2.6}$$

hold for every $x\in\Omega_0$.

Proof. (i) Let i be an arbitrary fixed number belonging to the set $\tilde{I}_{m-1}$. Then the functions $D^{\beta^j}f_{i-j}$ $(\beta^j\in N_0^n,\ |\beta^j|\leqslant 2m-2i+2j-2,\ j\in\tilde{I}_i)$ are defined in Ω_0. Therefore the functions $D^{\beta^i}g_i$ $(\beta^i\in N_0^n,\ |\beta^i|\leqslant 2m-2i-2)$ are defined in Ω_0 and, by the fact that i is an arbitrary fixed index, assertion (i) holds.

(ii) Let $i\in\tilde{I}_{m-1}$. By assertion (i) both sides of equations (2.6) together with their derivatives of order not greater that $2m-2i-2$ are defined in Ω_0. Therefore, to prove equations (2.6), denote the right-hand sides of those equations by $P_i(x)$ and substitute the functions g_j $(j\in\tilde{I}_i)$, defined by formulae (2.5), for $P_i(x)$. Hence we have

$$\begin{aligned} P_i(x) &= \sum_{j=0}^{i}(-1)^j\binom{i}{j}\Delta^{i-j}\Big(\sum_{r=0}^{j}(-1)^{j-r}\binom{j}{r}\Delta^r f_{j-r}(x)\Big) \\ &= f_i(x) + \sum_{r=0}^{i-1}\Big(\sum_{j=r}^{i}(-1)^{j+r}\binom{i}{j}\binom{j}{j-r}\Big)\Delta^{i-r}f_r(x) \quad \text{for } x\in\Omega_0. \end{aligned}$$

Consequently, in order to prove assertion (ii) it is sufficient to show that

$$\sum_{j=r}^{i}(-1)^{j+r}\binom{i}{j}\binom{j}{j-r}=0 \quad \text{for } r\in\tilde{I}_{i-1}. \tag{2.7}$$

To this purpose observe that

$$(w_1+w_2)^i=\sum_{j=0}^{i}\binom{i}{j}w_1^j\,w_2^{i-j} \quad \text{for } w_1, w_2\in\boldsymbol{R}.$$

Differentiating both sides of the above equations r times with respect to w_1, we get

$$i(i-1)\cdot\ldots\cdot(i-r+1)(w_1+w_2)^{i-r}=\sum_{j=r}^{i}\binom{i}{j}j(j-1)\cdot\ldots\cdot(j-r+1)\,w_1^{j-r}\,w_2^{i-j}$$

for $r\in\tilde{I}_i$ and therefore

$$\sum_{j=r}^{i}(-1)^{j+r}\binom{i}{j}j(j-1)\cdot\ldots\cdot(j-r+1)=0 \quad \text{for } r\in\tilde{I}_{i-1}. \tag{2.8}$$

Since

$$\sum_{j=r}^{i}(-1)^{j+r}\binom{i}{j}\binom{j}{j-r}=\frac{1}{r!}\sum_{j=r}^{i}(-1)^{j+r}\binom{i}{j}j(j-1)\cdot\ldots\cdot(j-r+1)$$

for $r\in\tilde{I}_i$, by (2.8) we get (2.7). This completes the proof of Lemma 2.1.

3. Formulation of the homogeneous with respect to the differential equation of Fourier's first linear iterated problem of type (C^m). A continuous function u in $\bar{D}$ is called a *(m)-regular* in D if the derivatives $D^{\alpha}_{x,t}u$ ($\alpha=(\tilde{\alpha},\alpha_*)$, $\tilde{\alpha}\in N_0^n$, $\alpha_*\in N_0$, $0<|\tilde{\alpha}|+2\alpha_*\leqslant 2m$) are continuous in D.

Given the functions f_k, $f_{i,k}^j$ ($i\in I_n$, $j\in I_2$, $k\in\tilde{I}_{m-1}$), the homogeneous with respect to the differential equation Fourier's first linear iterated problem of type (C^m) in D consists in finding a (*m*)-regular function u in D satisfying the equation

$$P^m u(x,t)=0 \quad \text{for } (x,t)\in D, \tag{3.1}$$

satisfying the initial conditions

$$D_t^k u(x,t)=\begin{cases} f_0(x) & \text{for } (x,t)\in\bar{S}_0,\ k=0,\\ f_k(x) & \text{for } (x,t)\in S_0,\ k\in I_{m-1},\end{cases} \tag{3.2}$$

and satisfying the boundary conditions(1)

$$P^k u(x,t)=\begin{cases} f_{i,0}^j(x^i,t) & \text{for } (x,t)\in\bar{S}_i^j,\ i\in I_n,\ j\in I_2,\ k=0,\\ f_{i,k}^j(x^i,t) & \text{for } (x,t)\in S_i^j,\ i\in I_n,\ j\in I_2,\ k\in I_{m-1}.\end{cases} \tag{3.3}$$

(1) The left-hand sides of equations (3.2) and (3.3) are meant in the limit sense.

A function u with the foregoing properties is called a *(m)-regular solution* in D of the problem above, and this problem is called shortly the *(C^m) linear problem.* If $m = 1$, then a (1)-regular solution in D of the (C^1) linear problem is called a *regular solution* in D of this problem.

4. The (m)-regular solution of the (C^m) linear problem. We shall prove in Sections 5 and 6 that, under certain assumptions concerning the functions f_k, $f_{i,k}^j$ $(i \in \boldsymbol{I}_n,\ j \in \boldsymbol{I}_2,\ k \in \tilde{\boldsymbol{I}}_{m-1})$, the function u of the form

$$u(x, t) = u^1(x, t) + u^2(x, t) \quad \text{for } (x, t) \in \bar{D}, \tag{4.1}$$

where

$$u^1(x, t) = \sum_{i=0}^{m-1} u_i(x, t) \quad \text{for } (x, t) \in \bar{D}, \tag{4.2}$$

$$u_i(x, t) = \begin{cases} \int\limits_{D_0} g_i(y) G^{i+1}(x, t, y, 0)\, dy & \text{for } (x, t) \in \bar{D}_0 \times (0, T],\ i \in \tilde{\boldsymbol{I}}_{m-1}, \\ l_i(x, t) & \text{for } (x, t) \in S_0, \quad i \in \tilde{\boldsymbol{I}}_{m-1}, \\ 0 & \text{for } (x, t) \in \partial D_0 \times \{0\}, \quad i \in \tilde{\boldsymbol{I}}_{m-1}, \end{cases} \tag{4.3}$$

$$g_i(y) = \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} \Delta^j f_{i-j}(y) \quad \text{for } y \in \bar{D}_0,\ i \in \tilde{\boldsymbol{I}}_{m-1}, \tag{4.4}$$

$$l_i(x, t) = \begin{cases} f_0(x) & \text{for } (x, t) \in S_0,\ i = 0, \\ 0 & \text{for } (x, t) \in S_0,\ i \in \boldsymbol{I}_{m-1}, \end{cases} \tag{4.5}$$

$$u^2(x, t) = \sum_{i=1}^{n} \sum_{k=0}^{m-1} \left(u_{i,k}^1(x, t) + u_{i,k}^2(x, t)\right) \quad \text{for } (x, t) \in \bar{D}, \tag{4.6}$$

$$u_{i,k}^j(x, t) = \begin{cases} -2a_i \int\limits_0^t \int\limits_{D_i} f_{i,k}^j(y^i, s)\, D_{y_i} G^{k+1}(x, t, y, s)\Big|_{y_i = (-1)^j c_i} dy^i\, ds & \\ \qquad\qquad \text{for } (x, t) \in (\bar{D}_0 \times (0, T]) \setminus \tilde{S}_i^j, \\ g_{i,k}^j(x^i, t) & \text{for } (x, t) \in \tilde{S}_i^j, \\ 0 & \text{for } (x, t) \in \bar{S}_0 \end{cases} \tag{4.7}$$

for $i \in \boldsymbol{I}_n$, $j \in \boldsymbol{I}_2$, $k \in \tilde{\boldsymbol{I}}_{m-1}$ and

$$g_{i,k}^j(x^i, t) = \begin{cases} f_{i,0}^j(x^i, t) & \text{for } (x^i, t) \in \bar{D}_i \times (0, T],\ i \in \boldsymbol{I}_n,\ j \in \boldsymbol{I}_2,\ k = 0, \\ 0 & \text{for } (x^i, t) \in \bar{D}_i \times (0, T],\ i \in \boldsymbol{I}_n,\ j \in \boldsymbol{I}_2,\ k \in \boldsymbol{I}_{m-1}, \end{cases} \tag{4.8}$$

is the (m)-regular solution in D of the (C^m) linear problem.

5. Properties of heat iterated potentials of the first kind. Let $i \in \tilde{\boldsymbol{I}}_{m-1}$, $\alpha = (\tilde{\alpha}, \alpha_*)$, $\tilde{\alpha} \in \boldsymbol{N}_0^n$, $\alpha_* \in \boldsymbol{N}_0$.

Let us consider the integrals

$$X_i^{\alpha}(x, t) = \int_{D_0} h_i(y) D_{x,t}^{\alpha} G^{i+1}(x, t, y, 0)\, dy,$$

where h_i are the given functions and G^{i+1} are the functions defined by formulae (2.2)–(2.4).

The integrals X_i^0 are called the *heat iterated potentials* of the first kind of the surface S_0.

Given a function h, let us introduce the notation

$$\omega_h(x, t) = \int_{D_0} h(y) G(x, t, y, 0)\, dy. \tag{5.1}$$

We shall use the following lemmas:

LEMMA 5.1 ([6]). *Assume that $i \in \tilde{I}_{m-1}$. If the functions h_i are measurable and bounded in the domain D_0, then:*

(i) *The integrals X_i^{α} ($|\tilde{\alpha}| + 2\alpha_* \leqslant 2m$) are locally uniformly convergent in the domain $\bar{D}_0 \times (0, T]$.*

(ii) *For every point $(x, t) \in \bar{D}_0 \times (0, T]$ there exist the derivatives $D_{x,t}^{\alpha} X_i^0$ ($0 < |\tilde{\alpha}| + 2\alpha_* \leqslant 2m$) and $D_{x,t}^{\alpha} X_i^0(x, t) = X_i^{\alpha}(x, t)$ for all $(x, t) \in \bar{D}_0 \times (0, T]$, $0 < |\tilde{\alpha}| + 2\alpha_* \leqslant 2m$.*

LEMMA 5.2 ([6]). *The following assertions are true:*

(i) *Let k be an arbitrary fixed number belonging to the set $\tilde{I}_{m-1}$. If the functions $D^{\alpha^i} h_i$ ($\alpha^i \in N_0^n$, $|\alpha^i| \leqslant 2k - 2i$, $i \in \tilde{I}_k$) are measurable and bounded in the domain D_0 and if these functions are continuous at every fixed point $x_0 \in D_0$, then*

$$D_t^{k-i} \omega_{h_i}(x, t) \to \Delta_x^{k-i} h_i(x_0) \quad \text{as } (x, t) \to (x_0, 0^+),\ (x, t) \in D \ (i \in \tilde{I}_k).$$

(ii) *Let x_0 be an arbitrary fixed point belonging to $\bar{D}_0$. If the function h is measurable and bounded in D_0, continuous on ∂D_0 and such that $h|_{\partial D_0} = 0$, and if h is continuous at the point x_0, when this point belongs to D_0, then*

$$\omega_h(x, t) \to h(x_0) \quad \text{as } (x, t) \to (x_0, 0^+),\ (x, t) \in \bar{D}_0 \times (0, T].$$

LEMMA 5.3. *Let k be an arbitrary fixed number belonging to $\tilde{I}_{m-1}$, and let u_i, g_i ($i \in \tilde{I}_{m-1}$) be the functions given by formulae (4.3)–(4.5). Assume that the functions $D^{\alpha^j} f_j$ ($\alpha^j \in N_0^n$, $|\alpha^j| \leqslant 2i - 2j$, $j \in \tilde{I}_i$, $i \in \tilde{I}_{m-1}$) are measurable and bounded in D_0. Then:*

(i) *The functions u_i ($i \in \tilde{I}_{m-1}$) satisfy the equations*

$$P^m u_i(x, t) = 0 \quad \text{for } (x, t) \in \bar{D}_0 \times (0, T] \ (i \in \tilde{I}_{m-1}) \tag{5.2}$$

and the boundary conditions

$$P^k u_i(x, t) = 0 \quad \text{for } (x, t) \in \tilde{S}_p^r \ (i \in \tilde{I}_{m-1},\ p \in I_n,\ r \in I_2). \tag{5.3}$$

(ii) *If, additionally, the functions $D^{\alpha^j} f_j$ ($\alpha^j \in N_0^n$, $|\alpha^j| \leqslant 2k - 2j$, $j \in \tilde{I}_i$, $i \in \tilde{I}_k$) are continuous at an arbitrary fixed point $x_0 \in D_0$, then the functions u_i*

($i \in \tilde{I}_{m-1}$) *satisfy the initial conditions*

$$(5.4)\qquad \lim_{(x,t)\to(x_0,0^+)} D_t^k u_i(x,t) = \begin{cases} 0 & \text{for } i > k,\\ (-1)^i \binom{k}{i} \Delta_x^{k-i} g_i(x_0) & \text{for } i \leqslant k,\end{cases}$$

where $(x, t) \in D$.

(iii) *If, additionally, the function* f_0 *is continuous on* ∂D_0 *and such that* $f_0|_{\partial D_0} = 0$, *and if* $x_0 \in \bar{D}_0$ *is an arbitrary fixed point at which* f_0 *is continuous, when* $x_0 \in D_0$, *then*

$$(5.5)\qquad \lim_{(x,t)\to(x_0,0^+)} u_i(x,t) = \begin{cases} f_0(x_0) & \text{for } i = 0,\\ 0 & \text{for } i \in I_{m-1},\end{cases}$$

where $(x, t) \in \bar{D}_0 \times (0, T]$.

Proof. (i) Since the functions g_i ($i \in \tilde{I}_{m-1}$) are measurable and bounded in D_0, by assertion (ii) of Lemma 5.1 we have

$$P^r u_i(x,t) = \int_{D_0} g_i(y)\, P^r G^{i+1}(x, t, y, 0)\, dy$$

$$\text{for } (x,t) \in \bar{D}_0 \times (0, T] \ (i \in \tilde{I}_{m-1},\ r \in I_m).$$

Consequently, by assertions (i) and (iii) of Lemma 2.2, and by assertion (i) of Lemma 5.1 applied to the equations above when $r = 0$, we obtain formulae (5.2) and (5.3), respectively.

(ii) Formula (5.1) and the Leibniz theorem on the differentiation imply the equations

$$D_t^k u_i(x,t) = \frac{(-1)^i}{i!} \sum_{j=0}^{l} \binom{k}{j} i(i-1)\cdot\ldots\cdot(i-j+1)\, t^{i-j} D_t^{k-j} \omega_{g_i}(x,t),$$

where $(x, t) \in \bar{D}_0 \times (0, T]$, $l = k$ for $i > k$ and $l = i$ for $i \leqslant k$.

Consequently,

$$\lim_{(x,t)\to(x_0,0^+)} D_t^k u_i(x,t) = \begin{cases} 0 & \text{for } i > k,\\ (-1)^i \binom{k}{i} \lim\limits_{(x,t)\to(x_0,0^+)} D_t^{k-i} \omega_{g_i}(x,t) & \text{for } i \leqslant k,\end{cases}$$

vhere $(x, t) \in D$.

Since the functions $D^{\alpha^i} g_i$ ($\alpha^i \in N_0^n$, $|\alpha^i| \leqslant 2k-2i$, $i \in \tilde{I}_k$) are measurable and bounded in D_0, and since these functions are continuous at the point x_0, by the above equations and by assertion (i) of Lemma 5.2 we get conditions (5.4).

(iii) Since the function g_0 is measurable and bounded in D_0, continuous on ∂D_0 and such that $g_0|_{\partial D_0} = 0$, and since this function is continuous at the point x_0, when $x_0 \in D_0$, it follows that by formula (5.1), by assertion (i) of Lemma 5.1 and by assertion (ii) of Lemma 5.2 we have

$$\lim_{(x,t)\to(x_0,0^+)} u_i(x, t) = \frac{(-1)^i}{i!} \lim_{(x,t)\to(x_0,0^+)} t^i \omega_{g_i}(x, t) = \begin{cases} g_0(x_0) & \text{for } i = 0, \\ 0 & \text{for } i \in \boldsymbol{I}_{m-1}, \end{cases}$$

where $(x, t) \in \bar{D}_0 \times (0, T]$.

Thus, by formula (4.4), we obtain conditions (5.5).

THEOREM 5.1. *Let the functions* $D^{\alpha^i} f_i$ $(\alpha^i \in N_0^n,\ |\alpha^i| \leqslant 2m-2i-2,\ i \in \tilde{\boldsymbol{I}}_{m-1})$ *be continuous and bounded in* D_0, *and let, additionally, the function* f_0 *be continuous in* $\bar{D}_0$ *and such that* $f_0|_{\partial D_0} = 0$. *Moreover, let* u^1 *be the function given by formula* (4.2). *Then*:

(A) *The function* u^1 *satisfies the equation*

$$P^m u^1(x, t) = 0 \quad \text{for } (x, t) \in \bar{D}_0 \times (0, T], \tag{5.6}$$

the initial conditions

$$D_t^k u^1(x, t) = f_k(x) \quad \text{for } (x, t) \in S_0 \ (k \in \boldsymbol{I}_{m-1}), \tag{5.7}$$

$$u^1(x, t) = f_0(x) \quad \text{for } (x, t) \in \bar{S}_0, \tag{5.8}$$

and the boundary conditions

$$P^k u^1(x, t) = 0 \quad \text{for } (x, t) \in \tilde{S}_p^r \ (k \in \tilde{\boldsymbol{I}}_{m-1},\ p \in \boldsymbol{I}_n,\ r \in \boldsymbol{I}_2). \tag{5.9}$$

(B) *The function* u^1 *is* (m)*-regular in* D.

Proof. (A) Formulae (5.6) and (5.9) are consequences of formula (4.2) and of assertion (i) of Lemma 5.3.

To prove (5.7) let us fix an arbitrary $x_0 \in D_0$. We have, by (4.2) and (5.4), the equations

$$\lim_{(x,t)\to(x_0,0^+)} D_t^k u^1(x, t) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \Delta_x^{k-i} g_i(x_0) \quad \text{for } (x, t) \in D \ (k \in \tilde{\boldsymbol{I}}_{m-1}).$$

Hence we get, from assertion (ii) of Lemma 2.3, conditions (5.7).

In a similar way, using (4.2) and (5.5), we obtain

$$\lim_{(x,t)\to(x_0,0^+)} u^1(x, t) = \sum_{i=0}^{m-1} \lim_{(x,t)\to(x_0,0^+)} u_i(x, t) = f_0(x_0),$$

where $(x, t) \in \bar{D}_0 \times (0, T]$, and x_0 is an arbitrary fixed point belonging to $\bar{D}_0$. Therefore condition (5.8) holds.

(B) This assertion is a consequence of assertion (i) of Lemma 5.1, and of formulae (4.2)–(4.5) and (5.8).

6. Properties of heat iterated potentials of the second kind. Let $i \in I_n$, $j \in I_2$, $q \in \tilde{I}_{m-1}$, $\alpha = (\tilde{\alpha}, \alpha_*)$, $\tilde{\alpha} = (\alpha_1, \ldots, \alpha_n) \in N_0^n$, $\alpha_* \in N_0$.

Let us consider the integrals

$$u^{\alpha}_{i,j,q}(x, t) = -2a_i \int_0^t \int_{D_i} f^j_{i,q}(y^i, s) D^{\alpha}_{x,t} D_{y_i} G^{q+1}(x, t, y, s)\big|_{y_i = (-1)^j c_i} dy^i ds,$$

$$Y^{\alpha}_{i,j,q}(x, t, s) = -2a_i \int_{D_i} f^j_{i,q}(y^i, s) D^{\alpha}_{x,t} D_{y_i} G^{q+1}(x, t, y, s)\big|_{y_i = (-1)^j c_i} dy^i,$$

where $f^j_{i,q}$ are the given functions and G^{q+1} are the functions defined by formulae (2.2)–(2.4).

The integrals $u^0_{i,j,q}$, which are equal to the functions $u^j_{i,q}$ for $(x, t) \in (\bar{D}_0 \times (0, T]) \setminus \tilde{S}^j_i$, respectively, are called the *heat iterated potentials of the second kind* of the surfaces S^j_i, respectively.

Lemma 6.1. *Assume that $i \in I_n$, $j \in I_2$, $q \in \tilde{I}_{m-1}$. If the functions $f^j_{i,q}$ are measurable and bounded in the domains $D_i \times (0, T]$, respectively, then:*

(i) *The integrals $u^{\alpha}_{i,j,q}$ ($|\tilde{\alpha}| + 2\alpha_* \leqslant 2m$) are locally uniformly convergent in the domains $(\bar{D}_0 \times (0, T]) \setminus \tilde{S}^j_i$, respectively.*

Moreover, the integrals $Y^{\alpha}_{i,j,q}$ ($|\tilde{\alpha}| + 2\alpha_* \leqslant 2m$) are locally uniformly convergent, as the functions of the variable (x, t), in the domains $(\bar{D}_0 \times (0, T]) \setminus \tilde{S}^j_i$, respectively.

(ii) $\lim_{s \to t} Y^{\alpha[r]}_{i,j,q}(x, t, s) = 0$, *respectively, for* $(x, t) \in (\bar{D}_0 \times (0, T]) \setminus \tilde{S}^j_i$, $0 \leqslant s < t$, *where* $\alpha[r] := (\tilde{\alpha}, \alpha_{*-r-1})$, $r \in \tilde{I}_{\alpha_*-1}$, $\alpha_* \in N$, $0 < |\tilde{\alpha}| + 2\alpha_* \leqslant 2m$.

(iii) *For every point $(x, t) \in (\bar{D}_0 \times (0, T]) \setminus \tilde{S}^j_i$ there exist the derivatives $D^{\alpha}_{x,t} u^j_{i,q}$ ($0 < |\tilde{\alpha}| + 2\alpha_* \leqslant 2m$), respectively, and $D^{\alpha}_{x,t} u^j_{i,q}(x, t) = u^{\alpha}_{i,j,q}(x, t)$ ($0 < |\tilde{\alpha}| + 2\alpha_* \leqslant 2m$) for all $(x, t) \in (\bar{D}_0 \times (0, T]) \setminus \tilde{S}^j_i$, respectively.*

Proof. (i) According to the argumentation given in the proof of assertion (i) of Lemma 5.1 (see [6], Theorem 1) to prove assertion (i) of this lemma for the integrals $u^{\alpha}_{i,j,q}$ ($i \in I_n$, $j \in I_2$, $q \in \tilde{I}_{m-1}$, $|\tilde{\alpha}| + 2\alpha_* \leqslant 2m$) it is sufficient to show that for arbitrary fixed $i \in I_n$, $j \in I_2$, $q \in \tilde{I}_{m-1}$, $\alpha = (\tilde{\alpha}, \alpha_*)$ such that $|\tilde{\alpha}| + 2\alpha_* \leqslant 2m$ and $0 \leqslant \alpha_* \leqslant q$ the integral on the right-hand side of the after-mentioned equation

$$u^{\alpha}_{i,j,q}(x, t) = \frac{2a_i(-1)^{q+1}}{q!} \sum_{p=0}^{\alpha_*} \binom{\alpha_*}{p} \sum_{|\alpha^p| = \alpha_* - p} \frac{(\alpha_* - p)!}{\alpha^p!} a^{\alpha^p} \times$$

$$\times \int_0^t \int_{D_i} f^j_{i,q}(y^i, s) D^p_t (t-s)^q D^{\beta^p}_x D_{y_i} G(x, t, y, s)\big|_{y_i = (-1)^j c_i} dy^i ds,$$

where $\alpha^p = (\alpha^p_1, \ldots, \alpha^p_n)$, $\beta^p = (\beta^p_1, \ldots, \beta^p_n)$, $\beta^p_r = \alpha_r + 2\alpha^p_r$ ($r \in I_n$), $|\beta^p| = |\tilde{\alpha}| + 2\alpha_* - 2p$ ($p \in \tilde{I}_{\alpha_*}$), is locally uniformly convergent in the domain $(\bar{D}_0 \times (0, T]) \setminus \tilde{S}^j_i$.

For this purpose, observe that by assertions (i), (iii) and (iv) of Lemma 2.1 we have

$$(6.1)\quad \left|D_x^{\beta^p} D_{y_i} G(x, t, y, s)\right|_{y_i=(-1)^j c_i}\Big| \leqslant C_1 (t-s)^{\varkappa/2}\left[1+|(-1)^j c_i - x_i|^{-\alpha_i - 2\alpha_i^p - 3} + |(-1)^j c_i + x_i + 2c_i|^{-\alpha_i - 2\alpha_i^p - 3} + |(-1)^j c_i + x_i - 2c_i|^{-\alpha_i - 2\alpha_i^p - 3}\right] \times \prod_{\substack{r=1\\ r\neq i}}^{n} \left[(t-s)^{-(\alpha_r + 2\alpha_r^p + 3)/2} + (t-s)^{1/2}\right]$$

for $(x, t) \in (D_0 \times (0, T]) \setminus \tilde{S}_i^j$, $(y^i, s) \in D_i \times [0, t)$, $p \in \tilde{I}_{\alpha_*}$, where

$$C_1 := \max_{i \in I_n, j \in \tilde{I}_{2m}} \left\{B_{j+1,\varkappa};\ 2(2c_i)^{-j-3} B_{j+1,\varkappa} \sum_{k=1}^{\infty} k^{-2}\right\} \times \left(\max_{i \in I_n, j \in \tilde{I}_{2m}} \left\{3A_j;\ 2(2c_i)^{-j-2} B_{j,1} \sum_{k=1}^{\varkappa} k^{-2}\right\}\right)^{n-1}.$$

Let now x_0 be an arbitrary fixed point belonging to the set $\bar{D}_0 \setminus \bar{D}_i^j$ and let $K_\eta(x_0)$, $K_\eta(x_i^0)$ be spheres with the centres at x_0 and x_i^0, respectively, and a radius $\eta > 0$ such that the set $\overline{K_\eta(x_0)} \cap (\bar{D}_0 \setminus \bar{D}_i^j)$ is closed. Then the set $\overline{K_\eta(x_i^0)} \cap ([-c_i, c_i] \setminus \{(-1)^j c_i\})$ is closed and since the function

$$\gamma(x_i) = 1 + |(-1)^j c_i - x_i|^{-\alpha_i - 2\alpha_i^p - 3} + |(-1)^j c_i + x_i + 2c_i|^{-\alpha_i - 2\alpha_i^p - 3} + |(-1)^j c_i + x_i - 2c_i|^{-\alpha_i - 2\alpha_i^p - 3}$$

is continuous in the set $R \setminus \{(-1)^j c_i,\ -2c_i - (-1)^j c_i,\ 2c_i - (-1)^j c_i\}$, which includes $\overline{K_\eta(x_i^0)}$, there exists a constant $C > 0$ (independent of η) such that $\gamma(x_i) \leqslant C$ for $x_i \in \overline{K_\eta(x_i^0)}$. Therefore, $\gamma(x_i) \leqslant C$ for $x \in \overline{K_\eta(x_0)}$ and consequently, by (6.1), we have

$$|J_p(x, t)| \leqslant C C_1 \sup_{D_i \times (0,T]} |f_{i,q}^j|\, \varphi(t) \int_{K_\eta(x_0^i)} dy^i$$

for $(x, t) \in (\bar{D}_0 \times (0, T]) \setminus \tilde{S}_i^j$, $|x - x_0| < \eta$, $p \in \tilde{I}_{\alpha_*}$, where

$$J_p(x, t) := \int_0^t \int_{D_i \cap K_\eta(x_0^i)} f_{i,q}^j(y^i, s)\, D_t^p (t-s)^q D_x^{\beta^p} D_{y_i} G(x, t, y, s)\Big|_{y_i = (-1)^j c_i} dy^i\, ds,$$

$$\varphi(t) := \int_0^t (t-s)^{\varkappa/2} D_t^p (t-s)^q \prod_{\substack{r=1\\ r\neq i}}^{n} \left[(t-s)^{-(\alpha_r + 2\alpha_r^p + 1)/2} + (t-s)^{1/2}\right] ds$$

and $K_\eta(x_0^i)$ is the sphere with the centre $x_0^i = (x_1^0, \ldots, x_{i-1}^0, x_{i+1}^0, \ldots, x_n^0)$ and the radius η. Choosing $\varkappa$ so large that exponent of $t-s$ is positive in the integral $\varphi(t)$, we obtain the inequality $\varphi(t) \leqslant \varphi(T)$ for $t \in [0, T]$. Therefore

$$|J_p(x, t)| \leqslant CC_1 \sup_{D_i \times (0,T]} |f^j_{i,q}| \varphi(T) \tau_{n-1} \eta^{n-1}$$

for $(x, t) \in (\bar{D}_0 \times (0, T]) \setminus \tilde{S}^j_i$, $|x - x_0| < \delta$, $p \in \tilde{I}_{\alpha_*}$, where τ_{n-1} is the volume of the $(n-1)$-dimensional unit sphere,

$$\delta = \eta < \left[\frac{\varepsilon}{CC_1 \sup\limits_{D_i \times (0,T]} |f^j_i, q| \varphi(T) \tau_{n-1}} \right]^{1/(n-1)}, \tag{6.2}$$

and ε is an arbitrary positive number. Then

$$|J_p(x, t)| \leqslant \varepsilon \quad \text{for } (x, t) \in (\bar{D}_0 \times (0, T]) \setminus \tilde{S}^j_i,\ |x - x_0| < \delta,\ p \in \tilde{I}_{\alpha_*}.$$

Since $\varepsilon > 0$, the indexes i, j, q, the multi-index α and the point x_0 are arbitrary, it follows that the proof of assertion (i) is complete for the integrals $u^\alpha_{i,j,q}$.

The proof of locally uniform convergence of the integrals $Y^\alpha_{i,j,q}$ is analogous to the proof of locally uniform convergence of the integrals $u^\alpha_{i,j,q}$. Indeed, applying all the notations from the above argument except formula (6.2), which now is of the form

$$\delta = \eta < \left[\frac{\varepsilon}{CC_1 \sup\limits_{D_i \times (0,T]} |f^j_{i,q}| \psi(T) \tau_{n-1}} \right]^{1/(n-1)},$$

where

$$\psi(t-s) := (t-s)^{\varkappa/2} D^p_t (t-s)^q \prod_{\substack{r=1 \\ r \neq i}}^{n} [(t-s)^{-(\alpha_r + 2\alpha^p_r + 1)/2} + (t-s)^{1/2}],$$

we obtain for so large $\varkappa$ for which exponent of $t-s$ is positive in $\psi(t-s)$ the following estimations:

$$\Big| \int\limits_{D_i \cap K_\eta(x^i_0)} f^j_{i,q}(y^i, s) D^p_t (t-s)^q D^{\beta^p}_x D_{y_i} G(x, t, y, s)\big|_{y_i = (-1)^j c_i} dy^i \Big|$$
$$\leqslant CC_1 \sup_{D_i \times (0,T]} |f^j_{i,q}| \psi(T) \tau_{n-1} \eta^{n-1} \leqslant \varepsilon$$

for $(x, t) \in (\bar{D}_0 \times (0, T]) \setminus \tilde{S}^j_i$, $|x - x_0| < \delta$, $p \in \tilde{I}_{\alpha_*}$. This completes the proof of assertion (i).

(ii) Fix indexes $i \in I_n$, $j \in I_2$, $q \in \tilde{I}_{m-1}$ and $\alpha = (\tilde{\alpha}, \alpha_*)$ such that $\alpha_* \in N$, $0 < |\tilde{\alpha}| + 2\alpha_* \leqslant 2m$. Observe now that by the Leibniz theorem on the differentiation and by assertion (ii) of Lemma 2.2 we get

$$Y^{\alpha[r]}_{i,j,q}(x, t, s) = \frac{2a_i(-1)^{q+1}}{q!} \sum_{p=0}^{l} \binom{\alpha_* - r - 1}{p} \sum_{|\alpha^p| = \alpha_* - r - p - 1} \frac{(\alpha_* - r - p - 1)!}{\alpha^p!} a^{\alpha^p} \times$$
$$\times \int\limits_{D_i} f^j_{i,q}(y^i, s) D^p_t (t-s)^q D^{\beta^p}_x D_{y_i} G(x, t, y, s)\big|_{y_i = (-1)^j c_i} dy^i,$$

where $(x, t) \in (\bar{D}_0 \times (0, T]) \setminus \tilde{S}_i^j$, $0 \leqslant s < t$, $l = \alpha_* - r - 1$ for $0 \leqslant \alpha_* - r - 1 \leqslant q$ and $l = q$ for $q < \alpha_* - r - 1 \leqslant m$, $\alpha^p = (\alpha_1^p, \ldots, \alpha_n^p)$, $\beta^p = (\beta_1^p, \ldots, \beta_n^p)$, $\beta_k^p = \alpha_k + 2\alpha_k^p$ $(k \in \boldsymbol{I}_n)$, $|\beta^p| = |\tilde{\alpha}| + 2\alpha_* - 2r - 2p - 2$ $(p \in \tilde{\boldsymbol{I}}_{\alpha_* - r - 1}$ or $p \in \tilde{\boldsymbol{I}}_q$, $r \in \tilde{\boldsymbol{I}}_{\alpha_* - 1})$.

Simultaneously, by assertions (ii), (iii) and (iv) of Lemma 2.1, we have

$$\Big| \int_{D_i} f_{i,q}^j(y^i, s) D_t^p (t-s)^q D_x^{\beta^p} D_{y_i} G(x, t, y, s)\big|_{y_i = (-1)^j c_i} dy^i \Big|$$

$$\leqslant q(q-1)\cdot\ldots\cdot(q-p+1) C_2 \sup_{D_i \times (0,T]} |f_{i,q}^j| \big[1 + |(-1)^j c_i - x_i|^{-\alpha_i - 2\alpha_i^p - \varkappa - 2} +$$

$$+ |(-1)^j c_i + x_i + 2c_i|^{-\alpha_i - 2\alpha_i^p - \varkappa - 2} + |(-1)^j c_i + x_i - 2c_i|^{-\alpha_i - 2\alpha_i^p - \varkappa - 2}\big] \times$$

$$\times (t-s)^{\varkappa/2 + q - p} \prod_{\substack{k=1 \\ k \neq i}}^{n} \big[(t-s)^{-(\alpha_k + 2\alpha_k^p)/2} + (t-s)^{1/2}\big],$$

where $(x, t) \in (\bar{D}_0 \times (0, T]) \setminus \tilde{S}_i^j$, $0 \leqslant s < t$, β^p satisfy the above properties, $p \in \tilde{\boldsymbol{I}}_{\alpha_* - r - 1}$ or $p \in \tilde{\boldsymbol{I}}_q$, $r \in \tilde{\boldsymbol{I}}_{\alpha_* - 1}$, the constant

$$(6.3) \qquad C_2 := \max_{i \in \boldsymbol{I}_n, j \in \tilde{\boldsymbol{I}}_{2m}} \{B_{j+1,\varkappa};\ 2(2c_i)^{-j-\varkappa-2} B_{j+1,\varkappa} \sum_{k=1}^{\infty} k^{-2}\} \times$$

$$\times \Big(\max_{i \in \boldsymbol{I}_n, j \in \tilde{\boldsymbol{I}}_{2m}} \{3\sqrt{8\pi A}\, A_j;\ 2(2c_i)^{-j-2} B_{j,1} \sum_{k=1}^{\infty} k^{-2}\}\Big)^{n-1}$$

and $\varkappa$ is a constant greater than -1.

Consequently, assertion (ii) holds since $\frac{1}{2}\varkappa + q - p - \frac{1}{2} \sum_{\substack{k=1 \\ k \neq i}}^{n} (\alpha_k + 2\alpha_k^p) > 0$ all possible p from both the cases where $\varkappa > 4m$.

(iii) First we shall show that

$$(6.4) \qquad D_{x,t}^{\alpha} u_{i,q}^j(x, t) = u_{i,j,q}^{\alpha}(x, t) + \sum_{r=0}^{\alpha_* - 1} D_t^r \big(\lim_{s \to t} Y_{i,j,q}^{\alpha[r]}(x, t, s)\big)$$

for all $(x, t) \in (\bar{D}_0 \times (0, T]) \setminus \tilde{S}_i^j$, $i \in \boldsymbol{I}_n$, $j \in \boldsymbol{I}_2$, $q \in \tilde{\boldsymbol{I}}_{m-1}$ and $\alpha = (\tilde{\alpha}, \alpha_*)$ such that $\alpha_* \in N$, $0 < |\tilde{\alpha}| + 2\alpha_* \leqslant 2m$.

Since, by assertion (i) of Lemma 6.1, the integrals $u_{i,j,q}^{\alpha}$ and $Y_{i,j,q}^{\alpha}$ $(i \in \boldsymbol{I}_n$, $j \in \boldsymbol{I}_2$, $q \in \tilde{\boldsymbol{I}}_{m-1}$, $|\tilde{\alpha}| + 2\alpha_* \leqslant 2m)$ are locally uniformly convergent, as the functions of the variable (x, t), in the domains $(\bar{D}_0 \times (0, T]) \setminus \tilde{S}_i^j$ $(i \in \boldsymbol{I}_n$, $j \in \boldsymbol{I}_2)$, respectively, and since the variable x is not in the integration limits, we can calculate the derivatives $D_x^{\tilde{\alpha}} u_{i,q}^j$ $(i \in \boldsymbol{I}_n$, $j \in \boldsymbol{I}_2$, $q \in \tilde{\boldsymbol{I}}_{m-1}$, $|\tilde{\alpha}| \leqslant 2m)$ in the usual sense by differentiating under the integral sign. Therefore, it is sufficient to prove formulae (6.4) by induction with respect to α_* under an arbitrary fixed $\tilde{\alpha} \in N_0^n$, i.e., it is sufficient to show that under an arbitrary fixed $\tilde{\alpha} \in N_0^n$ the following equations

$$(6.5) \qquad D_x^{\tilde{\alpha}} D_t^{\alpha_*} u_{i,q}^j(x, t) = u_{i,j,q}^{(\tilde{\alpha}, \alpha_*)}(x, t) + \sum_{r=0}^{\alpha_* - 1} D_t^r \big(\lim_{s \to t} Y_{i,j,q}^{(\tilde{\alpha}, \alpha_* - r - 1)}(x, t, s)\big)$$

hold, where $(x,t)\in(\bar{D}_0\times(0,T])\setminus\tilde{S}_i^j$, $i\in\boldsymbol{I}_n$, $j\in\boldsymbol{I}_2$, $q\in\tilde{\boldsymbol{I}}_{m-1}$ and $\alpha_*\in\boldsymbol{N}$, $0<|\tilde{\alpha}|+2\alpha_*\leqslant 2m$.

To prove formulae (6.5), fix $i\in\boldsymbol{I}_n$, $j\in\boldsymbol{I}_2$, $q\in\tilde{\boldsymbol{I}}_{m-1}$ and $\tilde{\alpha}\in\boldsymbol{N}_0^n$.

Put now $\alpha_*=1$. Then, applying the known formula about the differentiating under the integral sign (see [9], p. 329), we obtain

$$D_x^{\tilde{\alpha}}D_t u_{i,q}^j(x,t)=u_{i,j,q}^{(\tilde{\alpha},1)}(x,t)+\lim_{s\to t}Y_{i,j,q}^{(\tilde{\alpha},0)}(x,t,s) \tag{6.6}$$

for $(x,t)\in(\bar{D}_0\times(0,T])\setminus\tilde{S}_i^j$.

Assume next that formula (6.5) is true for an arbitrary fixed natural number α_*. Hence, by (6.6), we get the following sequence of the equations

$$\begin{aligned}D_x^{\tilde{\alpha}}D_t^{\alpha_*+1}u_{i,q}^j(x,t)&=u_{i,j,q}^{(\tilde{\alpha},\alpha_*+1)}(x,t)+\lim_{s\to t}Y_{i,j,q}^{(\tilde{\alpha},\alpha_*)}(x,t,s)+\\&\qquad+D_t\Big(\sum_{r=0}^{\alpha_*-1}D_t^r\big(\lim_{s\to t}Y_{i,j,q}^{(\tilde{\alpha},\alpha_*-r-1)}(x,t,s)\big)\Big)\\&=u_{i,j,q}^{(\tilde{\alpha},\alpha_*+1)}(x,t)+\sum_{r=0}^{\alpha_*}D_t^r\big(\lim_{s\to t}Y_{i,j,q}^{(\tilde{\alpha},\alpha_*-r-1)}(x,t,s)\big)\end{aligned}$$

for $(x,t)\in(\bar{D}_0\times(0,T])\setminus\tilde{S}_i^j$.

Then, by induction and by the fact that i, j, q, $\tilde{\alpha}$ are arbitrary, formulae (6.4) hold.

Observe now that if $\alpha_*=0$, assertion (iii) is a corollary from assertion (i) of Lemma 6.1, and if $\alpha_*\in\boldsymbol{N}$, assertion (iii) is a consequence of assertions (i) and (ii) of this lemma and of formulae (6.4).

This completes the proof of Lemma 6.1.

LEMMA 6.2 ([3]). *Let* $x\in\bar{D}_0\setminus\bar{D}_i^j$ $(i\in\boldsymbol{I}_n,\ j\in\boldsymbol{I}_2)$ *and* $s<t$. *Then*

$$a^{-1/2}(4\pi)^{-n/2}\big((-1)^j c_i-x_i\big)\times\\ \times\int_{-\infty}^{t}\int_{R^{n-1}}(t-s)^{-n/2-1}\exp\big(-(4(t-s))^{-1}K(x,y^i)\big)dy^i\,ds=1,$$

where

$$K(x,y^i):=\frac{((-1)^j c_i-x_i)^2}{a_i}+\sum_{\substack{k=1\\k\neq i}}^{n}\frac{(y_k-x_k)^2}{a_k},\qquad i\in\boldsymbol{I}_n,\ j\in\boldsymbol{I}_2.$$

LEMMA 6.3. *Let* $i,p\in\boldsymbol{I}_n$, $j,r\in\boldsymbol{I}_2$, $k,q\in\tilde{\boldsymbol{I}}_{m-1}$. *Moreover, let* $x_*\in\bar{D}_0$, $\tilde{x}_0\in\bar{D}_0$, $x_0\in\bar{D}_0\setminus\bar{D}_i^j$, $t_0\in(0,T]$ *and* $\tilde{t}_0\in[0,T]$ *be arbitrary fixed points, and let* $u_{i,q}^j$ *be the functions defined by formulae* (4.7), (4.8). *Then*:

(i) *If the functions* $f_{i,q}^j$ *are measurable and bounded in the domains* $D_i\times(0,T]$, *respectively, then the functions* $u_{i,q}^j$ *satisfy*:

(a) *The equations*

$$P^m u_{i,q}^j(x,t)=0\quad \textit{for } (x,t)\in(\bar{D}_0\times(0,T])\setminus\tilde{S}_i^j. \tag{6.7}$$

(b) *The initial conditions*

(6.8) $D_t^k u_{i,q}^j(x, t) \to 0$ *as* $(x, t) \to (x_0, 0^+)$, $(x, t) \in (\bar{D}_0 \times (0, T]) \setminus \tilde{S}_i^j$,

(6.9) $u_{i,q}^j(x, t) \to 0$ *as* $(x, t) \to (\tilde{x}_0, 0^+)$, $(x, t) \in \bar{D}_0 \times (0, T]$, $q \neq 0$.

(c) *The boundary conditions*

$$(6.10) \qquad \begin{cases} P^k u_{i,q}^j(x, t) = 0 & \textit{for } (x, t) \in S_p^r, \ (p, r, k) \neq (i, j, q), \\ u_{i,q}^j(x, t) = 0 & \textit{for } (x, t) \in \tilde{S}_p^r, \qquad q \neq 0, \\ u_{i,0}^j(x, t) = 0 & \textit{for } (x, t) \in \tilde{S}_p^r, \quad (p, r) \neq (i, j), \end{cases}$$

and moreover:

(d) *The functions* $u_{i,q}^j$ *together with their derivatives* $D_{x,t}^{\alpha} u_{i,q}^j$ $(\alpha = (\tilde{\alpha}, \alpha_*)$, $\tilde{\alpha} \in N_0^n$, $\alpha_* \in N_0$, $0 < |\tilde{\alpha}| + 2\alpha_* \leqslant 2m)$ *are continuous in the domains* $(\bar{D}_0 \times (0, T]) \setminus \tilde{S}_i^j$, *respectively. The functions* $u_{i,q}^j$ $(q \neq 0)$ *are continuous in* $\bar{D}$, *besides.*

(ii) *If the functions* $f_{i,k}^j$ *are continuous and bounded in the domains* $D_i \times (0, T]$, *respectively, and if* $f_{i,k}^j|_{Z_i} = 0$, *then the functions* $u_{i,k}^j$ *satisfy the boundary conditions*

(6.11) $P^k u_{i,k}^j(x, t) \to f_{i,k}^j(x_*^i, t_0)$

as $(x, t) \to (x_*^{i,j}, t_0)$, $(x, t) \in (\bar{D}_0 \times (0, T]) \setminus \tilde{S}_i^j$.

(iii) *If the functions* $f_{i,0}^j$ *are continuous in the domains* $D_i \times [0, T]$, *respectively, and if* $f_{i,0}^j|_{Z_i \cup (D_i \times \{0\})} = 0$, *then*:

(a) *The functions* $u_{i,0}^j$ *satisfy the boundary conditions*

(6.12) $u_{i,0}^j(x, t) \to f_{i,0}^j(x_*^i, \tilde{t}_0)$

as $(x, t) \to (x_*^{i,j}, \tilde{t}_0)$, $(x, t) \in (\bar{D}_0 \times (0, T]) \setminus \tilde{S}_i^j$.

(b) *The functions* $u_{i,0}^j$ *are continuous in* $\bar{D}$.

Proof (i) (a) This assertion is a consequence of assertion (iii) of Lemma 6.1 and of assertion (i) of Lemma 2.2.

(i) (b) To prove conditions (6.8) let us fix indexes $i \in \boldsymbol{I}_n$, $j \in \boldsymbol{I}_2$, $k, q \in \tilde{\boldsymbol{I}}_{m-1}$ and observe next that assertion (iii) of Lemma 6.1, the Leibniz theorem on the differentiation and assertion (ii) of Lemma 2.2 imply the equations

$$D_t^k u_{i,q}^j(x, t) = \frac{2a_i(-1)^{q+1}k!}{q!} \sum_{p=0}^{l} \frac{q(q-1)\cdot\ldots\cdot(q-p+1)}{p!} \sum_{|\alpha^p| = k-p} \frac{a^{\alpha^p}}{\alpha^p!} \times$$

$$\times \int_0^t \int_{D_i} \left| f_{i,q}^j(y^i, s)(t-s)^{q-p} D_{y_i} D_x^{2\alpha^p} G(x, t, y, s) \right|_{y_i = (-1)^j c_i} dy^i \, ds,$$

where $(x, t) \in (\bar{D}_0 \times (0, T]) \setminus \tilde{S}_i^j$, $l = k$ for $k \leqslant q$ and $l = q$ for $k > q$.

Simultaneously, by assertions (ii), (iii) and (iv) of Lemma 2.1, we get the estimations

$$\Big|\int_0^t \int_{D_i} f_{i,q}^j(y^i, s)(t-s)^{q-p} D_{y_i} D_x^{2\alpha^p} G(x, t, y, s)|_{y_i=(-1)^j c_i} dy^i\, ds\Big|$$
$$\leqslant C_2 \sup_{D_i \times (0,T]} |f_{i,q}^j| \big[1+|(-1)^j c_i - x_i|^{-2\alpha_i^p - \varkappa - 2} + |(-1)^j c_i + x_i + 2c_i|^{-2\alpha_i^p - \varkappa - 2} +$$
$$+|(-1)^j c_i + x_i - 2c_i|^{-2\alpha_i^p - \varkappa - 2}\big] \int_0^t (t-s)^{\varkappa/2+q-p} \prod_{\substack{r=1\\ r\neq i}}^{n} [(t-s)^{-\alpha_r^p} + (t-s)^{1/2}]\, ds,$$

where $(x, t) \in (\bar{D}_0 \times (0, T]) \setminus \tilde{S}_i^j$, $\alpha^p = (\alpha_1^p, \ldots, \alpha_n^p) \in N_0^n$, $|\alpha^p| = k-p$, $p = 0, 1, \ldots, k$ or $p = 0, 1, \ldots, q$; C_2 is the positive constants given by formula (6.3) and $\varkappa$ is a constant greater than -1.

Consequently, conditions (6.8) hold since $\frac{1}{2}\varkappa + q - p - \sum_{\substack{r=1\\ r\neq i}}^{n} \alpha_r^p > 0$ for all possible p from both cases where $\varkappa > \max\{0, k-q\}$.

Observe now that, by assertions (i), (ii) and (iv) of Lemma 2.1, we obtain the inequalities

$$(6.13)\qquad |u_{i,q}^j(x, t)|$$
$$\leqslant \frac{2a_i}{q!} C_3 \sup_{D_i \times (0,T]} |f_{i,q}^j| \int_0^t (t-s)^q \big((t-s)^{-1} + (t-s)^{1/2}\big)\big(1+(t-s)^{1/2}\big)^{n-1} ds,$$

where $(x, t) \in \bar{D}_0 \times (0, T]$, $i \in \boldsymbol{I}_n$, $j \in \boldsymbol{I}_2$, $q \in \tilde{\boldsymbol{I}}_{m-1}$ and

$$C_3 := \big(\max\{3A_1, 3\sqrt{8\pi A}\, A_0, \max_{i \in \boldsymbol{I}_n}\{2(2c_i)^{-3} B_{1,1}; c_i^{-1} B_{0,1}\} \sum_{k=1}^{\infty} k^{-2}\}\big)^n.$$

By the above inequalities we get conditions (6.9).

(i) (c) The first part and the third part of conditions (6.10) are consequences of assertions (i) and (iii) of Lemma 6.1 and of assertion (iv) of Lemma 2.2. Simultaneously, by (6.13), the integrals $u_{i,q}^j$ ($i \in \boldsymbol{I}_n$, $j \in \boldsymbol{I}_2$, $q \in \boldsymbol{I}_{m-1}$) are locally uniformly convergent in $\bar{D}_0 \times (0, T]$. Hence, from assertion (iv) of Lemma 2.2, we obtain the second part of conditions (6.10).

(i) (d) The first part of this assertion is a consequence of assertions (i) and (iii) of Lemma 6.1. The second part of assertion (i) (d) is a consequence of the first part of this assertion, of conditions (6.9), of the second part of conditions (6.10) and of formulae (4.7), (4.8).

(ii) To prove conditions (6.11) let us fix indexes $i \in \boldsymbol{I}_n$, $j \in \boldsymbol{I}_2$, $k \in \tilde{\boldsymbol{I}}_{m-1}$ and observe that, by assertion (iii) of Lemma 6.1 and by assertion (i) of Lemma 2.2, we obtain for $(x, t)\in(\bar{D}_0\times(0, T])\setminus\tilde{S}_i^j$ the equation

$$(6.14)\qquad P^k u_{i,k}^j(x, t) = a_{i,k}^j(x, t)+b_{i,k}^j(x, t)+c_{i,k}^j(x, t),$$

where

$$a_{i,k}^j(x, t) = -2a_i\int_0^t\int_{D_i} f_{i,k}^j(y^i, s)\, D_{y_i} U_i\big|_{y_i=(-1)^j c_i}\prod_{\substack{r=1\\ r\neq i}}^n U_r(x_r, t, y_r, s)\, dy^i\, ds,$$

$$b_{i,k}^j(x, t) = -2a_i\int_0^t\int_{D_i} f_{i,k}^j(y^i, s)\, D_{y_i} U_i\big|_{y_i=(-1)^j c_i}\Big(\prod_{\substack{r=1\\ r\neq i}}^n G_r(x_r, t, y_r, s)-$$
$$-\prod_{\substack{r=1\\ r\neq i}}^n U_r(x_r, t, y_r, s)\Big)\, dy^i\, ds,$$

$$c_{i,k}^j(x, t) = -2a_i\int_0^t\int_{D_i} f_{i,k}^j(y^i, s)\sum_{r=1}^{\infty}(-1)^r\big(D_{y_i} U_{i,r}^{(1)}+D_{y_i} U_{i,r}^{(2)}\big)\big|_{y_i=(-1)^j c_i}\times$$
$$\times\prod_{\substack{r=1\\ r\neq i}}^n G_r(x_r, t, y_r, s)\, dy^i\, ds.$$

First, we shall prove

$$(6.15)\quad a_{i,k}^j(x, t)\to f_{i,k}^j(x_*^i, t_0)\quad \text{as } (x, t)\to(x_*^{i,j}, t_0),\ (x, t)\in(\bar{D}_0\times(0, T])\setminus\tilde{S}_i^j.$$

Put

$$\bar{f}_{i,k}^j(y^i, s) := \begin{cases} f_{i,k}^j(y^i, s) & \text{for } (y^i, s)\in\bar{D}_i\times(0, T],\\ 0 & \text{for } (y^i, s)\in\boldsymbol{R}^{n-1}\times(-\infty, T]\setminus(\bar{D}_i\times(0, T]),\end{cases}$$

$$M_{i,k}^j := \sup_{\bar{D}_i\times(0,T]}|f_{i,k}^j|,\qquad K_i^{\varrho} := \{y^i\in D_i\colon |y_r-x_r^*|<\varrho\ (r\in\boldsymbol{I}_n,\ r\neq i)\},$$

where ϱ is an arbitrary positive number. Moreover, let ε denote an arbitrary chosen positive number.

By continuity of the function $\bar{f}_{i,k}^j$ at the point $(x_*^i, t_0)\in\bar{D}_i\times(0, T]$ (see assumptions from (ii) of this lemma) there exists a number $\delta_1>0$ such that

$$(6.16)\qquad |\bar{f}_{i,k}^j(y^i, s)-\bar{f}_{i,k}^j(x_*^i, t_0)|<\tfrac{1}{4}\varepsilon$$

for $y^i\in\bar{D}_i\cap K_i^{\delta_1}$, $0<t_0-\delta_1<s<t_0+\delta_1<T$.

Simultaneously, the set $\boldsymbol{R}^{n-1}\times(-\infty, t)$ may be represented as an union:

$$\boldsymbol{R}^{n-1}\times(-\infty, t) = \bigcup_{r=1}^{4} Z_{i,r}^{\delta},$$

where

$$Z^{\delta}_{i,1} = (\bar{D}_i \cap K^{\delta}_i) \times (t_0 - \delta, t), \qquad Z^{\delta}_{i,2} = (K^{\delta}_i \setminus \bar{D}_i) \times (t_0 - \delta, t),$$
$$Z^{\delta}_{i,3} = K^{\delta}_i \times (-\infty, t_0 - \delta), \qquad Z^{\delta}_{i,4} = (R^{n-1} \setminus K^{\delta}_i) \times (-\infty, t)$$

and

(6.17) $$\delta = \min\left\{\delta_1, \tfrac{1}{2}\pi\left(\frac{\varepsilon n\sqrt{a}}{4M^j_{i,k}}\right)^{2/n}\right\}.$$

Therefore, by Lemma 6.2, we have

(6.18) $$a^j_{i,k}(x, t) - f^j_{i,k}(x^i_*, t_0) = \sum_{r=1}^{4} I^r_{i,j,k}(x, t)$$

for $(x, t) \in (\bar{D}_0 \times (0, T]) \setminus \tilde{S}^j_i$, $|t - t_0| < \frac{1}{2}\delta$, where

$$I^r_{i,j,k}(x, t) = \int_{Z^{\delta}_{i,r}} \left(\bar{f}^j_{i,k}(y^i, s) - \bar{f}^j_{i,k}(x^i_*, t_0)\right) a^{-1/2} (4\pi)^{-n/2} (t-s)^{-n/2-1} \times$$
$$\times \left((-1)^j c_i - x_i\right) \exp\left(-(4(t-s))^{-1} K(x, y^i)\right) dy^i\, ds.$$

It is seen that formulae (6.16), (6.17) and Lemma 6.2 imply the inequality

(6.19) $$|I^1_{i,j,k}(x, t)| < \tfrac{1}{4}\varepsilon \quad \text{for } (x, t) \in (\bar{D}_0 \times (0, T]) \setminus \tilde{S}^j_i,\ |t - t_0| < \tfrac{1}{2}\delta.$$

Now

$$|I^2_{i,j,k}(x, t)| \leqslant 2M^j_{i,k} a^{-1/2} (4\pi)^{-n/2} |(-1)^j c_i - x_i| \times$$
$$\times \int_{R^{n-1} \setminus \bar{D}_i} \int_{-\infty}^{t} (t-s)^{-n/2-1} \exp\left(-(4(t-s))^{-1} K(x, y^i)\right) dy^i\, ds$$
$$\text{for } (x, t) \in (\bar{D}_0 \times (0, T]) \setminus \tilde{S}^j_i,\ |t - t_0| < \tfrac{1}{2}\delta.$$

Applying to the right-hand side of the above estimation the substitution

(6.20) $$v_i = (4(t-s))^{-1} K(x, y^i),$$

we get inequality

$$|I^2_{i,j,k}(x, t)| \leqslant 2M^j_{i,k} a^{-1/2} \pi^{-n/2} |(-1)^j c_i - x_i| \int_0^{\infty} v_i^{n/2-1} \exp(-v_i)\, dv_i \times$$
$$\times \int_{R^{n-1} \setminus \bar{D}_i} K(x, y^i)^{-n/2}\, dy^i \quad \text{for } (x, t) \in (\bar{D}_0 \times (0, T]) \setminus \tilde{S}^j_i,\ |t - t_0| < \tfrac{1}{2}\delta.$$

Simultaneously, the transformation

(6.21) $$y_r - x_r = \sqrt{(a_r/a_i)}\, |(-1)^j c_i - x_i|\, v_r \qquad (r \in I_n,\ r \neq i)$$

maps the domain $\bar{D}_i$ into the domain

$$Q_{i,j} = \{v^i \in \boldsymbol{R}^{n-1}\colon\ -x_r - c_r \leqslant \sqrt{(a_r/a_i)}\,|(-1)^j c_i - x_i|\, v_r$$
$$\leqslant -x_r + c_r\ (r \in \boldsymbol{I}_n,\ r \neq i)\},$$

and since we can assume that $|x_r - x_r^*| < \frac{1}{2}\delta$ $(r \in \boldsymbol{I}_n,\ r \neq i)$, so

$$\boldsymbol{R}^{n-1} \setminus Q_{i,j} \subset \boldsymbol{R}^{n-1} \setminus Q^\delta_{i,j},$$

where

$$Q^\delta_{i,j} = \{v^i \in \boldsymbol{R}^{n-1}\colon\ -x_r^* - \tfrac{1}{2}\delta - c_r \leqslant \sqrt{(a_r/a_i)}\,|(-1)^j c_i - x_i|\, v_r$$
$$\leqslant -x_r^* + \tfrac{1}{2}\delta + c_r\ (r \in \boldsymbol{I}_n,\ r \neq i)\}.$$

Consequently

$$|I^2_{i,j,k}(x, t)| \leqslant 2M^j_{i,k}\pi^{-n/2} \int_0^\infty v_i^{n/2-1} \exp(-v_i)\, dv_i \int_{\boldsymbol{R}^{n-1} \setminus Q^\delta_{i,j}} [1 + \sum_{\substack{r=1\\ r\neq i}}^n v_r^2]^{-n/2}\, dv^r$$

for $(x, t) \in (\bar{D}_0 \times (0, T]) \setminus \tilde{S}^j_i$, $|x_r - x_r^*| < \frac{1}{2}\delta$ $(r \in \boldsymbol{I}_n,\ r \neq i)$, $|t - t_0| < \frac{1}{2}\delta$.

Hence, by the convergence of the integrals

$$\int_0^\infty v_i^{n/2-1} \exp(-v_i)\, dv_i, \qquad \int_{\boldsymbol{R}^{n-1}} [1 + \sum_{\substack{r=1\\ r\neq i}}^n v_r^2]^{-n/2}\, dv^r$$

and by the fact that the edges of the cube $Q^\delta_{i,j}$ are equal with the interval $(-\infty, \infty)$, as $x_i \to (-1)^j c_i$, there exists a number δ_2 dependent on ε such that

(6.22) $$|I^2_{i,j,k}(x, t)| < \tfrac{1}{4}\varepsilon$$

for $(x, t) \in (\bar{D}_0 \times (0, T]) \setminus \tilde{S}^j_i$, $|x_r - x_r^*| < \frac{1}{2}\delta$ $(r \in \boldsymbol{I}_n,\ r \neq i)$, $|x_i - (-1)^j c_i| < \delta_2$, $|t - t_0| < \frac{1}{2}\delta$.

Next

$$|I^3_{i,j,k}(x, t)| \leqslant 2M^j_{i,k}\, a^{-1/2} (4\pi)^{-n/2} |(-1)^j c_i - x_i| \int_{K^\delta_i} dy^i \int_{-\infty}^{t_0 - \delta} (t-s)^{-n/2-1}\, ds$$

for $(x, t) \in (\bar{D}_0 \times (0, T]) \setminus \tilde{S}^j_i$, $|t - t_0| < \frac{1}{2}\delta$ and since, by (6.17),

$$\delta \leqslant \tfrac{1}{2}\pi \left(\frac{\varepsilon n \sqrt{a}}{4M^j_{i,k}}\right)^{2/n},$$

we have

(6.23) $$|I^3_{i,j,k}(x, t)| < \tfrac{1}{4}\varepsilon$$

for $(x, t) \in (\bar{D}_0 \times (0, T]) \setminus \tilde{S}^j_i$, $|x_i - (-1)^j c_i| < \frac{1}{2}\delta$, $|t - t_0| < \frac{1}{2}\delta$.

Finally, applying to the integral $I^4_{i,j,k}$ the substitutions (6.20) and (6.21) successively, we obtain the inequality

$$|I^4_{i,j,k}(x, t)| \leqslant 2M^j_{i,k}\pi^{-n/2} \int_0^\infty v_i^{n/2-1} \exp(-v_i)\, dv_i \int_{\boldsymbol{R}^{n-1} \setminus \tilde{Q}^\delta_{i,j}} [1 + \sum_{\substack{r=1\\ r\neq i}}^n v_r^2]^{-n/2}\, dv^r$$

for $(x, t) \in (\bar{D}_0 \times (0, T]) \setminus \tilde{S}_i^j$, $|x_r - x_r^*| < \frac{1}{2}\delta$ $(r \in I_n,\ r \neq i)$, $|t - t_0| < \frac{1}{2}\delta$, where

$$\tilde{Q}_{i,j}^{\delta} = \left\{ v^i \in R^{n-1}:\ |v_r| < \frac{\delta\sqrt{a_i}}{2|(-1)^j c_i - x_i|\sqrt{a_r}} \quad (r \in I_n,\ r \neq i) \right\}.$$

Hence, by an analogous argument as in the proof of the convergence of the integral $I_{i,j,k}^2$, we get that there exists a number δ_3 dependent on ε such that

$$|I_{i,j,k}^4(x, t)| < \tfrac{1}{4}\varepsilon \tag{6.24}$$

for $(x, t) \in (\bar{D}_0 \times (0, T]) \setminus \tilde{S}_i^j$, $|x_r - x_r^*| < \frac{1}{2}\delta$ $(r \in I_n,\ r \neq i)$, $|x_i - (-1)^j c_i| < \delta_3$, $|t - t_0| < \frac{1}{2}\delta$.

Conditions (6.18), (6.19), (6.22)–(6.24) imply the inequality

$$|a_{i,k}^j(x, t) - f_{i,k}^j(x_*^i, t_0)| < \varepsilon$$

for $|x_r - x_r^*| < \frac{1}{2}\delta$ $(r \in I_n,\ r \neq i)$, $|x_i - (-1)^j c_i| < \min(\frac{1}{2}\delta, \delta_2, \delta_3)$, $|t - t_0| < \frac{1}{2}\delta$ and therefore (6.15) holds.

Since $D_{y_i} U_i(x_i, t, (-1)^j c_i, s) \to 0$ as $x_i \to (-1)^j c_i$, $0 \leqslant s < t$, we have

$$b_{i,k}^j(x, t) \to 0 \quad \text{as} \quad (x, t) \to (x_*^{i,j}, t_0),\ (x, t) \in (\bar{D}_0 \times (0, T]) \setminus \tilde{S}_i^j. \tag{6.25}$$

Finally, analogously as in the proof of assertion 3° of Lemma 3 from [3], we obtain that $D_{y_i} R_i(x_i, t, (-1)^r c_i, s) \to 0$ as $x_i \to (-1)^j c_i$, $0 \leqslant s < t$ $(r \in I_2)$ and therefore

$$c_{i,k}^j(x, t) \to 0 \quad \text{as } (x, t) \to (x_*^{i,j}, t_0),\ (x, t) \in (\bar{D}_0 \times (0, T]) \setminus \tilde{S}_i^j. \tag{6.26}$$

By (6.14), (6.15), (6.25), (6.26), and by the fact that the indexes i, j, k are arbitrary, we get (6.11).

(iii)(a) Conditions (6.11) for $k = 0$ and an analogous argument as in [8] (see Section 59.5) imply assertion (iii)(a).

(iii)(b) This assertion is a consequence of assertion (i)(d), of conditions (6.8), (6.12) and of formulae (4.7), (4.8). Therefore the proof of Lemma 6.3 is finished.

Remark 6.1. It follows from the proof of assertion (ii) of Lemma 6.3 that to prove the following conditions:

$$P^k u_{i,k}^j(x, t) \to f_{i,k}^j(x_*^i, t_0)$$

as $(x, t) \to (x_*^{i,j}, t_0) \in S_i^j$, $(x, t) \in D$, $i \in I_n$, $j \in I_2$, $k \in I_{m-1}$, instead of conditions (6.11) for $k \in I_{m-1}$, it is sufficient to assume that the functions $f_{i,k}^j$ $(i \in I_n,\ j \in I_2,$ $k \in I_{m-1})$ are measurable and bounded in $D_i \times (0, T]$ $(i \in I_n)$ and continuous at the points (x_*^i, t_0) $(i \in I_n)$, respectively.

THEOREM 6.1. *Let the functions* $f_{i,q}^j$ $(i \in I_n,\ j \in I_2,\ q \in I_{m-1})$ *be continuous and bounded in the domains* $D_i \times (0, T]$, *respectively, and let the functions* $f_{i,0}^j$

$(i \in I_n,\ j \in I_2)$ be continuous in the domains $\bar{D}_i \times [0, T]$, respectively, and satisfy the equations

$$f^j_{i,0}(x^i, t) = 0 \quad \text{for } (x^i, t) \in Z_i \cup (\bar{D}_i \times \{0\}) \ (i \in I_n,\ j \in I_2). \tag{6.27}$$

Moreover, let u^2 be the function given by formula (4.6). *Then*:

(A) *The function u^2 satisfies the equation*

$$P^m u^2(x, t) = 0 \quad \text{for } (x, t) \in D, \tag{6.28}$$

the initial conditions

$$D^k_t u^2(x, t) = \begin{cases} 0 & \text{for } (x, t) \in \bar{S}_0,\ k = 0, \\ 0 & \text{for } (x, t) \in S_0,\ k \in I_{m-1}, \end{cases} \tag{6.29}$$

and the boundary conditions

$$P^k u^2(x, t) = \begin{cases} f^j_{i,0}(x^i, t) & \text{for } (x, t) \in \bar{S}^j_i,\ i \in I_n,\ j \in I_2,\ k = 0, \\ f^j_{i,k}(x^i, t) & \text{for } (x, t) \in S^j_i,\ i \in I_n,\ j \in I_2,\ k \in I_{m-1}. \end{cases} \tag{6.30}$$

(B) *The function u^2 is (m)-regular in D.*

Proof. Equation (6.28) is a consequence of formulae (6.7) and (4.6), and conditions (6.29) are a consequence of formulae (6.8), (6.9), (6.27), (6.12) and (4.6). Next, formulae (6.27), (6.9) and (6.10), Remark 6.1, and formulae (6.12) and (4.6) imply conditions (6.30). Finally, assertion (B) is a consequence of conditions (i) (d) and (iii) (b) of Lemma 6.3 and of formula (4.6).

7. Theorem on the existence of the (m)-regular solution of the (C^m) linear problem. As a consequence of Theorems 5.1 and 6.1 we get the following:

THEOREM 7.1. *If the functions f_k, $f^j_{i,k}$ $(i \in I_n,\ j \in I_2,\ k \in \tilde{I}_{m-1})$ satisfy the assumptions of Theorems* 5.1 *and* 6.1, *then the function u given by formulae* (4.1)–(4.8) *is the (m)-regular solution in D of the (C^m) linear problem.*

Remark 7.1. If all the assumptions of Theorem 7.1 are satisfied for $m = 1$, then Section 22.7 from [8] or Theorem 2.1 from [2] imply that the function u given by formulae (4.1)–(4.8) is the only one regular solution in D of the (C^1) linear problem.

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