ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XXVII (1988)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XXVII (1988)

MAREK WISŁA (Poznań)

Boundedness of the identity embedding of some Musielak–Orlicz spaces

Abstract. The aim of this paper is to extend a theorem proved by P. Turpin in [4] which concerns the boundedness of the identity embedding of Musielak–Orlicz spaces.

1. Introduction. P. Turpin [4] has studied in details the identity embedding of Musielak–Orlicz spaces of functions with real values. Theorem 4 presented in that paper which gives conditions concerning the boundedness of the embedding $i\colon L^{\varphi} \to L^{\psi}$, $i(f) = f$ (cf. Definition 1.3 below) we extend to the case of more generally Musielak–Orlicz spaces consisting of functions with values in linear-topological spaces.

After the preliminary comprehensions and auxiliary results of this section, in the second one we give fundamental theorems of this paper. In these theorems there are considered functions $\alpha_{u,c}$ satisfying the condition

$$\forall_{\varepsilon>0} \forall_{u>0} \exists_{c>0} \int_T \alpha_{u,c}(t)\,d\mu < \varepsilon.$$

In the third section we study conditions which must be assumed in order that the above condition could be replaced by a weaker one:

$$\forall_{u>0} \exists_{c>0} \int_T \alpha_{u,c}(t)\,d\mu < +\infty.$$

The last condition links the considerations included in this paper with the above-mentioned theorem of P. Turpin.

Let (T, Σ, μ) be a measure space, where T is an abstract set, Σ is a σ-algebra of subsets of T, μ is a non-negative, complete, atomless and σ-finite measure on Σ. (X, τ) will denote a real linear-topological space. By $\mathscr{B}_X$ we shall denote the σ-algebra of Borel subsets of X.

Let $\mathfrak{M}(T, X)$ be the set of all measurable functions $f\colon T \to X$, i.e., functions satisfying $f^{-1}(U) \in \Sigma$ for every $U \in \mathscr{B}_X$. We shall denote by $M(T, X)$ (or shortly M) an arbitrary linear subset of the set $\mathfrak{M}(T, X)$. In particular, $S(T, X)$ will denote the set of all simple functions, i.e., functions of the type $\sum_{k=1}^{n} x_k \chi_{A_k}$, where $x_k \in X$, $\mu(A_k) < +\infty$ for $k = 1, 2, \ldots, n$.

1.1. Definition. A function Φ: $X \times T \to [0, +\infty]$ is said to be a Φ-*function* if there is a set T_0 of measure 0 such that:

(a) Φ is $\mathscr{B}_X \times \Sigma$-measurable;

(b) $\Phi(0, t) = 0$ and $\Phi(x, t) = \Phi(-x, t)$ for every $x \in X$ and $t \in T \setminus T_0$;

(c) $\Phi(\cdot, t)$ is lower semicontinuous on X and continuous at 0 for every $t \in T \setminus T_0$;

(d) $\Phi(ux+vy, t) \leqslant \Phi(x, t)+\Phi(y, t)$ for every $u, v \geqslant 0$, $u+v = 1$, $x, y \in X$ and $t \in T \setminus T_0$.

If $\Phi(\cdot, t)$ is continuous on $\{x: \Phi(x, t) < +\infty\}$ (resp. convex on X) for almost every (a.e.) $t \in T$, we shall shortly write "Φ is continuous" (resp. convex). The functions which are Φ-functions at least we shall denote by Greek letters Φ, Ψ, ...

The functional I_Φ: $\mathfrak{M}(T, X) \to [0, +\infty]$ is defined by $I_\Phi(f) = \int_T \Phi(f(t), t)\,d\mu$. Let us note that $I_\Phi|_M$: $M \to [0, +\infty]$, $I_\Phi|_M(f) = I_\Phi(f)$ is a pseudomodular on $M = M(T, X)$ (see [1], [2]).

1.2. Definition. By the *Musielak–Orlicz space* $L^\Phi(M(T, X))$ (or shortly $L^\Phi(M)$) we mean the set of all functions $f \in M(T, X)$ such that $I_\Phi(af) < +\infty$ for some $a > 0$.

In the space $L^\Phi(M)$ we shall consider the F-seminorm defined by

$$|f|_\Phi = \inf\{u > 0: I_\Phi(f/u) \leqslant u\}.$$

In the sequel the following notations will be used:

(1) $$B_\Phi(r) = \{f \in L^\Phi(M): |f|_\Phi < r\}, \quad \text{for } r > 0,$$

(2) $$\operatorname{dom} I_\Phi = \{f \in \mathfrak{M}(T, X): I_\Phi(f) < +\infty\},$$

(3) $$P_{u,c}(x) = \{t \in T: u\Psi(cx, t) > \Phi(x, t) \text{ and } \Phi(x, t) < +\infty\} \quad \text{for } u, c > 0,\ x \in X,$$

(4) $$\alpha_{u,c}(t) = \sup_{x \in X} \Psi(cx, t)\chi_{P_{u,c}(x)}(t) \quad \text{for } u, c > 0,\ x \in X.$$

1.3. Definition. Let Y, Z be linear-topological spaces. A linear operator G: $Y \to Z$ is *bounded* if there is a neighbourhood U of 0 in Y such that the set $G(U)$ is bounded in Z (the set U may be not bounded in Y).

The following assertions will be quoted in the sequel:

1.4. [5] If X is a separable space and Φ is continuous, then the functions $\alpha_{u,c}$ are measurable and

$$\alpha_{u,c}(t) = \sup_{k \in N} \Psi(cx_k, t)\chi_{P_{u,c}(x_k)}(t) \quad \text{for a.e. } t \in T,$$

where the set $\{x_1, x_2, \ldots\}$ is dense in the space X.

1.5. [6] Assume that X is a separable space and Φ is continuous. If

$$\forall_{m\in N} \int_T \alpha_{u_m,c_m}(t)\,d\mu > 2^m$$

for some sequences (u_m) and (c_m) of positive numbers, then there are sequences (m_k) of numbers and (g_k) of simple functions such that

$$\{t\in T:\ g_k(t)\neq 0\}\cap\{t\in T:\ g_l(t)\neq 0\} = \emptyset \quad \text{for } k\neq l,$$

$$I_\Phi(g_k)\leqslant u_{m_k} \quad \text{and} \quad I_\Psi(c_{m_k} g_k)\geqslant 1 \quad \text{for } k = 1, 2, \ldots$$

1.6. [6] If the assumptions of 1.5 are satisfied and

$$\forall_{m\in N} \int_T \alpha_{u_m,c_m}(t)\,d\mu > \varepsilon$$

for some $\varepsilon > 0$ and some sequences (u_m), (c_m) of positive numbers, then there is a sequence (g_m) of simple functions such that $g_m \in \operatorname{dom} I_\Phi$, $I_\Phi(g_m)\leqslant u_m\varepsilon$ and $I_\Psi(c_m g_m)\geqslant \varepsilon$ for $m\in N$.

1.7. [4] The set $A\subset L^\Phi(M)$ is additively bounded, i.e., for every $s>0$ there is $n\in N$ such that $A\subset B_\Phi(s)+\overset{(n)}{\ldots}+B_\Phi(s)$ (n terms $B_\Phi(s)$, we shall denote this set by $+^n B_\Phi(s)$) if and only if

$$\exists_{a>0} \sup_{f\in A} \int_T \Phi(af(t), t)\,d\mu < +\infty.$$

2. The following theorem is a generalization of Theorem 4 (b) in [4]:

2.1. Theorem. (a) *If*

(B) *there is a set T_0 of measure* 0 *such that for all numbers $\varepsilon > 0$ and $K>0$ there are a number $c>0$ and a measurable function h: $T\to[0, +\infty]$ such that $\int_T h(t)\,d\mu < \varepsilon$ and*

$$\Psi(cx, t)\leqslant K\Phi(x, t)+h(t) \quad \textit{for all } x\in X \textit{ and } t\in T\setminus T_0,$$

then the embedding i: $L^\Phi(M)\to L^\Psi(M)$, $i(f) = f$, is bounded.

(b) *If X is a separable space, Φ is continuous on X, $S(T,X)\cap \cap\operatorname{dom} I_\Phi\subset M(T, X)$ and the embedding i: $L^\Phi(M)\to L^\Psi(M)$ is bounded, then condition* (B) *holds.*

Proof. (a) We shall show that the unit ball $B_\Phi(1) = \{f\in L^\Phi(M):\ |f|_\Phi < 1\}$ is bounded in the space $L^\Psi(M)$, i.e.,

$$\forall_{d>0}\ \exists_{a>0}\ aB_\Phi(1)\subset B_\Psi(d).$$

Let d be an arbitrary positive number. In virtue of (B) there are a set T_0 of measure 0, a number $c>0$ and a function $h\colon T\to[0,+\infty]$ such that $\int_T h(t)\,d\mu<\frac{1}{3}d$ and

(5) $$\Psi(cx,t)\leqslant\tfrac{1}{3}d\Phi(x,t)+h(t)\quad\text{for all } x\in X \text{ and } t\notin T_0.$$

Now, let $a=\frac{1}{3}cd$ and $f\in B_\Phi(1)$. Then, by (5),

$$I_\Psi\left(\frac{af}{\frac{2}{3}d}\right)\leqslant I_\Psi\left(\frac{af}{\frac{1}{3}d}\right)=I_\Psi(cf)\leqslant\tfrac{1}{3}d\cdot I_\Phi(f)+\tfrac{1}{3}d\leqslant\tfrac{2}{3}d.$$

Thus $|af|_\Psi\leqslant\frac{2}{3}d<d$, so $af\in B_\Psi(d)$.

(b) In virtue of 1.4 the functions $\alpha_{u,c}$ given by (4) are measurable. Consider

(6) $$\underset{\varepsilon>0}{\forall}\ \underset{u>0}{\forall}\ \underset{c>0}{\exists}\ \int_T\alpha_{u,c}(t)\,d\mu<\varepsilon.$$

Suppose that (6) does not hold. Then there are numbers $\varepsilon_0>0$ and $u>0$ such that $\int_T\alpha_{u,c}(t)\,d\mu\geqslant\varepsilon_0$ for every $c>0$. Since the embedding i is bounded, the ball $B_\Phi(b)$ is bounded in the space $L^\Psi(M)$ for some $b>0$. Now, if ε is a sufficiently small number, e.g. $0<\varepsilon<\min\{\varepsilon_0, b/2u\}$, then $\int_T\alpha_{u,c}(t)\,d\mu>\varepsilon$ for every $c>0$ and the ball $B_\Phi(2u\varepsilon)$ is bounded in the space $L^\Psi(M)$. Hence $a\bar{B}_\Phi(u\varepsilon)\subset B_\Psi(\varepsilon)$ for some $a>0$, where $\bar{B}_\Phi(r)=\{f\in L^\Phi(M)\colon |f|_\Phi\leqslant r\}$, $r>0$.

In virtue of 1.6 (putting $u_m=u$, $c_m=au$ for each $m=1,2,\ldots$) there is a simple function g such that $I_\Phi(g)\leqslant u\varepsilon$ and $I_\Psi(aug)>\varepsilon$. Write $f=\varepsilon\cdot ug$. Then $I_\Phi(f/u\varepsilon)=\mathrm{I}_\Phi(g)\leqslant u\varepsilon$, i.e., $f\in\bar{B}_\Phi(u\varepsilon)$.

On the other hand $I_\Psi(af/\varepsilon)=I_\Psi(aug)>\varepsilon$, so $af\notin B_\Psi(\varepsilon)$ and we get a contradiction. Since (6) $\Rightarrow$ (B) (with $K=u^{-1}$ and $h(t)=\alpha_{u,c}(t)$) the proof is finished.

2.2. Remark. Conditions (B) and (6) are equivalent by the assumptions of Theorem 2.1 (b).

2.3. COROLLARY. (a) *If*

(LB) *there is a set T_0 of measure 0 such that for arbitrary numbers $\varepsilon>0$ and $K>0$ we can find a number $c>0$ and a measurable function h: $T\to[0,+\infty]$ such that $\int_T h(t)\,d\mu<\varepsilon$ and*

$$\Phi(cx,t)\leqslant K\Phi(x,t)+h(t)\quad\textit{for every } x\in X \textit{ and } t\in T\setminus T_0,$$

then the Musielak–Orlicz space $(L^\Phi(M),|\cdot|_\Phi)$ is locally bounded.

(b) *If X is a separable space, Φ is continuous, $S(T,X)\cap\operatorname{dom}I_\Phi\subset M(T,X)$ and the space $(L^\Phi(M),|\cdot|_\Phi)$ is locally bounded, then the function Φ satisfies condition* (LB).

2.4. COROLLARY. *If*

(+) *there is a function* $\varkappa\colon [0, +\infty) \to [0, +\infty)$ *such that* $\varkappa(0) = 0$, $\lim_{c\to 0} \varkappa(c) = 0$ *and* $\Phi(cx, t) \leqslant \varkappa(c)\Phi(x, t)$ *for all* $c \geqslant 0$, $x \in X$ *and a.e.* $t \in T$

(in particular: if Φ is a p-convex function on X), *then the Musielak–Orlicz space* $(L^{\Phi}(M), |\cdot|_{\Phi})$ *is locally bounded.*

2.5. Remark. If the function Φ satisfies condition (+), then either $\sup_{x\in X}\Phi(x, t) = +\infty$ or $\Phi(\cdot, t) \equiv 0$ for a.e. $t \in T$.

Proof. Assume that Φ satisfies (+) for all $t \notin T_0$ ($\mu(T_0) = 0$). Moreover, let $t \notin T_0$ be fixed and suppose $\Phi(\cdot, t) \not\equiv 0$, i.e., $\Phi(y, t) > 0$ for some $y \in X$. By continuity of the function $\varkappa$ at 0 we obtain

$$\forall_{n\in N} \ \exists_{c_n>0} \ \forall_{x\in X} \ \Phi(c_n\cdot x, t) \leqslant \frac{1}{n}\Phi(x, t).$$

Let (d_n) be a sequence of positive numbers such that $1 \leqslant c_n d_n \leqslant c_{n+1} d_{n+1}$ for each $n \in N$. Therefore

$$\sup_{x\in X}\Phi(x, t) \geqslant \sup_{n\in N}\Phi(d_n\cdot y, t) \geqslant \sup_{n\in N}\big(n\cdot\Phi(c_n d_n y, t)\big) \geqslant \sup_{n\in N}\big(n\cdot\Phi(y, t)\big) = +\infty.$$

2.6. EXAMPLE. Denote by $C(\boldsymbol{R}, \boldsymbol{R})$ the space of all continuous functions $x\colon \boldsymbol{R} \to \boldsymbol{R}$ ($\boldsymbol{R}$ being the space of real numbers) with topology yielded by the family of sets of the form

$$V(a_1, \ldots, a_m; \varepsilon) = \{x \in C(\boldsymbol{R}, \boldsymbol{R})\colon \forall_{1\leqslant i\leqslant m} |x(a_i)| < \varepsilon\},$$

where $a_1, \ldots, a_m \in \boldsymbol{R}$ and $\varepsilon > 0$. As the space of parameters T we shall consider the space $\boldsymbol{R}$ with σ-algebra Σ which consists of all Lebesgue measurable subsets of $\boldsymbol{R}$ and with the Lebesgue measure μ on Σ. Define

$$\Xi\colon C(\boldsymbol{R}, \boldsymbol{R}) \times \boldsymbol{R} \to [0, +\infty), \qquad (x, t) \to |x(t)|.$$

Then Ξ is a convex Φ-function with finite values and, in virtue of Corollary 2.4, the Musielak–Orlicz space $\big(L^{\Xi}(M(\boldsymbol{R}, C(\boldsymbol{R}, \boldsymbol{R}))), |\cdot|_{\Xi}\big)$ is locally bounded.

2.7. P. Turpin in [4] has proved that in the case where $X = \boldsymbol{R}$, Φ is a Musielak–Orlicz function (cf. [4] Definition 1, p. 71) $M(T, \boldsymbol{R})$ is the space of all measurable functions from T into $\boldsymbol{R}$, the embedding $i\colon L^{\Phi}(M) \to L^{\Psi}(M)$ is bounded if and only if

(7) $$\forall_{u>0} \ \exists_{c>0} \ f_{c,u} \in L^{\Psi}(M),$$

where $f_{c,u}(t) = \sup\{x \geqslant 0\colon u\Psi(cx, t) > \Phi(x, t)\}$ (if the set on the right-hand side of the equality is empty, then we define $f_{c,u}(t) = 0$).

On the other hand, defining $\Psi(+\infty, t) = \lim_{x \to +\infty} \Psi(x, t)$, we have

$$\alpha_{u,c}(t) = \sup_{x \geqslant 0} \Psi(cx, t)\chi_{P_{u,c}(x)}(t) = \Psi(cf_{c,u}(t), t) \tag{8}$$

for a.e. $t \in T$. Hence condition (B) takes the form

$$\forall_{\varepsilon > 0} \forall_{u > 0} \exists_{c > 0} \int_T \Psi(cf_{c,u}(t), t)\, d\mu < \varepsilon. \tag{9}$$

Therefore, are conditions (7) and (9) equivalent?

The implication (9)$\Rightarrow$(7) is obvious. Conversely, let $u > 0$. By (7), $\int_T \Psi(sf_{c,u}(t), t)\, d\mu < +\infty$ for some $c, s > 0$. Since $f_{v,u}(t) \leqslant f_{c,u}(t)$ for every $0 < v \leqslant c$ and $t \in T$,

$$\Psi(vf_{v,u}(t), t) \leqslant \Psi(sf_{c,u}(t), t) \quad \text{for} \quad 0 < v \leqslant \min\{c, s\} \text{ and a.e. } t \in T.$$

Moreover, by continuity of the function $\Psi(\cdot, t)$ at 0,

$$\lim_{v \to 0} \Psi(vf_{v,u}(t), t) \leqslant \lim_{v \to 0} \Psi(vf_{c,u}(t), t) = 0 \quad \text{for a.e. } t \in T,$$

so, by the Lebesgue dominated convergence theorem, $\lim_{v \to 0} \int_T \Psi(vf_{v,u}(t), t)\, d\mu = 0$. Thus, condition (9) holds.

Let us note that in this case the assumption concerning the continuity of the function Φ on R may be omitted, since it is only used in the proof of the fact that the functions $\alpha_{u,c}$ are measurable. The measurability of these functions follows immediately from (8).

3. In this section we shall study a little more general problem than the above discussed, namely: what conditions must be fulfilled in order that conditions (B) and (B*), where

(B*) there is a set T_0 of measure 0 such that for every $K > 0$ there are a number $c > 0$ and a measurable function $h\colon T \to [0, +\infty]$ such that $\int_T h(t)\, d\mu < +\infty$ and

$$\Psi(cx, t) \leqslant K\Phi(x, t) + h(t) \quad \text{for every } x \in X \text{ and } t \in T \setminus T_0,$$

to be equivalent to each other?

To simplify the notations, in the sequel we shall assume that X is a separable space and Φ is a continuous Φ-function.

3.1. THEOREM. *The equivalence*

$$\text{(B)} \Leftrightarrow [\text{(B*) and } \forall_{u > 0} \lim_{c \to 0} \alpha_{u,c}(t) = 0 \text{ for a.e. } t \in T]$$

holds.

Proof. The implication (B) $\Rightarrow$ (B*) is obvious. Let $u > 0$ be an arbitrary fixed number. Write

$$A_n = \bigcap_{m=1}^{\infty} \{t \in T: \alpha_{u,1/m}(t) > 1/n\}, \qquad A = \bigcup_{n=1}^{\infty} A_n.$$

If $\mu(A) = 0$ then

$$\forall_{t \notin A} \ \forall_{n \in N} \ \exists_{m \in N} \ \alpha_{u,1/m}(t) \leqslant 1/n,$$

so, $\lim_{c \to 0} \alpha_{u,c}(t) = 0$ for a.e. $t \in T$.

Now, assume $\mu(A) > 0$. Then $\mu(A_n) > 0$ for some $n \in N$: and hence

$$\int_T \alpha_{u,1/m}(t)\, d\mu \geqslant \int_{A_n} \alpha_{u,1/m}(t)\, d\mu \geqslant (1/n)\, \mu(A_n)$$

for each $m = 1, 2, \ldots$ Therefore, (B) is not true for $0 < \varepsilon < (1/n)\,\mu(A_n)$ and we get a contradiction.

To prove the converse implication we shall show at first that condition (B*) is equivalent to the following one:

$$\forall_{u>0} \ \exists_{c>0} \ \int_T \alpha_{u,c}(t)\, d\mu < +\infty. \tag{10}$$

Suppose that (10) does not holds, i.e.,

$$\exists_{u>0} \ \forall_{m \in N} \ \int_T \alpha_{u,1/m}(t)\, d\mu = +\infty.$$

Then, by 1.5 (for sequences $u_m = u$, $c_m = 1/m$, $m = 1, 2, \ldots$), we can find a sequence of simple functions (g_k) such that

$$\begin{gathered} \{t \in T: g_k(t) \neq 0\} \cap \{t \in T: g_l(t) \neq 0\} = \emptyset \quad \text{for } k \neq l, \\ I_\Phi(g_k) \leqslant u \quad \text{and} \quad I_\Psi\left(\frac{1}{m_k} g_k\right) \geqslant 1 \quad \text{for } k = 1, 2, \ldots \end{gathered} \tag{11}$$

In virtue of (B*) (for $K = 1/2u$) there are a set T_0 of measure 0, a number $c > 0$ and a function $h: T \to [0, +\infty]$ such that $H = \int_T h(t)\, d\mu < +\infty$ and

$$\Psi(cx, t) \leqslant \frac{1}{2u} \Phi(x, t) + h(t) \quad \text{for every } x \in X \text{ and } t \in T \setminus T_0.$$

Let $m \in N$ be an arbitrary number. Then there is $k_0 \in N$ such that $1/m_k \leqslant c$ for $k \geqslant k_0$. Write

$$f_n = \sum_{k=k_0}^{k_0+n-1} g_k, \qquad n = 1, 2, \ldots$$

Then

$$I_\Psi(cf_n) \leqslant \frac{1}{2u} I_\Phi(f_n) + H = \frac{1}{2u} \sum_{k=k_0}^{k_0+n-1} I_\Phi(g_k) + H \leqslant \frac{1}{2} n + H.$$

On the other hand, by (11)

$$I_\Psi(cf_n) \geqslant \sum_{k=k_0}^{k_0+n-1} I_\Psi\left(\frac{1}{m_k} g_k\right) \geqslant n,$$

so $H \geqslant \frac{1}{2} n$ for all $n \in \mathbf{N}$, i.e., $\int_T h(t)\,d\mu = +\infty$.

The obtained contradiction ends the proof of the implication $(\mathrm{B}^*) \Rightarrow (10)$. Since the converse one is obvious, the equivalence $(\mathrm{B}^*) \Leftrightarrow (10)$ holds. Now, it suffices to use the Lebesgue dominated convergence theorem and Remark 2.2 to obtain the thesis.

3.2. LEMMA. *The following conditions are equivalent*:

(i) *the identity embedding* $i\colon L^\Phi(M) \to L^\Psi(M)$ *is bounded*,

(ii) *every ball* $B_\Phi(r)$ *is bounded in the space* $L^\Psi(M)$.

Proof. It suffices to prove the implication (i) $\Rightarrow$ (ii) only. Let $s > 0$ be such a number that the ball $B_\Phi(s)$ is bounded in the space $L^\Psi(M)$ and let $\varepsilon, r > 0$. By 1.7, the ball $B_\Phi(r)$ is additively bounded in $L^\Phi(M)$, i.e., $B_\Phi(r) \subset +^n B_\Phi(s)$ for some $n \in \mathbf{N}$. Moreover, we can find $a > 0$ such that $a \cdot B_\Phi(s) \subset B_\Psi(\varepsilon/n)$, so

$$a \cdot B_\Phi(r) \subset +^n a B_\Phi(s) \subset +^n B_\Psi(\varepsilon/n) \subset B_\Psi(\varepsilon).$$

Thus $B_\Phi(r)$ is a bounded set in $L^\Psi(M)$.

3.3. COROLLARY. *The Musielak–Orlicz space* $L^\Phi(M)$ *is locally bounded if and only if the base* $\{B_\Phi(r)\colon r > 0\}$ *of the topology of the space* $L^\Phi(M)$ *consists of bounded sets.*

Of course, the above lemma and corollary remain true for arbitrary space X and Φ-functions Φ and Ψ.

In the next theorem the following condition will be used:

(12) for every sequence (x_n) of elements of the space X there is a set T_0 of measure 0 such that the implication

$$\sup_{n \in \mathbf{N}} \Phi(x_n, t) < +\infty \Rightarrow \lim_{v \to 0} \sup_{n \in \mathbf{N}} \Psi(v x_n, t) = 0 \quad \text{for every } t \in T \setminus T_0$$

holds.

3.4. THEOREM. $(\mathrm{B}) \Leftrightarrow [(\mathrm{B}^*)$ *and* $(12)]$.

Proof. Assume that conditions (B*) and (12) are satisfied and let $u > 0$. Then $\int_T \alpha_{u,c}(t)\,d\mu < +\infty$ for some $c > 0$. Write (cf. (3), (4))

$$A_{u,c} = \{t \in T:\ 0 < \alpha_{u,c}(t) < +\infty\},$$

$$J_{t,u,c} = \{k \in N:\ t \in P_{u,c}(x_k)\},$$

where the set $\{x_k:\ k \in N\}$ is dense in the space X. Let T_0 be the set taken from condition (12) for the sequence (x_k). Then

$$\alpha_{u,c}(t) = \sup_{k \in J_{t,u,c}} \Psi(cx_k, t) \leqslant K_t < +\infty$$

for $t \in A_{u,c} \setminus T_0$ and some number $K_t > 0$. Hence

$$\sup_{k \in J_{t,u,c}} \Phi(x_k, t) \leqslant \sup_{k \in J_{t,u,c}} u\Psi(cx_k, t) \leqslant u \cdot K_t < +\infty$$

for every $t \in A_{u,c} \setminus T_0$. By (12)

$$\lim_{v \to 0} \alpha_{u,v}(t) = \lim_{v \to 0} \sup_{k \in J_{t,u,v}} \Psi(vx_k, t) \leqslant \lim_{v \to 0} \sup_{k \in J_{t,u,c}} \Psi(vx_k, t) = 0$$

for $t \in A_{u,c} \setminus T_0$.

If $t \in T \setminus A_{u,c}$ then $\alpha_{u,c}(t)$ is equal either 0 or $+\infty$. In the first case $\alpha_{u,v}(t) = 0$ for every $0 < v \leqslant c$. Moreover, the set $\{t \in T:\ \alpha_{u,c}(t) = +\infty\}$ is of measure 0, so $\lim_{v \to 0} \alpha_{u,v}(t) = 0$ for a.e. $t \in T$. Hence, in virtue of Theorem 3.1, condition (B) holds.

Now, we shall show that (B)$\Rightarrow$(12). Suppose that condition (12) is not satisfied, i.e., there is a sequence (x_k) of elements of X such that the set

$$G = \{t \in T:\ \sup_{k \in N} \Phi(x_k, t) < +\infty \text{ and } \lim_{c \to 0} \sup_{k \in N} \Psi(cx_k, t) > 0\}$$

is of positive measure. Write

$$G_n = \{t \in T_n:\ \sup_{k \in N} \Phi(x_k, t) \leqslant n \text{ and } \lim_{c \to 0} \sup_{k \in N} \Psi(cx_k, t) > 1/n\}$$

for $n = 1, 2, \ldots$, where (T_n) is a nondecreasing sequence of subsets of T of finite measure and such that $\mu(T \setminus \bigcup_{n=1}^{\infty} T_n) = 0$. Since $\mu(G \setminus \bigcup_{n=1}^{\infty} G_n) = 0$, there is a number n such that $0 < \mu(G_n) < +\infty$. Now, write $f_k = x_k \chi_{G_n}$ for $k = 1, 2, \ldots$ Then

$$I_\Phi(f_k) \leqslant n\mu(G_n) \quad \text{for each } k \in N. \tag{13}$$

Moreover,

$$\inf_{c > 0} \sup_{k \in N} \Psi(cx_k, t) > 1/n \quad \text{for a.e. } t \in G_n,$$

because $\Psi(c_1 x, t) \leqslant \Psi(c_2 x, t)$ for every $c_1 \leqslant c_2$, $x \in X$ and a.e. $t \in T$. Hence

$$\forall_{m \in N} \exists_{k_m \in N} I_\Psi\left(\frac{1}{m} f_{k_m}\right) \geqslant \frac{1}{n} \mu(G_n),$$

so

$$\inf_{m \in N} I_\Psi\left(\frac{1}{m} f_{k_m}\right) > 0.$$

Let $F = \{f_{k_m}: m \in N\}$. By (13) $F \subset B_\Phi(r)$, where

$$r = \begin{cases} 1 & \text{if } n\mu(G_n) \leqslant 1, \\ n \cdot \mu(G_n) & \text{otherwise.} \end{cases}$$

By Lemma 3.2, the ball $B_\Phi(r)$ is bounded in the space $L^\Psi(M)$, so

$$\forall_{\varepsilon > 0} \exists_{a > 0} a \cdot F \subset a \cdot B_\Phi(r) \subset B_\Psi(\varepsilon).$$

On the other hand, taking a number m_0 such that

$$\frac{1}{m_0} \leqslant a \quad \text{and} \quad \varepsilon \in \left(0, \inf_{m \in N} I_\Psi\left(\frac{1}{m} f_{k_m}\right)\right)$$

we obtain

$$\varepsilon < \inf_{m \in N} I_\Psi\left(\frac{1}{m} \cdot f_{k_m}\right) \leqslant I_\Psi\left(\frac{1}{m_0} \cdot f_{k_{m_0}}\right) \leqslant I_\Psi(a \cdot f_{k_{m_0}}) < \varepsilon.$$

This contradiction ends the proof.

Write

(14) for every sequence (x_n) there is a set T_0 of measure 0 such that

$$\sup_{n \in N} \Psi(x_n, t) < +\infty \Rightarrow \lim_{v \to 0} \sup_{n \in N} \Psi(v x_n, t) = 0$$

for every $t \in T \setminus T_0$.

3.5. LEMMA. *[(B*) and (14)]* $\Rightarrow$ (B).

Proof. In virtue of Theorem 3.1 it suffices to show that $\lim_{v \to 0} \alpha_{u,v}(t) = 0$ for a.e. $t \in T$ and every $u > 0$. Let $u > 0$ be fixed. By (B*), $\int_T \alpha_{u,c}(t)\, d\mu < +\infty$ for some $c > 0$. Let us put $A = \{t \in T: 0 < \alpha_{u,c}(t) < +\infty\}$ and $J_{t,u,c} = \{k \in N: t \in P_{u,c}(x_k)\}$, where (x_k) is a dense subset of X. Then

$$0 < \alpha_{u,c}(t) = \sup_{k \in N} \Psi(c x_k, t) \chi_{P_{u,c}}(x_k)(t)$$
$$= \sup_{k \in J_{t,u,c}} \Psi(c x_k, t) < +\infty \quad \text{for } t \in A.$$

Now let $\varepsilon > 0$ and $t \in A \setminus T_0$, where the set T_0 is taken from (14) for the sequence (x_k). By (14) we obtain the existence of a number $a_t \in (0, 1)$ such that $\Psi(a_t c x_k, t) < \varepsilon$ for $k \in J_{t,u,c}$. Therefore

$$\alpha_{u,v}(t) = \sup_{k \in J_{t,u,v}} \Psi(v x_k, t) < \varepsilon$$

for all $v \leqslant a_t c$ because $J_{t,u,v} \subset J_{t,u,a_t c}$. Thus $\lim_{v \to 0} \alpha_{u,v}(t) = 0$ for a.e. $t \in T$.

3.6. COROLLARY. *If the function Ψ satisfies condition $(+)$ introduced in Corollary 2.4 (for $\Phi = \Psi$), then conditions* (B) *and* (B*) *are equivalent.*

3.7. THEOREM. *Assume that the topology in the space X is generated by a p-homogeneous norm $\|\cdot\|$.*

(a) *If* $\liminf_{\|x\| \to +\infty} \Psi(x, t) > 0$ *for a.e.* $t \in T$, *then*

$$(\mathrm{B}) \Rightarrow \Big[(\mathrm{B}^*) \text{ and } \forall_{\|x_n\| \to \infty} \big(\sup_{n \in N} \Phi(x_n, t) = +\infty \text{ for a.e. } t \in T\big)\Big].$$

(b) $\Big[(\mathrm{B}^*) \text{ and } \forall_{\|x_n\| \to \infty} \big(\sup_{n \in N} \Psi(x_n, t) = +\infty \text{ for a.e. } t \in T\big)\Big] \Rightarrow (\mathrm{B}).$

Proof. (a) Suppose that (B) is satisfied and there are a sequence (x_n) and a set A of positive measure such that $\|x_n\| \to +\infty$ and $\sup_{n \in N} \Phi(x_n, t) < +\infty$ for $t \in A$. Then

$$\sup_{n \in N} \Psi(c x_n, t) \geqslant \liminf_{\|x\| \to +\infty} \Psi(x, t) > 0$$

for every $c > 0$ and $t \in A$. Hence condition (12) does not hold and we get a contradiction.

(b) We shall show that Ψ satisfies condition (14). Let (x_n) be an arbitrary sequence of elements of the space X.

(b1) Suppose that the sequence $(\|x_n\|)$ is bounded, i.e., $\sup_{n \in N} \|x_n\| \leqslant a$ for some $a > 0$. Now, let T_0 be the set taken from Definition 1.1 and assume that $\varepsilon > 0$, $t \notin T_0$. By the continuity of the function $\Psi(\cdot, t)$ at 0 there is a number $\delta > 0$ such that $\Psi(x, t) < \varepsilon$ for all $\|x\| < \delta$. Since $\|v x_n\| < \delta$ for $0 < v < (\delta/a)^{1/p}$, $\lim_{v \to 0} \sup_{n \in N} \Psi(v x_n, t) = 0$, so Ψ satisfies (14).

(b2) The sequence $(\|x_n\|)$ is not bounded. Then there is a subsequence $(\|x_{n_k}\|)$ such that $\|x_{n_k}\| \to +\infty$ as $k \to +\infty$. Write

$$A = \{t \in T\colon \sup_{k \in N} \Psi(x_{n_k}, t) < +\infty\}.$$

By assumption, the set A is of measure 0. Let $t \notin A$. Then

$$+\infty = \sup_{k \in N} \Psi(x_{n_k}, t) \leqslant \sup_{n \in N} \Psi(x_n, t).$$

Thus the condition on the left-hand side of implication (14) is false, so Ψ satisfies condition (14).

3.8. Remark. If $X=(R,|\cdot|)$, then conditions:

(15) $$\forall_{|x_n|\to\infty} \sup_{n\in N} \Psi(x_n, t) = +\infty \quad \text{for a.e. } t\in T$$

and

(16) $$\lim_{|x|\to+\infty} \Psi(x, t) = +\infty \quad \text{for a.e. } t\in T$$

are equivalent.

3.9. COROLLARY. *If the topology of the space X is generated by a p-homogeneous norm $\|\cdot\|$, $\liminf_{\|x\|\to+\infty} \Psi(x, t) > 0$ and $\sup_{x\in X} \Phi(x, t) < +\infty$ for all $t\in G$, where $G\in\Sigma$ is a set of positive measure, then the identity embedding i: $L^{\Phi}(M)\to L^{\Phi}(M)$ is not bounded. In particular, if $\Psi=\Phi$, then the Musielak–Orlicz space $(L^{\Phi}(M), |\cdot|_{\Phi})$ is not locally bounded.*

Write:

(LB*) there is a set T_0 of measure 0 such that for every numbers $\varepsilon>0$ and $K>0$ we can find a number $c>0$ and an integrable function h: $T\to[0, +\infty]$ such that

$$\Phi(cx, t)\leqslant K\Phi(x, t)+h(t) \quad \text{for every } x\in X \text{ and } t\in T\setminus T_0.$$

From the above theorems we obtain the following corollary:

3.10. COROLLARY. (a) (LB) $\Leftrightarrow$ [(LB*) *and* $\forall_{u>0} \lim_{c\to 0}\alpha^*_{u,c}(t) = 0$ *for a.e.* $t\in T$], *where the function* $\alpha^*_{u,c}$ *is defined by* (4) *with* $\Psi=\Phi$.

(b) (LB) $\Leftrightarrow$ [(LB*) *and* (13*)], *where by* (13*) *we denote condition* (13) *with* $\Psi=\Phi$.

(c) *If the topology of the space X is generated by a p-homogeneous norm* $\|\cdot\|$ *and* $\liminf_{\|x\|\to+\infty} \Phi(x, t) > 0$ *for a.e.* $t\in T$, *then*

$$\text{(LB)}\Leftrightarrow\big[\text{(LB*) and } \forall_{\|x_n\|\to\infty} \big(\sup_{n\in N}\Phi(x_n, t) = +\infty \text{ for a.e. } t\in T\big)\big].$$

3.11. Assume $X=R$ and let the functions Φ, Ψ do not depend on the parameter. Then condition (B) (which is equivalent to (B*)) takes the form

$$\forall_{0<K<1}\ \exists_{c>0}\ \exists_{a>0}\ \forall_{x\geqslant a}\ \Psi(cx)\leqslant K\Phi(x)$$

if $\mu(T)<+\infty$; and if $\mu(T)=+\infty$:

$$\forall_{0<K<1}\ \exists_{c>0}\ \forall_{x\geqslant 0}\ \Psi(cx)\leqslant K\Phi(x).$$

If, moreover, $0 < \Phi(x) < +\infty$ for $x > 0$, then the above conditions can be written in the following way ([3], [4]):

$$s(\Phi, \Psi) = \inf_{\lambda > 0} \limsup_{t} \frac{\Psi(\lambda t)}{\Phi(t)} = 0,$$

where $t \to +\infty$ if $\mu(T) < +\infty$ and

$$t \to \begin{cases} 0, \\ +\infty, \end{cases}$$

if $\mu(T) = +\infty$.

References

[1] J. Musielak, W. Orlicz, *On modular spaces*, Studia Math. 18 (1959), 49–65.

[2] H. Nakano, *Generalized modular spaces*, ibidem 31 (1968), 439–449.

[3] P. Turpin, *Opérateurs linéaires entre espaces d'Orlicz non localement convexes*, ibidem 46 (1973), 153–165.

[4] –, *Conditions de bornitude et espaces de fonctions mesurables*, ibidem 56 (1976), 69–91.

[5] M. Wisła, *Continuity of the identity embedding of some Orlicz spaces. I*, Comment. Math. 24 (1984), 171–184.

[6] –, *Continuity of the identity embedding of some Orlicz spaces. II*, Bull. Acad. Polon. Sci. 31 (1983), 143–150.

INSTYTUT MATEMATYKI
UNIWERSYTET im. A. MICKIEWICZA, POZNAŃ
INSTITUTE OF MATHEMATICS
A. MICKIEWICZ UNIVERSITY, POZNAŃ