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On the Cauchy problem for a semilinear system of parabolic equations in a Banach space

Abstract. We consider the Cauchy problem for a semilinear system of parabolic equations (in the Pietrowski's sense) in a Banach space. We prove two existence theorems for the above problem.

1. Introduction. We consider the Cauchy problem

$$(Lu)(x, t) \equiv D_t u(x, t) - \sum_{|k| \leqslant 2p} a_k(x, t) D_x^k u(x, t) \tag{1.1}$$
$$= F\big(x, t, [D_u^{k'} u(x, t)]\big), \quad (x, t) \in G = \boldsymbol{R}^n \times (0, T],$$

$$u(x, 0) = g(x), \qquad x \in \boldsymbol{R}^n, \tag{1.2}$$

where $T > 0$ is a constant, p is a fixed positive integer, and L is a uniformly parabolic operator in Pietrowski's sense [1]. Here u, $D_t u$ and $D_x^k u$ are column vectors with components u_i, $D_t u_i$ and $D_x^k u_i$ ($i = 1, \ldots, N$), respectively, N being a fixed positive integer and

$$D_t = \frac{\partial}{\partial t}, \quad D_x^k = \frac{\partial^{|k|}}{\partial x_{x_1}^{k_1} \ldots \partial x_n^{k_n}}, \quad |k| = k_1 + \ldots + k_n.$$

$[D_x^{k'} u]$ is a column vector whose components are all the derivatives $D_x^{k'} u$ with $|k'| \leqslant 2p-1$. The coefficients

$$a_k(x, t) = [a_k^{hj}(x, t)], \quad |k| \leqslant 2p$$

are square matrices of order N with complex-valued elements. Finally, u, g and the right-hand side of (1.1) take values in the product B^N, where B is a complex Banach space with a norm $\|\cdot\|_B$.

First we extend some properties of [1] to the functions

$$\bar{f}(x, t) = \int_0^t \int_{\boldsymbol{R}^n} \Gamma(x, t; \xi, \tau) f(\xi, \tau) \, d\xi \, d\tau, \tag{1.3}$$

$$\bar{g}(x, t) = \int_{\boldsymbol{R}^n} \Gamma(x, t; \xi, 0) g(\xi) \, d\xi, \tag{1.4}$$

where $\Gamma = [\Gamma^{hj}]$ is the fundamental matrix of the system $Lv = 0$ (see [1] or [2]), g is the function appearing in (1.2), and $f\colon G \to B^N$. Next we prove two existence theorems for problem (1.1), (1.2). The first theorem is a direct generalization of the appropriate theorem of [1] and is obtained with the aid of the Banach fixed point theorem. The proof of the second theorem is based on some Darbo type fixed point theorem using measures of noncompactness.

The results of the present paper involve, particularly, the random case. Namely, let $(\Omega, \mathscr{F}, P)$ be a complete probability space. Then B may be the complex Banach space of all complex random variables u with finite norm

$$\|u\|_B = \Big[\int_\Omega |u(\omega)|^r P(d\omega)\Big]^{1/r}, \quad r \in [1, \infty) \text{ being a constant}$$

or

$$\|u\|_B = \operatorname{ess\,sup}\{|u(\omega)|\colon u \in \Omega\}.$$

2. Properties of functions (1.3), (1.4). We introduce the following assumptions.

(2.I) The coefficients a_k^{hj} ($|k| \leqslant 2p$, $h, j = 1, \ldots, N$) are complex-valued functions defined, continuous and bounded in $\bar{G} = \boldsymbol{R}^n \times [0, T]$, and satisfy the Hölder condition

$$(2.1) \qquad |a_k^{hj}(x, t) - a_k^{hj}(x', t)| \leqslant N_1 |x - x'|^\alpha, \quad x, x' \in \boldsymbol{R}^n,\ t \in [0, T],$$

where

$$|y| = \Big(\sum_{i=1}^n |y_i|^2\Big)^{1/2}, \quad y \in \boldsymbol{R}^n,$$

$\alpha \in (0, 1)$ and $N_1 > 0$ being constants. Moreover, for $|k| = 2p$ the coefficients a_k^{hj} are continuous in t, uniformly with respect to $(x, t) \in \bar{G}$.

(2.II) The operator L is uniformly parabolic in Pietrowski's sense (see [1] or [2]).

Let us write

$$p_1 = (2p)^{-1}, \quad q = 2p(2p-1)^{-1}, \quad \|y\| = \Big(\sum_{i=1}^n |y_i|^q\Big)^{1/q}, \quad y \in \boldsymbol{R}^n.$$

By Theorem 2.1 of [1] (p. 73) there exists a fundamental matrix $\Gamma = [\Gamma^{hj}]$ of the system $Lv = 0$ which satisfies the inequalities

$$(2.2) \qquad |D_x^k \Gamma^{hj}(x, t; \xi, \tau)| \leqslant N_2 (t-\tau)^{-p_1(n+|k|)} \cdot \exp\left[-c\left(\frac{\|x-\xi\|}{(t-\tau)^{p_1}}\right)^q\right],$$

$$|k| \leqslant 2p,\ x, \xi \in \boldsymbol{R}^n,\ 0 \leqslant \tau < t \leqslant T,$$

$$(2.3)\qquad |D_t \Gamma^{hj}(x, t; \xi, \tau)| \leqslant N_2 (t-\tau)^{-p_1(n+2p)} \cdot \exp\left[-c\left(\frac{\|x-\xi\|}{(t-\tau)^{p_1}}\right)^q\right],$$

$$x, \xi \in \boldsymbol{R}^n, \; 0 \leqslant \tau < t \leqslant T$$

and

$$(2.4)\qquad |D_x^k \Gamma^{hj}(x, t; \xi, \tau) - D_x^k \Gamma^{hj}(x', t; \xi, \tau)| \leqslant N_2 |x-x'|^\alpha (t-\tau)^{-p_1(n+|k|+\alpha)} +$$

$$+\left\{\exp\left[-c\left(\frac{\|x-\xi\|}{(t-\tau)^{p_1}}\right)^q\right] + \exp\left[-c\left(\frac{\|x'-\xi\|}{(t-\tau)^{p_1}}\right)^q\right]\right\},$$

$|k| \leqslant 2p-1$, $\xi, x, x' \in \boldsymbol{R}^n$, $0 \leqslant \tau < t \leqslant T$, where $N_2, c > 0$ are some constants.

Now, replace assumption (2.I) by the following one.

(2.III) Assumption (2.I) with condition (2.1) replaced by

$$|a_k^{hj}(x, t) - a_k^{hj}(x', t')| \leqslant N_1 [|x-x'|^\alpha + |t-t'|^{p_1\alpha}], \qquad (x, t), (x', t') \in \bar{G}.$$

Then from [1] (Property 8, p. 104) the estimate follows

$$(2.5)\qquad |D_x^k \Gamma^{hj}(x, t; \xi, \tau) - D_x^k \Gamma^{hj}(x, t'; \xi, \tau)|$$

$$\leqslant N_2 (t'-t)^{p_1\alpha} \cdot (t-\tau)^{-p_1\alpha} \left\{(t-\tau)^{-p_1(n+|k|)} \exp\left[-c\left(\frac{\|x-\xi\|}{(t-\tau)^{p_1}}\right)^q\right] + \right.$$

$$\left. + (t'-\tau)^{-p_1(n+|k|)} \exp\left[-c\left(\frac{\|x-\xi\|}{(t'-\tau)^{p_1}}\right)^q\right]\right\},$$

$$|k| \leqslant 2p-1, \; x, \xi \in \boldsymbol{R}^n, \; 0 \leqslant \tau < t \leqslant t' \leqslant T.$$

In the present paper, B is a complex Banach space with a norm $\|\cdot\|_B$. We also use the product B^m (m being a positive integer) consisting of all column vectors with m components belonging to B. It is clear that B^m with norm defined by

$$\|b\|_B^{(m)} = \sum_{i=1}^{m} \|b_i\|_B, \qquad b \in B^m$$

is a complex Banach space too. For functions of real variables with values in B (or in B^m) the limit, continuity and partial derivatives are taken in the strong sense, and integrals in the Bochner sense.

As in [1] (p. 42), we use the function

$$\varrho(t, a, c) = ac(c^{2p-1} - a^{2p-1} t)^{1/(1-2p)}, \qquad 0 \leqslant t \leqslant T,$$

where $a \in (0, cT^{1/(1-2p)})$ is a constant (c being the constant appearing in (2.2)–(2.5)). Let $\varepsilon \in (0, c)$ be a constant such that

$$(c-\varepsilon)\, T^{1/(1-2p)} > a.$$

Then Lemma 6.1 of [1] (p. 41) implies the following one.

LEMMA 2.1. *There holds true the estimate*

$$\int_{R^n} \exp\left[-c\left(\frac{\|x-\xi\|}{t^{p_1}}\right)^q + a\|\xi\|^q\right]d\xi \leqslant N_3\varepsilon^{-p_1 n} t^{p_1 n}\exp[\varrho(t, a, c-\varepsilon)\|x\|^q]$$

for any $x \in \boldsymbol{R}^n$, $t \in [0, T]$, $N_3 > 0$ *being a constant.*

Lemma 2.1 and the equality

$$\varrho(t-\tau, \varrho(\tau, a, c), c) = \varrho(t, a, c)$$

imply the following lemma, immediately.

LEMMA 2.2. *There holds true the estimate*

$$\int_{R^n} \exp\left[-c\left(\frac{\|x-\xi\|}{(t-\tau)^{p_1}}\right)^q + \varrho(\tau, a, c-\varepsilon)\|\xi\|^q\right]d\xi$$
$$\leqslant N_3\varepsilon^{-p_1 n}(t-\tau)^{p_1 n}\exp[\varrho(t, a, c-\varepsilon)\|x\|^q], \quad x \in \boldsymbol{R}^n,\ 0 \leqslant \tau < t \leqslant T.$$

We introduce the following assumption.

(2.IV) The function g: $\boldsymbol{R}^n \to B^N$ is continuous and

$$\|g(x)\|_B^{(N)} \leqslant N_g \exp(a\|x\|^q), \quad x \in \boldsymbol{R}^n,$$

where $N_g > 0$ is a constant.

Arguing as in [1] (Sec. 1, Chapter III) or [2], and using estimates (2.2)–(2.5) and Lemma 2.1, one can obtain the following theorem.

THEOREM 2.1. *Let assumptions* (2.I), (2.II) *and* (2.IV) *be satisfied. Then the function* $\bar{g}$ *defined by* (1.3) *has the following properties*:

(a) *All the derivatives* $D_x^k \bar{g}$, $|k| \leqslant 2p$ *and* $D_t \bar{g}$ *are continuous in* G.

$$(L\bar{g})(x, t) = 0, \quad (x, t) \in G$$

and

$$\lim_{t \searrow 0} \bar{g}(x, t) = g(x), \quad x \in \boldsymbol{R}^n,$$

where the convergence is uniform in every bounded domain of $\boldsymbol{R}^n$.

(b) *There hold the inequalities*

$$\|D_x^k \bar{g}(x, t)\|_B^{(N)} \leqslant N_4 N_g t^{-p_1|k|}\exp[\varrho(t, a, c-\varepsilon)\|x\|^q], \quad (x, t) \in G,\ |k| \leqslant 2p$$

and

$$\|D_x^k \bar{g}(x, t) - D_x^k \bar{g}(x', t)\|_B^{(N)} \leqslant N_4 N_g |x-x'|^\alpha t^{-p_1(|k|+\alpha)}\exp[\varrho(t, a, c-\varepsilon)\|x'\|^q]$$

for any $x, x' \in \boldsymbol{R}^n$, $\|x\| \leqslant \|x'\|$, $0 < t \leqslant T$, $|k| \leqslant 2p-1$, *where* $N_4 > 0$ *is a constant.*

(c) *If assumption* (2.I) *is replaced by* (2.III), *then*

$$\|D_x^k \bar{g}(x, t) - D_x^k \bar{g}(x, t')\|_B^{(N)} \leqslant N_4 N_g (t'-t)^{p_1 \alpha} t^{-p_1(|k|+\alpha)} \cdot \exp[\varrho(t', a, c-\varepsilon)\|x\|^q],$$
$$x \in \boldsymbol{R}^n,\ 0 < t \leqslant t' \leqslant T,\ |k| \leqslant 2p-1.$$

Now we need the following assumption.

(2.V) The function $f\colon G \to B^N$ is continuous and satisfies the inequality

$$\sup\{t^\beta \|f(x, t)\|_B^{(N)} \exp[-\varrho(t, a, c-\varepsilon)\|x\|^q]\colon (x, t) \in G\} < \infty$$

and the generalized Hölder condition [1] (p. 68) with respect to x in every cylinder

$$G_{r,\tau} = \{(x, t)\colon |x| \leqslant r,\ t \in [\tau, T]\}, \quad r > 0,\ \tau \in (0, T), \tag{2.6}$$

$\beta \in (0, 1)$ being a constant.

Using estimates (2.2)–(2.5) and Lemma 2.2, and arguing like in [1] (p. 238), we get the following theorem.

Theorem 2.2 *Let assumptions* (2.I), (2.II) *and* (2.V) *be satisfied. Then the function* $\bar{f}$ *defined by* (1.4) *has the properties*:

(i) *All the derivatives* $D_x^k \bar{f}$, $|k| \leqslant 2p$ *and* $D_t \bar{f}$ *are continuous in* G,

$$(L\bar{f})(x, t) = f(x, t), \quad (x, t) \in G$$

and

$$\lim_{t \searrow 0} \bar{f}(x, t) = 0, \quad x \in \boldsymbol{R}^n,$$

where the convergence is uniform in every bounded domain of $\boldsymbol{R}^n$.

(ii) *For any* $\gamma \geqslant 1$ *write*

$$N(\gamma, f) = \sup_{(x,t) \in G} \{t^\beta \|f(x, t)\|_B^{(N)} \exp[-\gamma t - \varrho(t, a, c-\varepsilon)\|x\|^q]\}.$$

Then for any $|k| \leqslant 2p-1$ *there hold the estimates*

$$e^{-\gamma t}\|D_x^k \bar{f}(x, t)\|_B^{(N)} \leqslant N_5 N(\gamma, f) t^{\delta-\beta-p_1|k|} \gamma^{-1+\delta} \cdot \exp[\varrho(t, a, c-\varepsilon)\|x\|^q], \quad (x, t) \in G$$

and

$$e^{-\gamma t}\|D_x^k \bar{f}(x, t) - D_x^k \bar{f}(x', t)\|_B^{(N)} \leqslant N_5 N(\gamma, f) \gamma^{-1+\delta} \cdot t^{\delta-\beta-p_1(|k|+\alpha)} |x-x'|^\alpha \exp[\varrho(t, a, c-\varepsilon)\|x\|^q]$$

for any $x, x' \in \boldsymbol{R}^n$, $\|x\| \leqslant \|x'\|$, $t \in [0, T]$, *where*

$$\delta \in (\max\{\beta, p_1(2p-1+\alpha)\}, 1), \quad N_5 > 0$$

are constants independent of γ *and* $N(\gamma, f)$.

(iii) *If assumption* (2.I) *is replaced by* (2.III), *then for any* $|k| \leqslant 2p-1$ *there holds the estimate*

$$e^{-\gamma t'}\|D_x^k \bar{f}(x, t)-D_x^k \bar{f}(x, t')\|_B^{(N)}$$
$$\leqslant N_5 N(\gamma, f)\gamma^{-1+\delta}\cdot t^{\delta-\beta-p_1(|k|+\alpha)}(t'-t)^{p_1\alpha}\exp[\varrho(t', a, c-\varepsilon)\|x\|^q]$$

for any $x \in \mathbf{R}^n$, $0 < t \leqslant t' \leqslant T$.

Note that the generalized Hölder condition for f is used only in the proof of the existence of the derivatives $D_x^k \bar{f}$, $|k| = 2p$, and $D_t \bar{f}$. Therefore the condition above is superfluous in assertions (ii) and (iii).

3. The first existence theorem for problem (1.1), (1.2). In [1] (pp. 241–250) there was considered the "scalar" problem (1.1), (1.2) with a generalized condition (1.2). With the aid of the method of successive approximations applied to an appropriate system of Volterra's integral equations there was proved the existence of solutions of the above problem in various classes of functions. All those results can be extended to a Banach space case. In this paper we restrict ourselves to proving the existence of classical solutions of problem (1.1), (1.2) in a Banach space B (1). In the present section we obtain an existence theorem for that problem with the aid of the Banach fixed point theorem applied to the same system of integral equations as in [1].

First, retaining the notation of the previous sections, we introduce some functional spaces needed in our further consideration. By $Z(\beta)$, $\beta \in [0, 1)$, we denote the Banach space consisting of all continuous functions $z\colon G \to B$ with finite norm

$$\|z\|_{B,\beta} = \sup\{\|t^\beta z(x, t)\exp[-\varrho(t, a, c-\varepsilon)\|x\|^q]\|_B\colon (x, t) \in G\}.$$

It will also be used the norm

$$\|z\|_{B,\beta,\gamma} = \sup\{\|t^\beta z(x, t)\exp[-\gamma t-\varrho(t, a, c-\varepsilon)\|x\|^q]\|_B\colon (x, t) \in G\}$$

($\gamma \in R$ being a constant) equivalent to the above norm. By $Z(\beta, m)$ (m being a positive integer) we denote the Banach space of all continuous functions $z\colon G \to B^m$ with norm

$$\|z\|_{B,\beta}^{(m)} = \sum_{i=1}^{m} \|z_i\|_{B,\beta}.$$

We shall also use the norm

$$\|z\|_{B,\beta,\gamma}^{(m)} = \sum_{i=1}^{m} \|z_i\|_{B,\beta,\gamma}$$

(1) By the *classical solution* of problem (1.1), (1.2) we mean a continuous function $u\colon \bar{G} \to B^N$ possessing the derivatives $D_x^k u$, $|k| \leqslant 2p$, $D_t u$ continuous in G and satisfying (1.1), (1.2) pointwise.

equivalent to the above norm. Let us denote

$$\bar{\beta} = (\beta_0, \beta_1, \ldots, \beta_{2p-1}), \qquad \beta_i \in [0, 1), \qquad \bar{m} = (m_0, m_1, \ldots, m_{2p-1}),$$

where $m_0 = N$ and $m_i N^{-1}$ $(i = 1, \ldots, 2p-1)$ is the number of all multi-indices $k = (k_1, \ldots, k_n)$ such that $|k| = i$. By $Z(\bar{\beta}, \bar{m})$ we denote the product of the spaces $Z(\beta_i, m_i)$ $(i = 0, 1, \ldots, 2p-1)$ consisting of all column vectors with components $z_i \in Z(\beta_i, m_i)$. The above product with the norm

$$\|z\|_{B,\bar{\beta}}^{(\bar{m})} = \sum_{i=0}^{2p-1} \|z_i\|_{B,\beta_i}^{(m_i)}$$

is, of course, a Banach space. That norm is equivalent to the norm

$$\|z\|_{B,\bar{\beta},\gamma}^{(\bar{m})} = \sum_{i=0}^{2p-1} \|z_i\|_{B,\beta_i,\gamma}^{(m_i)}.$$

We need the following assumption.

(3.I) The function $F\colon G \times B^m \to B^N$ is continuous and satisfies the Lipschitz condition

$$\|F(x, t, z) - F(x, t, z')\|_B^{(N)} \leqslant K_1 \|z - z'\|_B^{(m)}, \qquad (x, t) \in G, \ z, z' \in B^m$$

and the inequality

$$\|F(x, t, 0)\|_B^{(N)} \leqslant K_2 t^{-\beta} \exp[-\varrho(t, a, c-\varepsilon)\|x\|^q], \qquad (x, t) \in G,$$

where $m = m_0 + m_1 + \ldots + m_{2p-1}$ and $K_1, K_2 > 0$, $\beta \in (0, 1)$ are some constants. Moreover, $F(x, t, z)$ satisfies the generalized Hölder condition with respect to x in every bounded domain

$$G_{r,\tau} \times H \subset G \times B^m \qquad \text{(see (2.6))}.$$

THEOREM 3.1. *If assumptions* (2.I), (2.II), (2.IV) *and* (3.I) *are satisfied, then there exists a solution of problem* (1.1), (1.2).

Proof. As in [1] (p. 242), we consider the system of integral equations written in the form

(3.1) $$v(x, t) = \int_{R^n} \bar{\Gamma}(x, t; \xi, 0) g(\xi) d\xi + \int_0^t d\tau \int_{R^n} \bar{\Gamma}(x, t; \xi, \tau) F(\xi, \tau, v(\xi, \tau)) d\xi,$$

where $\bar{\Gamma} = [D_x^k \Gamma]$ is the appropriate $m \times N$ matrix and v is a column vector with m components. We write that system shortly in the operator form

(3.2) $$v = Ev.$$

With the aid of the Banach fixed point theorem it will be proved the existence of a unique solution of equation (3.2) in the space $Z = Z(\bar{\beta}, \bar{m})$ with

$\beta_i = p_1 i$ $(i = 0, 1, \ldots, 2p-1)$. In this space we shall use the norm $\|\cdot\|^{(\bar{m})}_{B,\bar{\beta},\gamma}$, where the constant $\gamma \geq 1$ will be specified later.

It follows from Theorem 2.1 that the function w_1 defined by the formula

$$w_1(x, t) = \int_{R^n} \bar{\Gamma}(x, t; \xi, 0) g(\xi) d\xi \tag{3.3}$$

belongs to Z. By assumption (3.I), for any $v \in Z$ the function $F(x, t, v(x, t))$ is continuous in G and satisfies the inequality

$$\|F(x, t, v(x, t))\|^{(N)}_B \leq K_2 t^{-\beta} \exp[\varrho(t, a, c-\varepsilon)\|x\|^q] + K_1 \|v(x, t)\|^{(m)}_B, \quad (x, t) \in G. \tag{3.4}$$

Setting $\delta_0 = \max\{\beta, \beta_{2p-1}\} = \max\{\beta, 1-p_1\}$, it follows from (3.4) that

$$t^{\delta_0} \|F(x, t, v(x, t))\|^{(N)}_B \exp[-\gamma t - \varrho(t, a, c-\varepsilon)\|x\|^q] \leq K_3 + K_4 \|v\|^{(\bar{m})}_{B,\bar{\beta},\gamma}, \quad (x, t) \in G,$$

where K_3, K_4 are positive constants independent of $\gamma \geq 1$. Hence, by Theorem 2.2, the function w_2 defined by the formula

$$w_2(x, t) = \int_0^t d\tau \int_{R^n} \bar{\Gamma}(x, t; \xi, \tau) F(\xi, \tau, v(\xi, \tau)) d\xi \tag{3.5}$$

belongs to Z. Thus we have proved that E maps Z into itself.

Using assumption (3.I), we get

$$t^{\delta_0} \|F(x, t, v(x, t)) - F(x, t, v'(x, t))\|^{(N)}_B \exp[-\gamma t - \varrho(t, a, c-\varepsilon)\|x\|^q] \leq K_5 \|v - v'\|^{(\bar{m})}_{B,\bar{\beta},\gamma}, \quad v, v' \in Z, \ (x, t) \in G,$$

$K_5 > 0$ being a constant. Hence, taking into consideration Theorems 2.1 and 2.2 and relations (3.1), (3.2), we find that

$$\|Ev - Ev'\|^{(\bar{m})}_{B,\bar{\beta},\gamma} \leq K_6 \gamma^{-1+\delta_1} \|v - v'\|^{(\bar{m})}_{B,\bar{\beta},\gamma}, \quad v, v' \in Z, \tag{3.6}$$

where $\delta_1 \in (0, 1)$ and $K_6 > 0$ are some constants independent of $\gamma \geq 1$. Now choose

$$\gamma = \max\{1, (2K_6)^{1/(1-\delta_1)}\}.$$

Then it follows from (3.6) that

$$\|Ev - Ev'\|^{(\bar{m})}_{B,\bar{\beta},\gamma} \leq \tfrac{1}{2} \|v - v'\|^{(\bar{m})}_{B,\bar{\beta},\gamma}, \quad v, v' \in Z.$$

Consequently, by the Banach fixed point theorem, there exists a unique solution $v \in Z$ of equation (3.2). Note that v is a column vector with components v_k, $|k| \leq 2p-1$, where each component v_k is a column vector with N components. Let us introduce the function

$$u(x, t) = \begin{cases} v_{(0,\ldots,0)}(x, t), & (x, t) \in G, \\ g(x), & x \in R^n, \ t = 0. \end{cases}$$

As in the scalar case, using assumptions (2.IV), (3.I) and Theorems 2.1 and 2.2, one can show that u is a solution of problem (1.1), (1.2). This completes the proof.

4. The second existence theorem for problem (1.1), (1.2). We use the notation of the previous sections. Moreover, we introduce further notation. Let μ be the Hausdorff measure of noncompactness in B (see, e.g. [3]). For any bounded set $V \subset B^m$ we define

$$\mu^{(m)}(V) = \sum_{i=1}^{m} \mu(V_i),$$

where

$$V_i = \{v_i\colon v_i \text{ is the } i\text{th component of some } v \in V\}. \tag{4.1}$$

By $C_0(\bar{G}, B)$ we denote the Banach space of all continuous and bounded functions $u\colon \bar{G} \to B$ with the norm

$$\|u\|_{B,G} = \sup\{\|u(x, t)\|_B\colon (x, t) \in G\}.$$

We shall also use the norm

$$\|u\|_{B,G,\gamma} = \sup\{\|e^{-\gamma t} u(x, t)\}\|_B\colon (x, t) \in G\}$$

($\gamma \in R$ being a constant) equivalent to the above norm. By $\mu_{0,\gamma}$ (2) we denote the Hausdorff measure of noncompactness in $C_0(\bar{G}, B)$ with respect to the norm $\|\cdot\|_{B,G,\gamma}$. $M_{\beta,\gamma}$, $\beta \in [0, 1)$, denotes the Hausdorff measure of noncompactness in $Z(\beta)$ with respect to the norm $\|\cdot\|_{B,\beta,\gamma}$. Finally, for any bounded set $V \subset Z(\beta, m)$ (resp. $V \subset Z(\bar{\beta}, \bar{m})$) we define

$$M_{\beta,\gamma}^{(m)}(V) = \sum_{i=1}^{m} M_{\beta,\gamma}(V_i) \quad \left(\text{resp. } M_{\bar{\beta},\gamma}^{(\bar{m})}(V) = \sum_{i=1}^{2p-1} M_{\beta_i,\gamma}^{(m_i)}(V_i)\right),$$

where V_i is given by (4.1).

Now we state some lemmas needed in our consideration concerning problem (1.1), (1.2).

LEMMA 4.1. *The function $M_{\bar{\beta},\gamma}^{(\bar{m})}$ satisfies assumption* (4.I) *of* [4] *(with B and μ_1 replaced by $Z(\bar{\beta}, \bar{m})$ and $M_{\bar{\beta},\gamma}^{(\bar{m})}$, respectively).*

The lemma follows easily from Lemma 2 of [3] and Lemma 4.5 of [4].

LEMMA 4.2. *If U is a bounded set of $C_0(\bar{G}, B)$, then $\mu_{0,\gamma}(U) = \mu_0(U_\gamma)$, where*

$$U_\gamma = \{v \in C_0(\bar{G}, B)\colon v(x, t) = e^{-\gamma t} u(x, t),\ u \in U\}.$$

(2) In the case $\gamma = 0$ we omit that subscript in all introduced symbols.

LEMMA 4.3. *Let U be a bounded set of $Z(\beta)$, $\beta \in [0, 1)$ and assume that for any $u \in U$ we have*

$$\|e^{-\gamma t} u(x, t)\|_B \leqslant C_1 t^{-\beta+\varepsilon_2} \exp[\varrho(t, a, c-\varepsilon_1)\|x\|^q], \quad (x, t) \in G,$$

where $\gamma \in R$, $C_1 > 0$, $\varepsilon_1 \in (0, \varepsilon)$ and $\varepsilon_2 > 0$ are constants. Denote by V the set of all functions

$$v(x, t) = \begin{cases} t^\beta u(x, t) \exp[-\varrho(t, a, c-\varepsilon)\|x\|^q], & (x, t) \in G, \\ 0, \quad x \in \boldsymbol{R}^n, \ t = 0, \end{cases}$$

where $u \in U$. Then we have $M_{\beta,\gamma}(U) = \mu_{0,\gamma}(V)$.

Lemmas 4.2 and 4.3 can be proved in the standard manner.

LEMMA 4.4. *Let assumptions of Lemma 4.3 be satisfied and suppose that in every cylinder $G_{r,\tau}$ (defined by (2.6)) all the functions of U are equicontinuous. Then*

$$M_{\beta,\gamma}(U) = \sup\{t^\beta \mu(U(x, t)) \exp[-\gamma t - \varrho(t, a, c-\varepsilon)\|x\|^q]: (x, t) \in G\},$$

where $U(x, t) = \{u(x, t): u \in U\}$.

Proof. Let us consider the set W of all functions

$$w(x, t) = \begin{cases} t^\beta u(x, t) \exp[-\gamma t - \varrho(t, a, c-\varepsilon)\|x\|^q], & (x, t) \in G, \\ 0, \quad x \in \boldsymbol{R}^n, \ t = 0, \end{cases}$$

where $u \in U$. It follows from Lemmas 4.2 and 4.3 that

$$M_{\beta,\gamma}(U) = \mu_{0,\gamma}(V) = \mu_0(W). \tag{4.2}$$

For any $w \in W$ we have

$$\|w(x, t)\|_B \leqslant C_1 t^{\varepsilon_2} \exp[-\varepsilon_3 \|x\|^q], \quad (x, t) \in \bar{G},$$

where $\varepsilon_3 > 0$ is a constant such that

$$-\varrho(t, a, c-\varepsilon) + \varrho(t, a, c-\varepsilon_1) \leqslant -\varepsilon_3, \quad t \in [0, T].$$

Moreover, all the functions of W are equicontinuous in every cylinder $G_{r,\tau}$. Therefore, arguing further like in the proof of Lemma 4.8 of [5], one can show that

$$\mu_0(W) = \sup\{\mu(W(x, t)): (x, t) \in \bar{G}\}.$$

Hence by (4.2), the assertion of Lemma 4.4 follows.

Now we introduce the following assumptions.

(4.I) Assumption (3.I) with the constant ε (appearing in the function ϱ) replaced by a constant $\varepsilon_1 \in (0, \varepsilon)$ and with the Lipschitz condition replaced by the condition

$$\|F(x, t, z)-F(x, t, z')\|_B^{(N)} \leqslant K_1 K\left(\|z-z'\|_B^{(m)}\right), \qquad (x, t)\in G,\ z, z'\in B^m,$$

where $K_1 > 0$ is a constant and

$$K(s) = \begin{cases} s^\nu, & 0 \leqslant s \leqslant 1, \\ s, & s > 1, \end{cases}$$

$\nu\in(0, 1)$ being a constant. Moreover, for any bounded set $G_{r,\tau}\times H$ of $G\times B^m$ the function $F(x, t, z)$ is continuous in $t\in[\tau, T]$, uniformly with respect to (x, z), $|x|\leqslant r$, $z\in H$.

(4.II) There is a constant $K_1' > 0$ such that for any bounded set $V\subset B^m$ we have

$$\mu^{(N)}\left(F(x, t, V)\right) \leqslant K_1' \mu^{(m)}(V), \qquad (x, t)\in G,$$

where $F(x, t, V) = \{F(x, t, v)\colon v\in V\}$.

THEOREM 4.1. *If assumptions* (2.I), (2.III), (2.IV), (4.I) *and* (4.II) *are satisfied, then there exists a solution of problem* (1.1), (1.2).

Proof. Like in the proof of Theorem 3.1 we consider equation (3.2) in the Banach space $Z = Z(\bar\beta, \bar m)$, where $\bar m$ has the same meaning as in Section 3 and

$$\bar\beta = (\beta_0, \beta_1, \ldots, \beta_{2p-1}), \qquad \beta_i = p_1 i+\theta, \qquad i = 0, 1, \ldots, 2p-1,$$

$\theta\in(0, p_1\alpha)$ being a constant. Denote by $K_0 = K_0(r_0, \gamma)$, $r_0 \geqslant 1$, $\gamma\geqslant 1$ the set of all functions $v\in Z$ such that for $i = 0, 1, \ldots, 2p-1$ we have

$$e^{-\gamma t}\|v_i(x, t)\|_B^{(m_i)} \leqslant r_0 t^{-p_1 i}\exp[\varrho(t, a, c-\varepsilon_1)\|x\|^q], \qquad (x, t)\in G,$$

$$e^{-\gamma t'}\|v_i(x, t)-v_i(x', t')\|_B^{(m_i)}$$
$$\leqslant r_0 t^{-p_1(i+\alpha)}\cdot[|x-x'|^\alpha+|t-t'|^{p_1\alpha}]\exp[\varrho(t', a, c-\varepsilon_1)\|x\|^q]$$

for any $(x, t), (x', t')\in G$, $\|x\|\leqslant\|x'\|$, $t\leqslant t'$, where r_0 and γ are constants which will be specified later. Note that K_0 is a closed, convex and bounded set of Z.

It follows from (4.I) that for any $v\in Z$ the function $F\left(x, t, v(x, t)\right)$ is continuous in G and there holds (3.4) with ε replaced by ε_1 and with K_2 replaced by some other positive constant. Hence we have

$$t^{\delta_0}\|F(x, t, v(x, t))\|_B^{(N)}\exp[-\gamma t-\varrho(t, a, c-\varepsilon_1)\|x\|^q] \leqslant C_2 r_0, \qquad v\in K_0,\ (x, t)\in G, \tag{4.3}$$

where $\delta_0 = \max\{\beta, 1-p_1(1-\alpha)\}$ and $C_2 > 0$ is a constant independent of γ and r_0. Consequently, by Theorem 2.2, every function w_2 defined by (3.5) for any $v \in K_0$ satisfies the inequalities

$$\|w_{2i}(x, t)\|_B^{(m_i)} \exp[-\gamma t - \varrho(t, a, c-\varepsilon_1)\|x\|^q] \leqslant C_3 r_0 \gamma^{-1+\delta} t^{-p_1 i}, \quad (x, t) \in G,$$

$$\|w_{2i}(x, t) - w_{2i}(x', t')\|_B^{(m_i)} \exp[-\gamma t' - \varrho(t', a, c-\varepsilon_1)\|x'\|^q]$$
$$\leqslant C_3 r_0 \gamma^{-1+\delta} t^{-p_1(i+\alpha)} [|x-x'|^\alpha + |t-t'|^{p_1 \alpha}]$$

for any $(x, t), (x', t') \in G$, $\|x\| \leqslant \|x'\|$, $t \leqslant t'$, where $i = 0, 1, \ldots, 2p-1$, $\delta = 2^{-1}(1+\delta_0)$, and $C_3 > 0$ is a constant independent of γ and r_0. It follows from Theorem 2.1 that the function w_1 defined by (3.3) satisfies the above inequalities with $C_3 r_0 \gamma^{-1+\delta}$ replaced by some other constant $C_4 > 0$ independent of γ and r_0.

Now let us choose

$$r_0 = \max\{1, 2C_4\}, \tag{4.4}$$

$$\gamma \geqslant \gamma_0 = \max\{1, (2C_3)^{1/(\delta-1)}\}. \tag{4.5}$$

Then it follows from the above consideration that $w = w_1 + w_2 = Ev \in K_0$ for any $v \in K_0$, i.e., E maps K_0 into itself. Using assumption (4.I) and Theorem 2.2 and arguing like in the proof of inequality (3.6), we get

$$\|Ev - Ev'\|_{B,\bar{\beta},\gamma}^{(\bar{m})} \leqslant C_5 K(\|v - v'\|_{B,\bar{\beta},\gamma}^{(\bar{m})}), \qquad v, v' \in K_0,$$

$C_5 > 0$ being a constant. This proves the continuity of E in K_0.

Now let us take any set $V \subset K_0$ and put

$$W_2 = \{w_2 \text{ defined by (3.5)}: v \in V\},$$

$$H_v(x, t) = F(x, t, v(x, t)), \qquad v \in V, \ (x, t) \in G,$$

$$H_V = \{H_v: v \in V\}, \qquad H_V(x, t) = \{H_v(x, t): v \in V\}.$$

Let $\delta' \in (\delta_0, \delta)$ be a constant. Inequality (4.3) implies that H_V is a bounded set of $Z(\delta', N)$ and for any $v \in V$ we have

$$\|H_v(x, t)\|_B^{(N)} \leqslant C_6 t^{-\delta_0} \exp[-\gamma t - \varrho(t, a, c-\varepsilon_1)\|x\|^q], \quad (x, t) \in G,$$

where $C_6 = C_2 r_0$. Recalling the definition of K_0 and assumption (4.I), it follows that all the functions H_v, $v \in V$, are equicontinuous in every cylinder $G_{r,\tau}$ (defined by (2.6)). Therefore, using Lemma 4.4 and assumption (4.II), we get

$$M_{\delta',\gamma}^{(N)}(H_V) = \sum_{i=1}^{N} M_{\delta',\gamma}((H_V)_i)$$
$$\leqslant C_7 \sup\{t^{\delta'} \mu^{(m)}(V(x, t)) \exp[-\gamma t - \varrho(t, a, c-\varepsilon)\|x\|^q]\}$$

and

$$\mu^{(m)}(V(x,t)) = \sum_{i=0}^{2p-1} \mu^{(m_i)}(V_i(x,t))$$
$$\leqslant \sum_{i=0}^{2p-1} t^{-\beta_i} M^{(m_i)}_{\beta_i,\gamma}(V_i)\exp[\gamma t+\varrho(t,a,c-\varepsilon)\|x\|^q],$$

$C_7 > 0$ being a constant independent of γ. Hence, in view of $\beta_i < \delta_0$, $i = 0, 1, \ldots, 2p-1$, we have

$$M^{(N)}_{\delta',\gamma}(H_V) \leqslant C_8 M^{(\bar{m})}_{\bar{\beta},\gamma}(V), \tag{4.6}$$

$C_8 > 0$ being a constant independent of γ.

Let us take any $\eta > 0$. Then there exist functions $\varphi_s \in Z(\delta', N)$, $s = 1, \ldots, s_0$, such that for any $v \in V$ we have

$$\|H_v - \varphi_{s_1}\|^{(N)}_{B,\delta'\gamma} \leqslant M^{(N)}_{\delta',\gamma}(H_V)+\eta \tag{4.7}$$

for some $s_1 \in \{1, \ldots, s_0\}$ (depending on v). According to Theorem 2.2 the functions

$$\Phi_s(x,t) = \int_0^t d\tau \int_{\mathbf{R}^n} \bar{\Gamma}(x,t;\xi,\tau)\varphi_s(\xi,\tau)\,d\xi, \qquad (x,t)\in G,\ s = 1, \ldots, s_0$$

(see (3.1)) belong to $Z(\bar{\beta}, \bar{m})$ and moreover, by (4.7) and (4.6), we have

$$\|w_2 - \Phi_{s_1}\|^{(\bar{m})}_{B,\bar{\beta},\gamma} \leqslant C_9\gamma^{-1+\delta} M^{(\bar{m})}_{\bar{\beta},\gamma}(V)+C_9\eta,$$

where w_2 is defined by (3.5) and $C_9 > 0$ is a constant independent of γ. Thus we have proved that

$$M^{(\bar{m})}_{\bar{\beta},\gamma}(W_2) \leqslant C_9\gamma^{-1+\delta} M^{(\bar{m})}_{\bar{\beta},\gamma}(V).$$

Hence, taking into account the equality $M^{(\bar{m})}_{\bar{\beta},\gamma}(\{w_1\}) = 0$ (w_1 being defined by (3.3)), it follows that

$$M^{(\bar{m})}_{\bar{\beta},\gamma}(EV) = M^{(\bar{m})}_{\bar{\beta},\gamma}(\{w_1\}+W_2) \leqslant C_9\gamma^{-1+\delta} M^{(\bar{m})}_{\bar{\beta},\gamma}(V).$$

Consequently, choosing

$$\gamma = \max\{\gamma_0, (2C_9)^{1/(\delta-1)}\} \qquad \text{(see (4.5))}$$

we get

$$M^{(\bar{m})}_{\bar{\beta},\gamma}(EV) \leqslant \tfrac{1}{2}M^{(\bar{m})}_{\bar{\beta},\gamma}(V), \qquad V\subset K_0 = K_0(r_0,\gamma),$$

where r_0 is defined by (4.4). Moreover, by the previous consideration, the operator E maps $K_0(r_0, \gamma)$ into itself and is continuous. Therefore, using Lemma 4.1 and Lemma 4.2 of [4] (the Darbo type fixed point theorem), we find that there exists a solution v of equation (3.2) in K_0. The function v

determines a solution u of problem (1.1), (1.2) in the same manner as in the proof of Theorem 3.1. This completes the proof.

Remark. A simple example of a function F satisfying assumption (4.II) is the case $F = F^{(1)} + F^{(2)}$, where (roughly speaking) $F^{(1)}$ is a Lipschitz function with respect to the functional argument and $F^{(2)}$ is a completely continuous function with respect to that argument.

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